Contents 25

partial differential **equations**

Learning **outcomes**

needs doing

Time **allocation**

You are expected to spend approximately thirteen hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Partial Differential Equations

25.1

Introduction

A **partial differential equation** (PDE) is one which involves one dependent variable and two or more independent variables. The independent variables may be space variables only or one or more space variables and time. PDEs arise in many situations involving natural phenomena.

The subject of PDEs is a very large one. We shall discuss only a few special PDEs which occur in a wide range of applied problems.

Prerequisites

Before starting this Section you should ...

① be able to carry out partial differentiation

② have met constant coefficient ordinary differential equations

Learning Outcomes

After completing this Section you should be able to ...

✓ verify solutions of given partial differential equations arising in engineering and science

1. Introduction

You have already studied Ordinary Differential Equations (ODEs) and have learnt how to obtain the solution of certain types. Since a knowledge of the solution of certain ODEs (viz those with constant coefficients) will be required in solving Partial Differential Equations (PDEs), we will begin this unit reminding you of some important results:

Key Point

The first order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

has general solution

$$y = Ae^{kx}$$

Here k is a constant which can be positive or negative.

In the above Key Point the quantity A in the general solution is a constant. To obtain the value of A we would have to know the value of y at a given value of x, perhaps $x = 0$. In other words we need to know an **initial condition**.

Find y as a function of x if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2y$$

and if $y(0) = 3$.

Your solution

From the Key Point with $k = -2$ we have the general solution

$$y = Ae^{-2x}$$

Putting $x = 0$ and $y = 3$ into this we obtain $3 = Ae^0$ i.e. $A = 3$ so the solution to the given initial value problem is

$$y = 3e^{-2x}$$

We shall also need to be familiar with solutions to second order, homogeneous, constant coefficient ODEs.

Key Point

An equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \ldots \quad (1)$$

(where a, b, c are constants) has an auxiliary equation

$$am^2 + bm + c = 0 \qquad \ldots \quad (2)$$

(obtained by the trial solution $y = e^{mx}$.)

The general solution of (1) then depends on the solutions (or roots) of the quadratic equation (2).

(a) If (2) has real, distinct roots $m = m_1$ and $m = m_2$ then

$$y = Ae^{m_1x} + Be^{m_2x}$$

(b) If (2) has a repeated root $m = m_1$ then

$$y = (A + Bx)e^{m_1x}$$

(c) If (2) has complex conjugate roots

$$m = \alpha \pm j\beta$$

then

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

In particular if in (1) in the Key Point we put $b = 0$ and then divide through by a we have an ODE of the form

$$\frac{d^2y}{dx^2} + \frac{c}{a}y = 0$$

The quantity $\dfrac{c}{a}$ can be positive (equal, say, to n^2) or negative (say $-n^2$) and we shall require the solution in both cases.

Find the general solution of

(i) $\dfrac{d^2y}{dx^2} - 4y = 0$ (ii) $\dfrac{d^2y}{dx^2} + 9y = 0$

Your solution to the ODE in (i)

Your solution

The auxiliary equation to (i) is

$$m^2 - 4 = 0$$

The roots of the auxiliary equation are

Your solution

$m = \pm 2$

The general solution to the ODE in (i) is

Your solution

$y = Ae^{2x} + Be^{-2x}$ (Since the roots of the auxiliary equation are real and distinct.)

Your solution to the ODE in (ii)

Your solution

The auxiliary equation to (ii) is

$m^2 + 9 = 0$

The roots of this auxiliary equation are

Your solution

$m = \pm 3j$

The general solution to the ODE (ii) is

Your solution

$y = A\cos 3x + B\sin 3x$

(Since the roots of the auxiliary equation are complex conjugates with real part $\alpha = 0$ and imaginary part $\beta = 3$.)

The two above problems with the numerical parameters $n^2 = -4$ or $n^2 = +9$ can be generalised as in the following Key Point.

Key Point

The general solution to: $\dfrac{d^2y}{dx^2} - n^2y = 0$ is

$$y = Ae^{nx} + Be^{-nx}$$

or, equivalently using hyperbolic functions,

$$y = C\cosh nx + D\sinh nx$$

The general solution to: $\dfrac{d^2y}{dx^2} + n^2y = 0$ is

$$y = A\cos nx + B\sin nx$$

Those of you who are familiar with elementary dynamics will recognise this second differential equation as modelling simple harmonic motion.

2. Partial Differential Equations

In all the above examples we had a function y of a single variable x, y being the solution of an **ordinary** differential equation.
In engineering and science ODEs arise as models for systems where there is **one** independent variable (our x) and one dependent variable (our y). Obvious examples are lumped electrical circuits where the current i is a function only of time t (and not of position in the circuit) and lumped mechanical systems (such as the simple harmonic oscillator referred to above) where the displacement of a moving particle depends only on t.
However in problems where one variable, say u, depends on **more than one** independent variable, say x and t, then any derivatives of u will be **partial** derivatives such as $\dfrac{\partial u}{\partial x}$ or $\dfrac{\partial^2 u}{\partial t^2}$ and any differential equation arising will be known as a **Partial Differential Equation.**
In particular, one-dimensional (1-d) time-dependent problems where u depends on a position coordinate x and the time t and two-dimensional (2-d) time-independent problems where u is a function of the two position coordinates x and y both give rise to PDEs involving **two** independent variables. This is the case we shall concentrate on. A two-dimensional time-dependent problem would involve 3 independent variables x, y, t as would a three-dimensional time-independent problem where x, y, z would be the independent variables.

Example Show that $u = \sin x \cosh y$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This PDE is known as **Laplace's Equation in two dimensions** and it arises in many applications e.g. electostatics, fluid flow, heat conduction.

