Cauchy's Theorem

26.5

Introduction

In this Section we introduce Cauchy's Theorem which allows us to simplify the calculation of certain contour integrals. A second result, known as Cauchy's Integral Formula, allows us to evaluate some integrals of the form $\oint_C \dfrac{f(z)}{z - z_0}\mathrm{d}z$ where z_0 lies inside C.

Prerequisites

Before starting this Section you should ...

① be familiar with the basic ideas of functions of a complex variable as in Section 26.1

② be familiar with line integrals

Learning Outcomes

After completing this Section you should be able to ...

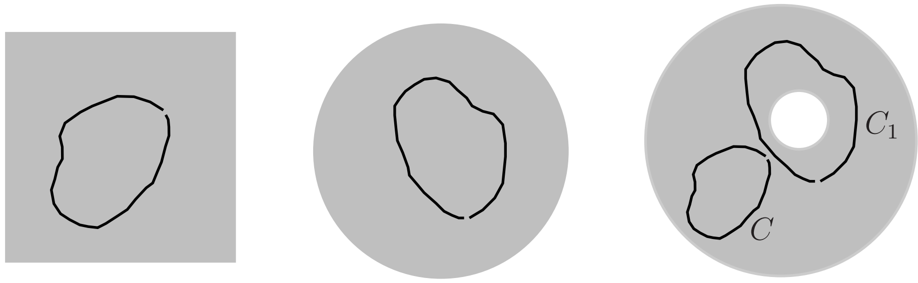
✓ state and use Cauchy's Theorem

✓ state and use Cauchy's Integral Formula

1. Cauchy's Theorem

Simply-Connected Regions

By a simply-connected region we mean that any closed curve in that region can be shrunk to a point without any part of it leaving a region. The interior of a square or a circle are examples of simply connected regions, see Figure 1 (a) and (b). In Figure 1 (c) we see that the region between the two circles is **not** simply-connected. Curve C will be able to shrink to a point but curve C_1 will not, due to the hole in its centre.



(a) (b) (c)

Figure 1

Key Point

Cauchy's Theorem

This is perhaps the most important theorem in the area of complex analysis. The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then:

$$\oint_C f(z)\mathrm{d}z = 0$$

for every simple closed path C lying in the region.

As a straightforward example note that $\oint_C z^2\mathrm{d}z = 0$, where C is the unit circle, since z^2 is analytic everywhere (see Workbook 26.1). Indeed $\oint_C z^2\mathrm{d}z = 0$ for *any* simple contour: it need not be circular.

Consider the contour shown in Figure 2 and let $f(z)$ be assumed to be analytic everywhere on and inside the contour C.

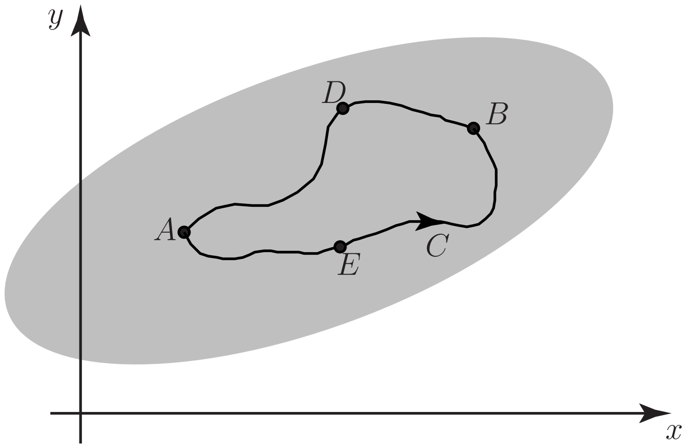


Figure 2

Then by analogy with real line integrals

$$\int_{AEB} f(z)\mathrm{d}z + \int_{BDA} f(z)\mathrm{d}z = \oint_C f(z)\mathrm{d}z = 0 \qquad \text{by Cauchy's Theorem}$$

Therefore

$$\int_{AEB} f(z)\mathrm{d}z = -\int_{BDA} f(z)\mathrm{d}z = \int_{ADB} f(z)\mathrm{d}z$$

(since reversing the direction of integration reverses the sign of the integral).

