Solution using Separation of Variables 25.3

Introduction

The main topic of this Section is the solution of PDEs using the method of separation of variables. In this method a PDE involving n independent variables is converted into n ordinary differential equations. (In this introductory account n will always be 2).

You should be aware that other analytical and also numerical methods are available for solving PDEs. However, the separation of variables technique does give some useful solutions to important PDEs.

Prerequisites

Before starting this Section you should ...

① some first and second order constant coefficient ordinary differential equations

Learning Outcomes

After completing this Section you should be able to ...

✓ apply the separation of variables method to obtain solutions of the heat conduction equation, wave equation and 2-D Laplace equation for specified boundary and/or initial conditions

1. Solution of Important PDEs

We shall just consider two analytic solution techniques for PDEs:

(a) Direct integration

(b) The method of separation of variables

The latter method is the more important and we will study it in detail shortly.
You should note that many practical problems involving PDEs have to be solved by **numerical** methods but that is another story.
The method of direct integration is a straightforward extension of solving very simple ODEs by direct integration.

Solve the ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^2 + 2$$

given that $y = 1$ when $x = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by integrating once, not forgetting the arbitrary constant of integration

Your solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3}{3} + 2x + A$$

Now find y by integrating again, not forgetting to include another arbitrary constant.

Your solution

$$y = \frac{x^4}{12} + x^2 + Ax + B$$

Now find A and B by inserting the two given initial conditions:

Your solution

$y(0) = 1$ gives $B = 1$ $\qquad$ $y'(0) = 2$ gives $A = 2$

so the required solution is

$$y = \frac{x^4}{12} + x^2 + 2x + 1$$

Consider now a similar type of PDE i.e. one that can also be solved by direct integration. Suppose we require the general solution of

$$\frac{\partial^2 u}{\partial x^2} = 2xe^t$$

where u is a function of x and t.
Integrating with respect to x gives us

$$\frac{\partial u}{\partial x} = x^2 e^t + f(t)$$

where the arbitrary function $f(t)$ replaces the normal "arbitrary constant" of ordinary integration. This function of t only is needed because we are integrating "partially" with respect to x i.e. we are reversing a partial differentiation with respect to x at constant t.
Integrating again with respect to x gives the general solution:

$$u = \frac{x^3}{3}e^t + x\ f(t) + g(t)$$

where $g(t)$ is a second arbitrary function. We have now obtained the general solution of the given PDE but to find the arbitrary function we must know two initial conditions.
Suppose, for the sake of example, that these conditions are

$$u(0,t) = t\ , \qquad \frac{\partial u}{\partial x}(0,t) = e^t$$

Inserting the first of these conditions into the general solution gives

$$g(t) = t$$

Inserting the second condition into the general solution gives

$$f(t) = e^t$$

so the final solution is

$$u = \frac{x^3}{3}e^t + xe^t + t.$$

Solve the PDE

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \ \cos y$$

subject to the conditions

$$\frac{\partial u}{\partial x} = 2x \quad \text{at} \quad y = \frac{\pi}{2}, \quad u = 2\sin y \quad \text{at} \quad x = \pi.$$

First integrate the PDE with respect to y: (it is equally valid to integrate first with respect to x). Don't forget the appropriate arbitrary function.

Your solution

Recall that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

Hence integration with respect to y gives

$$\frac{\partial u}{\partial x} = \sin x \sin y + f(x)$$

Since one of the given conditions is on $\dfrac{\partial u}{\partial x}$ we impose this condition at this stage to determine the arbitrary function $f(x)$:

Your solution

At $y = \pi/2$ the condition gives

$$\sin x \sin \pi/2 + f(x) = 2x \qquad \text{i.e.} \qquad f(x) = 2x - \sin x$$

So $\dfrac{\partial u}{\partial x} = \sin x \sin y + 2x - \sin x$

Now integrate again to determine u

Your solution

Integrating now with respect to x gives

$$u = -\cos x \sin y + x^2 + \cos x + g(y)$$

Finally, obtain the arbitrary function $g(y)$

Your solution

The condition $u(\pi, y) = 2\sin y$ gives

$$-\cos\pi \sin y + \pi^2 + \cos\pi + g(y) = 2\sin y$$

$$\therefore \quad \sin y + \pi^2 - 1 + g(y) = 2\sin y$$

$$\therefore \quad g(y) = \sin y + 1 - \pi^2$$

Now write down the final answer for $u(x, y)$

Your solution

$$u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$$

2. Method of Separation of Variables

In the previous Section we showed that

(a) $u(x, y) = \sin x \cosh y$

is **a** solution of the two-dimensional Laplace Equation

(b) $u(x, t) = e^{-2\pi^2 t} \sin \pi x$

is **a** solution of the one-dimensional heat conduction equation

(c) $u(x, t) = u_0 \sin\left(\frac{\pi x}{\ell}\right) \cos\left(\frac{\pi c t}{\ell}\right)$

is a solution of the one-dimensional wave equation.