Solution

$$u = \sin x \cosh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \qquad \frac{\partial^2 u}{\partial y^2} = \sin x \cosh y$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0$$

so the given function $u(x, y)$ is indeed a solution.

Show that

$$u = e^{-2\pi^2 t} \sin \pi x$$

is a solution of the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

First find $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$

Your solution

$$\frac{\partial u}{\partial t} = -2\pi^2 e^{-2\pi^2 t} \sin \pi x \qquad \frac{\partial u}{\partial x} = \pi e^{-2\pi^2 t} \cos \pi x$$

Now find $\frac{\partial^2 u}{\partial x^2}$

Your solution

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x.$$

We see that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

as required.

The PDE in the above exercise has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k is a positive constant. This equation is referred to as the **one-dimensional heat conduction equation** (or sometimes as the diffusion equation). In a heat conduction context the dependent variable u represents the temperature $u(x,t)$.
The third important PDE involving two independent variables is known as the **one-dimensional wave equation**. This has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

(Note that both partial derivatives in the wave equation are second-order in contrast to the heat conduction equation where the time derivative is first order.)

Example

(a) Verify that

$$u(x,t) = u_0 \sin\left(\frac{\pi x}{\ell}\right)\ \cos\left(\frac{\pi c t}{\ell}\right)$$

(where u_0, ℓ and c are constants) satisfies the one-dimensional wave equation.

(b) Verify that $u(0,t) = u(\ell,t) = 0$

(c) Verify that $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = u_0 \sin\left(\frac{\pi x}{\ell}\right)$

(d) Give a physical interpretation of this problem.

(a) By straightforward partial differentiation of the given function $u(x,t)$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= u_0\frac{\pi}{\ell}\cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)\\
\frac{\partial^2 u}{\partial x^2} &= -u_0\left(\frac{\pi}{\ell}\right)^2\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)\\
\frac{\partial u}{\partial t} &= -u_0\left(\frac{\pi c}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{\pi c t}{\ell}\right)\\
\frac{\partial^2 u}{\partial t^2} &= -u_0\left(\frac{\pi c}{\ell}\right)^2\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)
\end{aligned}$$

We see that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2}$ which completes the verification.

(b) Putting $x = 0$, and leaving t arbitrary, in the given solution for $u(x,t)$ gives

$$u(x,0) = u_0 \sin 0\ \cos\left(\frac{\pi c t}{\ell}\right) = 0 \quad \text{for all } t$$

Similarly putting $x = \ell$, t arbitrary:

$$u(\ell,0) = u_0 \sin \pi\ \cos\left(\frac{\pi c t}{\ell}\right) = 0 \quad \text{for all } t$$

(c) Evaluating $\frac{\partial u}{\partial t}$ firstly for general x and t

$$\frac{\partial u}{\partial t} = -u_0 \left(\frac{\pi c}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{\pi c t}{\ell}\right)$$

as already obtained.
Now putting $t = 0$ leaving x arbitrary

$$\frac{\partial u}{\partial t}(x, 0) = -u_0 \left(\frac{\pi c}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) \sin 0 = 0$$

Also, putting $t = 0$ in the expression for $u(x, t)$.

$$u(x, 0) = u_0 \sin\left(\frac{\pi x}{\ell}\right) \cos 0 = u_0 \sin\left(\frac{\pi x}{\ell}\right)$$

(d) Mathematically we have now proved that the given function $u(x,t)$ satisfies the $1-$d wave equation, the two **boundary conditions** specified in (b) and the two **initial conditions** specified in (c).

One possible physical interpretation of this problem is that $u(x,t)$ represents the displacement of a string stretched between two points at $x=0$ and $x=\ell$. Clearly the position of any point P on the vibrating string will depend upon its distance x from one end and on the time t.

The boundary conditions (b) represent the fact that the string is fixed at these end-points.

The initial condition

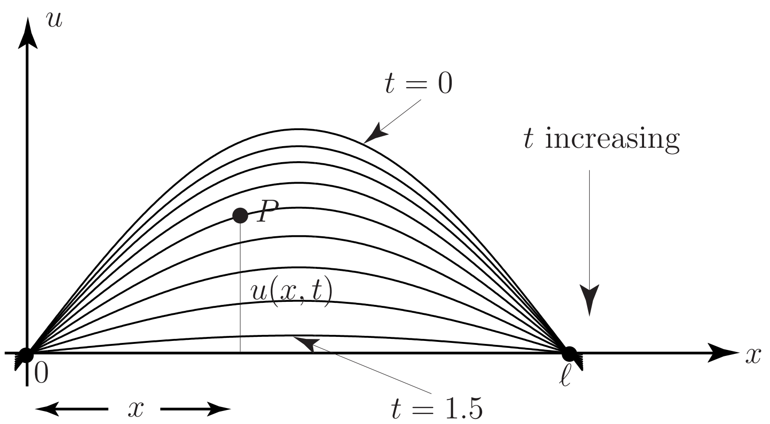
$$u(x,0) = u_0 \sin\left(\frac{\pi x}{\ell}\right)$$

represents the displacement of the string at $t=0$.

The initial condition

$$\frac{\partial u}{\partial t}(x,0) = 0$$

tells us that the string is at rest at $t=0$.



Note that it can be proved formally that if T is the tension in the string and if ρ is the mass per unit length of the string then u does, under certain conditions, satisfy the $1-$d wave equation with $c^2 = \dfrac{T}{\rho}$.

Key Point

The three PDEs with which we shall be mostly concerned are

(a)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the two-dimensional Laplace equation

(b)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

the one-dimensional heat conduction equation

(c)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

the one-dimensional wave equation.