This implies that we may choose any path between A and B and the integral will have the same value **providing $f(z)$ is analytic in the region concerned.**

Integrals of analytic functions only depend on the positions of the points A and B, not on the path connecting them. This explains the ‘coincidences’ referred to in Section 26.4 (page 7).

Using ‘simple’ integration evaluate $\displaystyle\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\ \mathrm{d}z$, explaining why this is valid.

Your solution

$$\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\ \mathrm{d}z = [\sin z]_{\mathrm{i}}^{1+2\mathrm{i}} = \sin(1+2\mathrm{i}) - \sin \mathrm{i}.$$

This way of determining the integral is allowed because $\cos z$ is analytic (everywhere).

We now investigate what occurs when the closed path of integration does not necessarily lie within a simply-connected region. Consider the situation described in Figure 3.

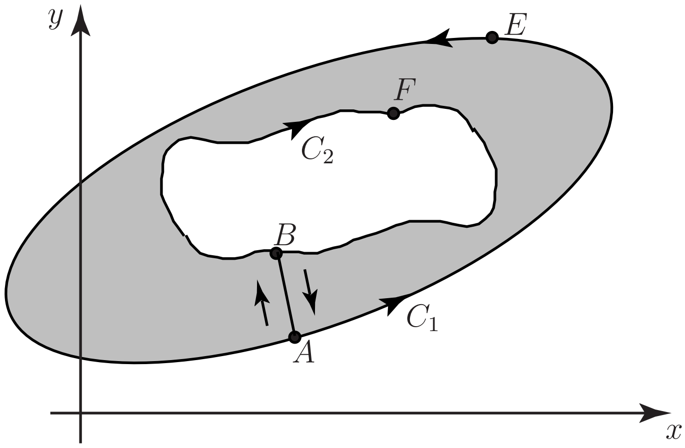


Figure 3

Let $f(z)$ be analytic in the region bounded by the closed curves C_1 and C_2. The region is cut by the line segment joining A and B.

Consider now the closed curve $AEABFBA$ travelling in the direction indicated by the arrows. No line can cross the cut AB and be regarded as remaining in the region. Because of the cut the shaded region is simply connected. Cauchy's theorem therefore applies (see the second Key Point).

Therefore

$$\oint_{AEABFBA} f(z)\mathrm{d}z = 0 \quad \text{since } f(z) \text{ is analytic within and on the curve } AEABFBA.$$

Note that

$$\int_{AB} f(z)\mathrm{d}z = -\int_{BA} f(z)\mathrm{d}z, \qquad \text{being a simple change of direction.}$$

Also, we can divide the closed curve into smaller sections:

$$\begin{aligned}\oint_{AEABFBA} f(z)\mathrm{d}z &= \int_{AEA} f(z)\mathrm{d}z + \int_{AB} f(z)\mathrm{d}z + \int_{BFB} f(z)\mathrm{d}z + \int_{BA} f(z)\mathrm{d}z \\ &= \int_{AEA} f(z)\mathrm{d}z + \int_{BFB} f(z)\mathrm{d}z = 0.\end{aligned}$$

i.e.

$$\oint_{C_1} f(z)\mathrm{d}z - \oint_{C_2} f(z)\mathrm{d}z = 0$$

(since we assume that closed paths are travelled anticlockwise).

Therefore $\displaystyle\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_2} f(z)\mathrm{d}z.$

This allows us to evaluate $\displaystyle\oint_{C_1} f(z)\mathrm{d}z$ by replacing C_1 by any curve C_2 such that the region between them contains no singularities (see Workbook 26.1) of $f(z)$. Often we choose a circle for C_2.

Example Determine $\displaystyle\oint_C \frac{6}{z(z-3)}\mathrm{d}z$ where C is the curve $|z-3| = 5$ shown in Figure 4.