All three solutions here have a specific form: in (a) $u(x, y)$ is a product of a function of x alone, viz $\sin x$, and a function of y alone, viz $\cosh y$.
Similarly in both (b) and (c) $u(x, t)$ is a product of a function of x alone and a function of t alone.
The method of separation of variables involves finding solutions of PDEs which are of this product form. In the method we assume that a solution to a PDE has the form. In a similar manner to second order differential equations (25.1 p3) if two or more different solutions are possible, their sum is also a solution.

e.q. if $u(x, y) = X_1(x)Y_1(y)$ and $u(x, y) = X_2(x)Y_2(y)$ are solutions then so is $u(x, t) = X_1(x)Y_1(y) + X_2(x)Y_2(y)$

$$u(x, t) = X(x)T(t)$$

$$(\text{or} \quad u(x, y) = X(x)Y(y))$$

where $X(x)$ is a function of x only, $T(t)$ is a function of t only and $Y(y)$ is a function y only.
You should note that not all solutions to PDEs are of this type; for example, it is easy to verify that

$$u(x, y) = x^2 - y^2$$

(which is **not** of the form $u(x, y) = X(x)Y(y)$) is a solution of the Laplace Equation.
However many interesting and useful solutions of PDEs are obtainable which are of the product form.
We shall firstly consider the types of solution obtainable for our three basic PDEs using trial solutions of the product form.

Heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad k > 0 \qquad \ldots(1)$$

Assuming that

$$\begin{aligned} u &= X(x)T(t) \qquad \ldots(2) \\ &= XT \quad \text{for short} \end{aligned}$$

then

$$\frac{\partial u}{\partial x} = \frac{\mathrm{d}X}{\mathrm{d}x}T = X'T \quad \text{for short}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}T = X''T \quad \text{for short}$$

$$\frac{\partial u}{\partial t} = X\frac{\mathrm{d}T}{\mathrm{d}t} = XT' \quad \text{for short}$$

Substituting into the original PDE (1)

$$X''T = \frac{1}{k}XT'$$

which can be re-arranged as

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} \qquad \ldots(3)$$

Now the left hand side of (3) involves functions of x only and the right hand side expression contain functions of t only.
Thus altering the value of t cannot change the left hand side of (3) i.e. it stays constant. Hence so must the r.h.s. be constant. We conclude that $T(t)$ is a function such that

$$\frac{1}{k}\frac{T'}{T} = K \qquad \ldots(4)$$

where K is a constant whose sign is yet to be determined.
By a similar argument, altering the value of x cannot change the right hand side of (3) and consequently the left hand side must be a constant

$$\text{i.e.} \qquad \frac{X''}{X} = K \qquad \ldots(5)$$

We see that the effect of assuming a product trial solution of the form (2) converts the PDE (1) into the two ODEs (4) and (5).
Both these ODEs are types whose solution we revised at the beginning of this Unit but we shall not attempt to solve them yet. In particular the solution of (5) depends on whether the constant K is positive or negative.

By following a similar procedure to the above, assume a product solution

$$u(x,t) = X(x)T(t)$$

for the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \ldots(6)$$

and find the two ODEs satisfied by $X(x)$ and $T(t)$ respectively.

First obtain $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$.

Your solution

$u = X(x)T(t)$ gives $\dfrac{\partial^2 u}{\partial x^2} = X''T$ and $\dfrac{\partial^2 u}{\partial t^2} = XT''$

Now substitute these results into (6) and transpose so the variables are separated i.e. all functions of x are on the left hand side, all funtions of t on the right hand side.

