Singularities and Residues

26.6

Introduction

Taylor's series for functions of a real variable is generalised to Laurent series for a function of a complex variable, which includes terms of the form $(z - z_0)^{-n}$.

The different types of singularity of a complex function $f(z)$ are discussed and the definition of a residue at a pole is given. The residue theorem is used to evaluate contour integrals where the only singularities of $f(z)$ inside the contour are poles.

Prerequisites

Before starting this Section you should ...

① be familiar with Binomial and Taylor series

Learning Outcomes

After completing this Section you should be able to ...

✓ understand the concept of a Laurent series

✓ find residues and use the residue theorem

1. Taylor and Laurent Series

Many of the results in the area of series of real variables can be extended into complex variables. As an example the concept of radius of convergence of a series is extended to the concept of a **circle of convergence**. If the circle of convergence of a series of complex numbers is $|z-z_0| = \rho$ then the series will converge if $|z - z_0| < \rho$.
For example, consider the function

$$f(z) = \frac{1}{1-z}$$

It has a singularity at $z = 1$. We can obtain the Maclaurin series, centered at $z = 0$, as

$$f(z) = 1 + z + z^2 + z^3 + \ldots$$

The circle of convergence is $|z| = 1$.
The radius of convergence for a series centred on $z = z_0$ is the distance between z_0 and the nearest singularity.

Laurent series

One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic.

As an example, the series

$$1 + z + z^2 + z^3 + \ldots \quad \text{converges to} \quad f(z) = \frac{1}{1-z}$$

only inside the circle $|z| = 1$ even though $f(z)$ is analytic everywhere **except** at $z = 1$.

The **Laurent series** is an attempt to represent $f(z)$ as a series at as many points as possible. We expand the series around a point of singularity up to, but not including, the singularity itself.

Figure 1 shows an **annulus of convergence** $r_1 < |z - z_0| < r_2$ within which the Laurent series (which is an extension of the Taylor series) will converge. The extension includes negative powers of $(z - z_0)$.

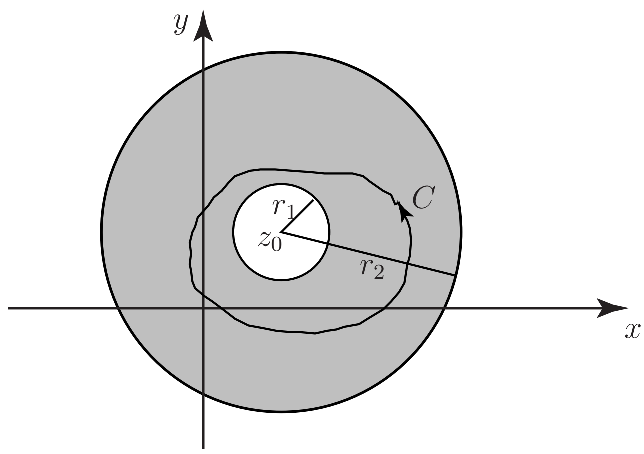


Figure 1

First, we state **Laurent's theorem**.

If $f(z)$ is analytic through a closed annulus D centred at $z = z_0$ then at any point z inside D we can write

$$\begin{aligned} f(z) = a_0 \quad &+ \quad a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \\ &+ \quad b_1(z - z_0)^{-1} + b_2(z - z_0)^{-2} + \ldots \end{aligned}$$

where the coefficients a_n and b_n (for each n) is given by

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z, \qquad b_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{1-n}} \, \mathrm{d}z,$$

the integral being taken around any simple closed path C lying inside D and encircling the inner boundary. Refer to Figure 1.

Example Expand $f(z) = \dfrac{1}{1 - z}$ in terms of negative powers of z which will be valid if $|z| > 1$.

Solution

First note that $1 - z \equiv -z\left(1 - \dfrac{1}{z}\right)$ so that

$$\begin{aligned} f(z) \quad &\equiv \quad -\frac{1}{z\left(1 - \frac{1}{z}\right)} \equiv -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right) \\ &= \quad -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \ldots \end{aligned}$$

This is valid for $\left|\dfrac{1}{z}\right| < 1$, that is, $\dfrac{1}{|z|} < 1$ or $|z| > 1$. Note that we used a binomial expansion rather than the theorem itself. Also note that together with the earlier result we are now able to expand $f(z) = \dfrac{1}{1 - z}$ everywhere, except for $|z| = 1$.

Consider $f(z) = \dfrac{1}{1 + z}$.

Using the binomial series, expand $f(z)$ in terms of non-negative power of z.

Your solution

$f(z) = (1 + z)^{-1} = 1 - z + z^2 - z^3 + \ldots$

For which values of z is this expansion valid?