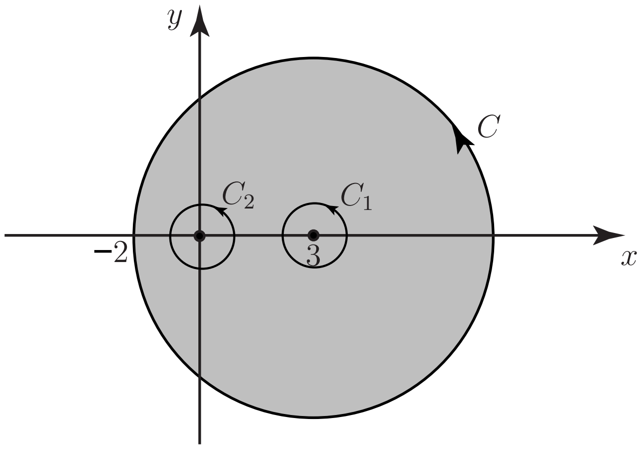


Figure 4

Solution

We observe that $f(z) = \dfrac{6}{z(z-3)}$ is analytic everywhere except at $z = 0$ and $z = 3$.

Let C_1 be the circle of unit radius centred at $z = 3$ and C_2 be the unit circle centered at the origin. By analogy with the previous example we state that

$$\oint_C \frac{6}{z(z-3)}\mathrm{d}z = \oint_{C_1} \frac{6}{z(z-3)}\mathrm{d}z + \oint_{C_2} \frac{6}{z(z-3)}\mathrm{d}z.$$

(To show this you would need two cuts: from C to C_1 and from C to C_2).

The remaining parts of this example are presented as guided exercises.

Expand $\dfrac{6}{z(z-3)}$ into partial functions and then use the Key Point to integrate each partial fraction separately.

Your solution

Let

$$\frac{6}{z(z-3)} \equiv \frac{A}{z} + \frac{B}{z-3} \equiv \frac{A(z-3)+Bz}{z(z-3)}.$$

Then $A(z-3)+Bz \equiv 6$.

If $z=0$ $\quad A(-3)=6 \quad \therefore \quad A=-2.$

If $z=3$ $\quad B.3=6 \quad \therefore \quad B=2.$

$\therefore$

$$\frac{6}{z(z-3)} \equiv -\frac{2}{z} + \frac{2}{z-3}.$$

Thus:

$$\begin{aligned}\oint_C \frac{6}{z(z-3)}\mathrm{d}z &= \oint_{C_1} \frac{2}{z-3}\mathrm{d}z - \oint_{C_1} \frac{2}{z}\mathrm{d}z + \oint_{C_2} \frac{2}{z-3}\mathrm{d}z - \oint_{C_2} \frac{2}{z}\mathrm{d}z \\ &= I_1 - I_2 + I_3 - I_4.\end{aligned}$$

Now the function $\frac{1}{z}$ is analytic inside and on C_1 so that $I_2 = 0$.

What is the value of I_3?

Your solution

The function $\frac{1}{z-3}$ is analytic inside and on C_2 so that $I_3 = 0$.

Using the first Key Point we find that $I_1 = 2 \times 2\pi\mathrm{i} = 4\pi\mathrm{i}$.

What is the value of I_4?

Your solution

$I_4 = 4\pi\mathrm{i}$ $\quad$ again using the first Key Point (page 9).

Hence, collecting these results together:

$$\oint_C \frac{6\mathrm{d}z}{z(z-3)} = 4\pi\mathrm{i} - 0 + 0 - 4\pi\mathrm{i} = 0.$$

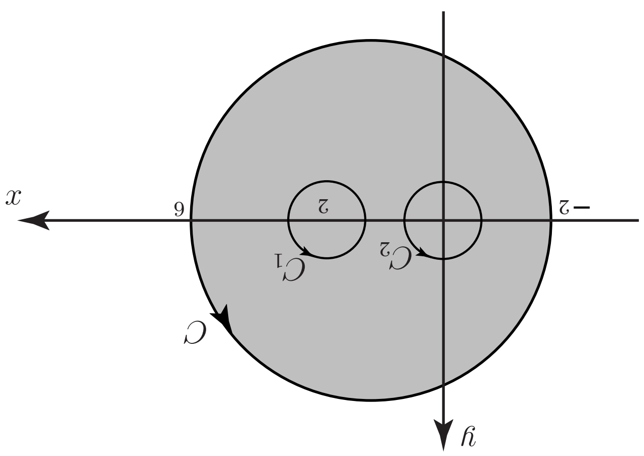
Exercises

1. Evaluate $\displaystyle\int_{1+\mathrm{i}}^{2+3\mathrm{i}} \sin z\,\mathrm{d}z$.

2. Determine $\displaystyle\oint_C \frac{4}{z(z-2)}\mathrm{d}z$ where C is the contour $|z-2| = 4$.

1. $\displaystyle\int_{1+\mathrm{i}}^{2+3\mathrm{i}} \sin z\,\mathrm{d}z = [-\cos z]_{1+\mathrm{i}}^{2+3\mathrm{i}} = \cos(1+\mathrm{i}) - \cos(2+3\mathrm{i})$ since $\sin z$ is analytic everywhere.