Your solution

We get $X''T = \dfrac{1}{c^2}XT''$ and, transposing, $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$

Finally, write down the required ODE

Your solution

Equating both sides to the same constant K gives

$$\frac{X''}{X} = K \quad \text{or} \quad \frac{d^2X}{dx^2} - KX = 0 \qquad \ldots (7)$$

and

$$\frac{1}{c^2}\frac{T''}{T} = K \qquad \text{or} \qquad \frac{d^2T}{dt^2} - Kc^2T = 0 \qquad \ldots (8)$$

Again, the solution of the ODEs (7) and (8) has been revised earlier and in both cases the solution will depend on the sign of K.

Separating the variables for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

follows similar lines. Obtain the ODEs satisfied by $X(x)$ and $Y(y)$.

Your solution

Assuming $u(x, y) = X(x)Y(y)$

leads to: $\dfrac{\partial^2 u}{\partial x^2} = X''Y \qquad \dfrac{\partial^2 u}{\partial y^2} = XY''$ so

$$X''Y + XY'' = 0 \qquad \text{or} \qquad \frac{X''}{X} = -\frac{Y''}{Y}$$

Equating each side to a constant K

$$\frac{X''}{X} = K \quad \text{or} \quad \frac{d^2X}{dx^2} - KX = 0 \qquad \ldots (9)$$

$$\frac{Y''}{Y} = -K \quad \text{or} \quad \frac{d^2Y}{dy^2} + KY = 0 \qquad \ldots (10)$$

(Note carefully the different signs in the two ODEs. Yet again the sign of the "separation constant" K will determine the solutions.)

We shall now study some specific problems which can be fully solved by the separation of variables method.

Example Solve the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

over $\quad 0 < x < 3, \quad t > 0 \quad$ for the boundary conditions

$$u(0, t) = u(3, t) = 0$$

and the initial condition

$$u(x, 0) = 5 \sin 4\pi x.$$

Solution

Assuming $u(x, t) = X(x)T(t)$ gives rise to the differential equations (4) and (5) with the parameter $k = 2$:

$$\frac{dT}{dt} = 2KT$$

$$\frac{d^2X}{dx^2} = KX$$

Now the T equation has general solution

$$T = Ae^{2Kt}$$

which will increase exponentially with increasing t if K is positive and decrease with t if K is negative. In any physical problem the latter is the meaningful situation. To emphasise that K

is being taken as negative we put

$$K = -\lambda^2$$

so

$$T = Ae^{-2\lambda^2 t}.$$

The X equation then becomes

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\lambda^2 X$$

which has solution

$$X(x) = B\cos\lambda x + C\sin\lambda x.$$

Hence

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= (D\cos\lambda x + E\sin\lambda x)e^{-2\lambda^2 t} \qquad \ldots(11) \end{aligned}$$

where $D = AB$ and $E = AC$.

(You should always try to keep the number of arbitrary constants down to an absolute minimum by multiplying them together in this way.)
We now insert the initial and boundary conditions to obtain the constant D and E and also the separation constant λ.
The initial condition $u(0,t) = 0$ gives

$$(D\cos 0 + E\sin 0)e^{-2\lambda^2 t} = 0 \qquad \text{for all } t.$$

Since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $D = 0$.
The other initial condition $u(3,t) = 0$ then gives

$$E\sin(3\lambda)e^{-2\lambda^2 t} = 0 \qquad \text{for all } t.$$

We cannot deduce that the constant E has to be zero because then the solution (11) would be the trivial solution $u \equiv 0$.
The only sensible deduction is that

$\sin 3\lambda = 0$ i.e. $3\lambda = n\pi$ (where n is some integer).

Hence solutions of the form (11) satisfying the 2 boundary conditions have the form

$$u(x,t) = E_n \sin\left(\frac{n\pi x}{3}\right) e^{-\dfrac{2n^2\pi^2 t}{9}}$$

where we have written E_n for E to allow for the possibility of a different value for the constant for each different value of n.
We obtain the value of n by using the initial condition $u(x,0) = 5\sin 4\pi x$ and forcing this solution to agree with it.
That is,

$$u(x,0) = E_n \sin\left(\frac{n\pi x}{3}\right) = 5\sin 4\pi x$$

so we must choose $n = 12$ with $E_{12} = 5$.
Hence, finally,

$$\begin{aligned} u(x,t) &= 5\sin\left(\frac{12\pi x}{3}\right) e^{-\dfrac{2}{9}(12)^2\pi^2 t} \\ &= 5\sin(4\pi x)\, e^{-32\pi^2 t}. \end{aligned}$$

Solve the 1−dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{16}\frac{\partial^2 u}{\partial t^2}$$

for $0 < x < 2, \quad t > 0$

The boundary conditions are

$$u(0, t) = u(2, t) = 0$$

(as in the previous example).