Your solution

$|z| < 1$ (standard result for a GP).

Using the identity $1 + z = z\left(1 + \frac{1}{z}\right)$

now expand $f(z)$ in terms of negative powers of z and state the values of z for which your expansion is valid.

Your solution

$$f(z) \equiv \frac{1}{z\left(1+\frac{1}{z}\right)} \equiv \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} \equiv \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$$

Valid for $\left|\frac{1}{z}\right| < 1$ i.e. $|z| > 1$.

2. Classifying Singularities

If the function $f(z)$ has a singularity at $z = z_0$ and in a **neighbourhood** of z_0 (i.e. a region of the complex plane which contains z_0) there are no other singularities then z_0 is an **isolated singularity** of $f(z)$.
The **principal part** of the Laurent series is the part containing negative powers of $(z - z_0)$. If the principal part has a finite number of terms say

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m} \quad \text{and } b_m \neq 0$$

then $f(z)$ has a **pole** of **order** m at $z = z_0$ (we have written b_1 for a_{-1}, b_2 for a_{-2} etc for simplicit.) Note that if $b_1 = b_2 = \ldots = 0$ and $b_m \neq 0$ the pole is still of order m.

A pole of order 1 is called a **simple pole** whilst a pole of order 2 is called a **double pole**. If the principal part of the Laurent series has an infinite number of terms then $z = z_0$ is called an **isolated essential singularity** of $f(z)$.

The function

$$f(z) = \frac{\mathrm{i}}{z(z-\mathrm{i})} \equiv \frac{1}{z-\mathrm{i}} - \frac{1}{z}$$

has a simple pole at $z = 0$ and another simple pole at $z = \mathrm{i}$. The function $e^{\frac{1}{z-2}}$ has an isolated essential singularity at $z = 2$. Some complex functions have non-isolated singularities called *branch points*. An example of such a function is $\sqrt{z}$.

Classify the singularities of the function

$$f(z) = \frac{2}{z} - \frac{1}{z^2} + \frac{1}{z+\mathrm{i}} + \frac{3}{(z-\mathrm{i})^4}.$$

Your solution

A pole of order 2 at $z = 0$, a simple pole at $z = -\mathrm{i}$ and a pole of order 4 at $z = \mathrm{i}$.

Exercises

1. Expand $f(z) = \dfrac{1}{2-z}$ in terms of negative powers of z to give a series which will be valid if $|z| > 2$.

2. Classify the singularities of the function:

$$f(z) = \frac{1}{z^2} + \frac{1}{(z+\mathrm{i})^2} - \frac{2}{(z+\mathrm{i})^3}.$$

1. $2 - z = -z(1 - \frac{2}{z})$ so that:

$$\begin{aligned} f(z) &\equiv \frac{-1}{z(1-\frac{2}{z})} \equiv -\frac{1}{z}(1-\frac{2}{z})^{-1} \\ &\equiv -\frac{1}{z}(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots) \\ &\equiv -\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\frac{8}{z^3}-\dots \end{aligned}$$

This is valid for $\left|\frac{2}{z}\right| < 1$ or $|z| > 2$.

2. A double pole at $z = 0$ and a pole of order 3 at $z = -\mathrm{i}$.

3. The Residue Theorem

Suppose $f(z)$ is a function which is analytic inside and on a closed contour C, except for a pole of order m at $z = z_0$, which lies inside C.

To evaluate $\oint_C f(z)\mathrm{d}z$ we can expand $f(z)$ in a Laurent series in powers of $(z - z_0)$.

If we let Γ be a circle of centre z_0 lying inside C then, as we saw in Section 26.2,

$$\oint_C f(z)\mathrm{d}z = \int_\Gamma f(z)\mathrm{d}z.$$

From the Key Point result in Section 26.1 we know that the integral of each of the positive and negative powers of $(z - z_0)$ is zero with the exception of $\dfrac{b_1}{z - z_0}$ and the key result shows this has value $2\pi b_1$. Since it is the only coefficient remaining after the integration it is called the **residue** of $f(z)$ at $z = z_0$. It is given by

$$b_1 = \frac{1}{2\pi\mathrm{i}}\oint_C f(z)\mathrm{d}z.$$

Calculating the residue, for any given function $f(z)$ is an important task and we examine some results concerning its determination for functions with simple poles, double poles and poles of order m.