2.



$f(z) = \dfrac{4}{z(z-2)}$ is analytic everywhere except at $z = 0$ and $z = 2$.

Call $I = \displaystyle\oint_C \frac{4}{z(z-2)}\mathrm{d}z = \oint_{C_1} \frac{4}{z(z-2)}\mathrm{d}z + \oint_{C_2} \frac{4}{z(z-2)}\mathrm{d}z.$

Now $\dfrac{4}{z(z-2)} \equiv -\dfrac{2}{z} + \dfrac{2}{z-2}$ so that

$$\begin{aligned} I &= \oint_{C_1} \frac{2}{z-2}\mathrm{d}z - \oint_{C_1} \frac{2}{z}\mathrm{d}z + \oint_{C_2} \frac{2}{z-3}\mathrm{d}z - \oint_{C_2} \frac{2}{z}\mathrm{d}z \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

I_2 and I_3 are zero because of analyticity.

$I_1 = 2 \times 2\pi\mathrm{i} = 4\pi\mathrm{i}$, by the first Key Point (page 9) and $I_4 = -4\pi\mathrm{i}$ likewise.

Hence $I = 4\pi\mathrm{i} - 0 + 0 - 4\pi\mathrm{i} = 0$.

2. Cauchy's Integral Formula

This is a generalization of the result in the earlier keypoint:

Key Point

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C,

$$\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = 2\pi \mathrm{i}\ f(z_0).$$

Example As an example we evaluate

$$\oint_C \frac{z}{z^2+1}\mathrm{d}z$$

where C is the path (refer to Figure 5)

(i) $C_1: \quad |z - \mathrm{i}| = \frac{1}{2}$ (ii) $C_2: \quad |z + \mathrm{i}| = \frac{1}{2}$ (iii) $C_3: \quad |z| = 2.$

First note that $z^2 + 1 \equiv (z + \mathrm{i})(z - \mathrm{i})$.

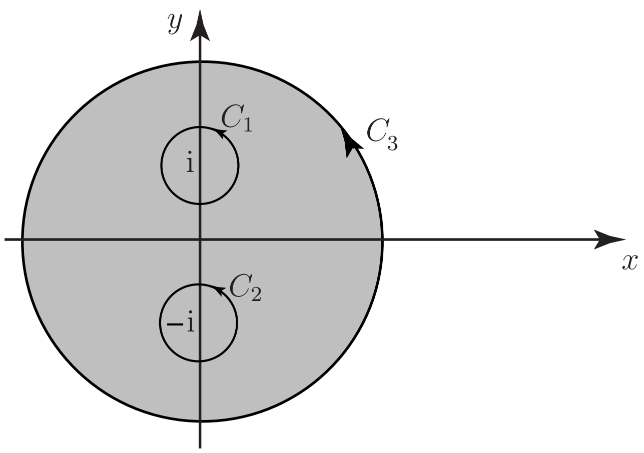


Figure 5

Solution

(i) Let $\dfrac{z}{z^2+1} = \dfrac{z}{(z+\mathrm{i})(z-\mathrm{i})} = \dfrac{z/(z+\mathrm{i})}{z-\mathrm{i}}$.

The numerator $z/(z+\mathrm{i})$is analytic inside and on the path C_1 so putting $z_0 = \mathrm{i}$ in the Cauchy Integral Formula (Key Point above)

$$\oint_{C_1} \frac{z}{z^2+1}\mathrm{d}z = 2\pi\mathrm{i}\left[\frac{\mathrm{i}}{\mathrm{i}+\mathrm{i}}\right] = 2\pi\mathrm{i}.\frac{1}{2} = \pi\mathrm{i}.$$

Use the Cauchy Integral Formula to find an expression for (ii) $\displaystyle\oint_{C_2} \frac{z}{z^2+1}\mathrm{d}z$.