The initial conditions are

(i) $u(x, 0) = 6 \sin \pi x - 3 \sin 4\pi x$

(which is similar to, but slightly more complex, than in the previous example.)

(ii) $\dfrac{\partial u}{\partial t}(x, 0) = 0$

(which has no counterpart in the previous example.)

Firstly using (7) and (8) (or by working from first principles from the product solution $u(x, t) = X(x)T(t)$) write down the ODEs satisfied by $X(x)$ and $T(t)$.

Your solution

$$\frac{X''}{X} = K \qquad \frac{T''}{16T} = K$$

Now decide on the appropriate sign for K and then write down the solution to these equations.

Your solution

Choosing K as negative (say $K = -\lambda^2$) will produce Sinusoidal solutions for X and T which are appropriate in the context of the wave equation where oscillatory solutions can be expected.

Then $X'' = -\lambda^2 X$ gives

$$X = A \cos \lambda x + B \sin \lambda x$$

Similarly $T'' = -16\lambda^2 T$ gives

$$T = C \cos 4\lambda t + D \sin 4\lambda t$$

Now obtain the general solution $u(x, t)$ by multiplying $X(x)$ by $T(t)$ and insert the two boundary conditions to obtain information about two of the constants.

Your solution

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos 4\lambda t + D\sin 4\lambda t)$$

$u(0,t) = 0$ for all t gives

$$A(C\cos 4\lambda t + D\sin 4\lambda t) = 0$$

which implies that $A = 0$.

$u(2,t) = 0$ for all t gives

$$B\sin 2\lambda(C\cos 4\lambda t + D\sin 4\lambda t) = 0$$

so, for a non-trivial solution,

$$\sin 2\lambda = 0 \quad \text{i.e.} \quad \lambda = \frac{n\pi}{2} \quad \text{for some integer } n.$$

At this stage we write the solution as

$$u(x,t) = \sin\left(\frac{n\pi x}{2}\right)(E\cos 2n\pi t + F\sin 2n\pi t)$$

where we have multiplied constants and put $E = BC$ and $F = BD$.

Now insert the initial condition

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{for all } x \qquad 0 < x < 2.$$

and deduce the value of F.

Your solution

Differentiating partially with respect to t

$$\frac{\partial u}{\partial t} = \sin\left(\frac{n\pi x}{2}\right)(-2n\pi E\sin 2n\pi t + 2n\pi F\cos 2n\pi t)$$

so at $t = 0$

$$\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{n\pi x}{2}\right)2n\pi F = 0$$

from which we must have that $F = 0$.

Using the other the initial condition

$$u(x,0) = 6\sin(\pi x) - 3\sin(4\pi x)$$

deduce the form of $u(x,t)$.

Your solution

At this stage the solution reads

$$u(x,t) = E\sin\left(\frac{n\pi x}{2}\right)\cos(2n\pi t) \qquad \ldots(12)$$

We now have to insert the final condition viz. the initial condition

$$u(x,0) = 6\sin\pi x - 3\sin 4\pi x \qquad \ldots(13)$$

This seems strange because, putting $t = 0$ in our solution (12) suggests

$$u(x,0) = E\sin\left(\frac{n\pi x}{2}\right)$$

At this point we seem to have incompatability because no single value of n will enable us to satisfy (13).

However in the solution (12), **any** positive integer value of n is acceptable and we can in fact, **superpose** solutions of the form (12) and still have a valid solution to the PDE Hence we first write, instead of (12)

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right)\cos(2n\pi t) \qquad \ldots(14)$$

from which

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right) \qquad \ldots(15)$$

(which looks very much like, and indeed is, a Fourier Series.)
To make the solution (15) fit the initial condition (13) we do not require all the terms in the infinite Fourier Series.

We need only the terms with $n = 2$ with coefficient $E_2 = 6$ and the term for which $n = 8$ with $E_8 = -3$. All the other coefficients E_n have to be chosen as zero.

Using these results in (14) we obtain the solution

$$u(x,t) = 6\sin\pi x\cos 4\pi t - 3\sin 4\pi x\cos 16\pi t$$

The above solution perhaps seems rather involved but there is a definite sequence of logical steps which can be readily applied to other similar problems.