Finding the residue

If $f(z)$ has a simple pole at $z = z_0$ then

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

so that

$$(z - z_0)f(z) = b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + \dots$$

Taking limits as $z \to z_0$,

$$\lim_{z \to z_0} \{(z - z_0)f(z)\} = b_1.$$

For example, let

$$f(z) = \frac{1}{z^2 + 1} \equiv \frac{1}{(z + \mathrm{i})(z - \mathrm{i})} \equiv \frac{-\frac{1}{2i}}{z + \mathrm{i}} + \frac{\frac{1}{2\mathrm{i}}}{z - \mathrm{i}}.$$

There are simple poles at $z = -\mathrm{i}$ and $z = \mathrm{i}$. The residue at $z = \mathrm{i}$ is

$$\lim_{z \to \mathrm{i}} \left\{ (z - \mathrm{i}) \frac{1}{(z + \mathrm{i})(z - \mathrm{i})} \right\} = \lim_{z \to \mathrm{i}} \left(\frac{1}{z + \mathrm{i}} \right) = \frac{1}{2\mathrm{i}}.$$

Similarly, the residue at $z = -\mathrm{i}$ is

$$\lim_{z \to \mathrm{i}} \left\{ (z + \mathrm{i}) \frac{1}{(z + \mathrm{i})(z - \mathrm{i})} \right\} = \lim_{z \to -\mathrm{i}} \left(\frac{1}{z - \mathrm{i}} \right) = \frac{-1}{2\mathrm{i}}.$$

Identify the singularities of $f(z) = \dfrac{1}{z^2 + 4}$.

Your solution

$$f(z) = \frac{1}{(z + 2\mathrm{i})(z - 2\mathrm{i})} = \frac{-\frac{1}{4\mathrm{i}}}{z + 2\mathrm{i}} + \frac{\frac{1}{4\mathrm{i}}}{z - 2\mathrm{i}}.$$

There are simple poles at $z = -2\mathrm{i}$ and $z = 2\mathrm{i}$.

Now find the residues of $f(z)$ at $z = 2\mathrm{i}$ and at $z = -2\mathrm{i}$.

Your solution

$$\lim_{z\to 2\mathrm{i}}\left\{(z-2\mathrm{i})\frac{1}{(z+2\mathrm{i})(z-2\mathrm{i})}\right\} = \lim_{z\to 2\mathrm{i}}\left(\frac{1}{z+2\mathrm{i}}\right) = \frac{1}{4\mathrm{i}}.$$

Similarly at $z=-2\mathrm{i}$.

$$\lim_{z\to -2\mathrm{i}}\left\{(z+2\mathrm{i})\frac{1}{(z+2\mathrm{i})(z-2\mathrm{i})}\right\} = \lim_{z\to -2\mathrm{i}}\left(\frac{1}{z-2\mathrm{i}}\right) = -\frac{1}{4\mathrm{i}}.$$

In general the residue at a pole of order m at $z = z_0$ is

$$b_1 = \frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}.$$

As an example, if $f(z) = \dfrac{z^2+1}{(z+1)^3}$, $f(z)$ has a pole of order 3 at $z=-1$ $(m=3)$.

We need first

$$\frac{d^2}{\mathrm{d}z^2}\left[(z+1)^3\frac{(z^2+1)}{(z+1)^3}\right] = \frac{d^2}{\mathrm{d}z^2}[z^2+1] = \frac{d}{\mathrm{d}z}[2z] = 2.$$

Then $b_1 = \dfrac{1}{2!}\times 2 = 1$.

We have a useful result which allows us to evaluate contour integrals quickly when $f(z)$ has only poles inside the contour.

Key Point

The Residue Theorem

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\times(\text{sum of the residues at the poles inside } C).$$

Example Let $f(z) = \dfrac{1}{z^2+1}$. Find the integrals $\oint_{C_1} \mathrm{d}z$, $\oint_{C_2} \mathrm{d}z$ and $\oint_{C_3} \mathrm{d}z$ in which C_1 is the circle $|z - \mathrm{i}| = 1$, C_2 is the circle $|z + \mathrm{i}| = 1$, and C_3 is any path enclosing both $z = \mathrm{i}$ and $z = -\mathrm{i}$. See Figure 2.

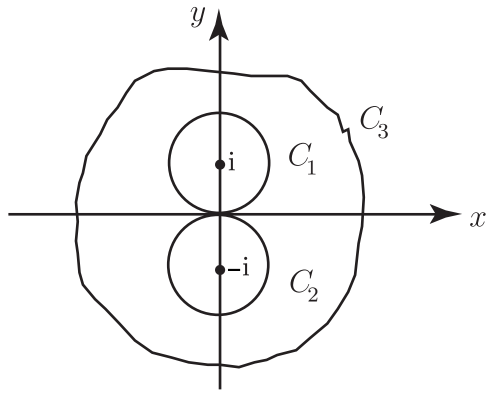


Figure 2

Solution

Figure 2 shows that only the pole at $z = \mathrm{i}$ lies inside C_1. The residue at this pole is $\dfrac{1}{2\mathrm{i}}$, as we found earlier. Hence $\oint_{C_1} f(z)\mathrm{d}z = 2\pi\mathrm{i} \times \dfrac{1}{2\mathrm{i}} = \pi$.