Your solution

Now let $\dfrac{z}{z^2+1} = \dfrac{z/(z-\mathrm{i})}{z+\mathrm{i}}$. The numerator is analytic inside and on the path C_2 so putting $z_0 = -\mathrm{i}$ in the Cauchy Integral Formula

$$\oint_{C_2} \frac{z}{z^2+1}\mathrm{d}z = 2\pi\mathrm{i}\left[\frac{-\mathrm{i}}{-2\mathrm{i}}\right] = \pi\mathrm{i}.$$

Now find (iii) $\displaystyle\oint_{C_3} \frac{z}{z^2+1}\mathrm{d}z$.

Your solution

By analogy with the previous example,

$$\oint_{C_3} \frac{z^2+1}{z}\mathrm{d}z = \oint_{C_1} \frac{z^2+1}{z}\mathrm{d}z + \oint_{C_2} \frac{z^2+1}{z}\mathrm{d}z$$

$$= \pi\mathrm{i} + \pi\mathrm{i} = 2\pi\mathrm{i}.$$

The Derivative of an Analytic Function

If $f(z)$ is analytic in a simply-connected region then at any interior point of the region, z_0 say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!).
The derivatives at the point z_0 are given by Cauchy's Integral Formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$$

where C is any simple closed curve, in the region, which encloses z_0.
Note the case $n = 1$:

$$f'(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^2}\mathrm{d}z.$$

Example Evaluate the contour integral

$$\oint_C \frac{z^3}{(z-1)^2}\mathrm{d}z$$

where C is a contour which encloses the point $z = 1$

Solution

Since $f(z) = \dfrac{z^3}{(z-1)^2}$ has a pole of order 2 at $z = 1$ then

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C'} \frac{z^3}{(z-1)^2}\,\mathrm{d}z$$

where C' is a circle centered at $z = 1$.

If $g(z) = z^3$ then

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C'} \frac{g(z)}{(z-1)^2}\,\mathrm{d}z$$

Since $g(z)$ is analytic within and on the circle C' we use Cauchy's Integral Formula for derivatives to show that

$$\oint_C \frac{z^3}{(z-1)^2}\,\mathrm{d}z = 2\pi\mathrm{i} \times \frac{1}{1!}\left[g'(z)\right]_{z=1} = 2\pi\mathrm{i}\left[3z^2\right]_{z=1} = 6\pi\mathrm{i}.$$

Exercises

1. Evaluate $\displaystyle\oint_C \frac{z}{z^2+9}\,\mathrm{d}z$ where C is the path:

   (a) $C_1: \quad |z-3\mathrm{i}|=1$ (b) $C_2: \quad |z+3\mathrm{i}|=1$ (c) $C_3: \quad |z|=6$.

1. (a) Let $\displaystyle\frac{z}{z^2+9} \equiv \frac{z}{(z+3\mathrm{i})(z-3\mathrm{i})} = \frac{z/(z+3\mathrm{i})}{z-3\mathrm{i}}$

The numerator $\displaystyle\frac{z}{z+3\mathrm{i}}$ is analytic inside and on the path C_1 so putting $z_0 = 3\mathrm{i}$ in the Cauchy Integral Formula (CIF)

$$\oint_{C_1} \frac{z}{z^2+9}\,\mathrm{d}z = 2\pi\mathrm{i}\left[\frac{3\mathrm{i}}{3\mathrm{i}+3\mathrm{i}}\right] = 2\pi\mathrm{i} \times \frac{1}{2} = \pi\mathrm{i}.$$

(b) Here $\displaystyle\frac{z/(z-3\mathrm{i})}{z+3\mathrm{i}}$

The numerator is analytic inside and on the path C_2 so putting $z = -3\mathrm{i}$ in CIF:

$$\oint_{C_2} \frac{z}{z^2+9}\,\mathrm{d}z = 2\pi\mathrm{i}\left[\frac{-3\mathrm{i}}{-3\mathrm{i}-3\mathrm{i}}\right] = \pi\mathrm{i}.$$

(c) The integral is the sum of the two previous integrals and has value $2\pi\mathrm{i}$.