Also, the residue at $z = -\mathrm{i}$, the only pole inside C_2, is $-\dfrac{1}{2\mathrm{i}}$. Hence

$$\oint_{C_2} f(z)\mathrm{d}z = -2\pi\mathrm{i} \times \frac{1}{2\mathrm{i}} = -\pi.$$

Note that the contour C_3 encloses both poles so that $\oint_{C_3} f(z)\mathrm{d}z = 2\pi\mathrm{i}\left(\dfrac{1}{2\mathrm{i}} - \dfrac{1}{2\mathrm{i}}\right) = 0$.

Exercises

1. Identify the singularities of $f(z) = \dfrac{1}{z^2(z^2+9)}$ and find the residue at each of them.

2. Find the integral $\oint_C f(z)\mathrm{d}z$ where $f(z) = \dfrac{1}{z^2+4}$ and C is

   (a) the circle $|z - 2\mathrm{i}| = 1$;

   (b) the circle $|z + 2\mathrm{i}| = 1$;

   (c) any closed path enclosing both $z = 2\mathrm{i}$ and $z = -2\mathrm{i}$.

1. Double pole at $z = 0$, simple poles at $z = 3\text{i}$ and $z = -3\text{i}$.

Residue at $z = 3\text{i}$

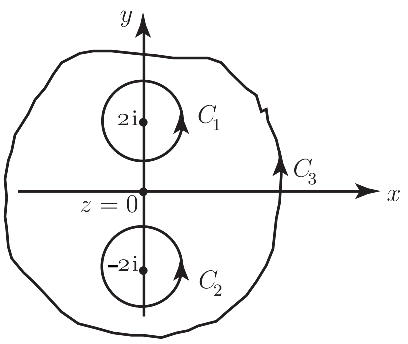
$$= \lim_{z\to 3\text{i}} \left\{ (z-3\text{i})\frac{1}{z^2(z+3\text{i})(z-3\text{i})} \right\} = \lim_{z\to 3\text{i}} \left\{ \frac{1}{z^2(z+3\text{i})} \right\} = \frac{1}{9\text{i}^2} \times \frac{1}{6\text{i}} = -\frac{1}{54\text{i}} = \frac{1}{54}\text{i}.$$

Residue at $z = -3\text{i}$

$$= \lim_{z\to -3\text{i}} \left\{ (z+3\text{i})\frac{1}{z^2(z+3\text{i})(z-3\text{i})} \right\} = \lim_{z\to -3\text{i}} \left\{ \frac{1}{z^2(z-3\text{i})} \right\} = \frac{1}{9\text{i}^2} \times \frac{1}{-6\text{i}} = -\frac{1}{54}\text{i}.$$

For the double pole at $z = 0$ we find $\dfrac{\text{d}}{\text{d}z}\left\{(z-0)^2 f(z)\right\} = \dfrac{\text{d}}{\text{d}z}\left(\dfrac{1}{z^2+9}\right) = \dfrac{-2z}{(z^2+9)^2}.$

Then, $\displaystyle\lim_{z\to 0}\left(\frac{-2z}{(z^2+9)^2}\right) = 0.$

2. 

$$f(z) = \frac{1}{(z+2\text{i})(z-2\text{i})}$$

(a) Only the pole at $z = 2\text{i}$ lies inside C_1. The residue there is $\displaystyle\lim_{z\to 2\text{i}}\left(\frac{1}{z+2\text{i}}\right) = \frac{1}{4\text{i}}.$

Hence $\displaystyle\oint_{C_1} f(z)\text{d}z = 2\pi\text{i} \times \frac{1}{4\text{i}} = \frac{\pi}{2}.$

(b) Only the pole at $z = -2\text{i}$ lies inside C_2. The residue there is $\displaystyle\lim_{z\to -2\text{i}} \frac{1}{z-2\text{i}} = -\frac{1}{4\text{i}}.$

Hence $\displaystyle\oint_{C_2} f(z)\text{d}z = 2\pi\text{i} \times (-\frac{1}{4\text{i}}) = -\frac{\pi}{2}.$

(c) The contour C_3 encloses both poles so that

$$\oint_{C_3} f(z)\text{d}z = 2\pi\text{i}\left(\frac{1}{4\text{i}} - \frac{1}{4\text{i}}\right) = 0.$$