# Changing Coordinates 27.4

## Introduction

We have seen how changing the variable of integration of a single integral or changing the coordinate system for multiple integrals can make integrals easier to evaluate. In this section we introduce the Jacobian. The Jacobian gives a general method for transforming the coordinates of any multiple integral.

## Prerequisites

Before starting this Section you should ...

① have a thorough understanding of the various techniques of integration

② be familiar with the concept of a function of several variables

③ be able to evaluate the determinant of a matrix

④ have completed the previous block

## Learning Outcomes

After completing this Section you should be able to ...

✓ decide which coordinate transformation simplify an integral

✓ determine the Jacobian for a coordinate transformation

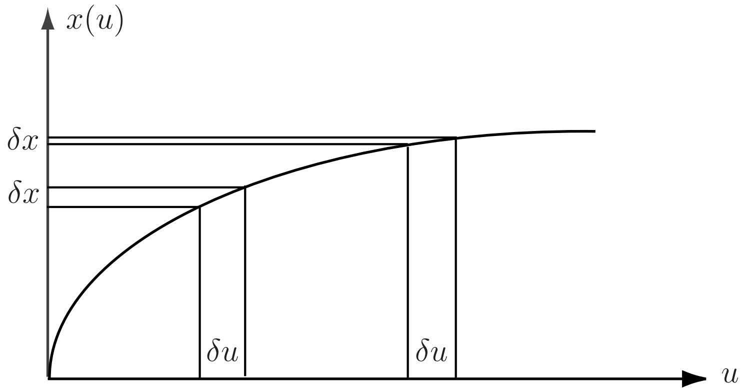
✓ evaluate multiple integrals using a transformation of coordinates

# 1. You have seen the Jacobian before

When the method of substitution is used to solve an integral of the form

$$\int_a^b f(x)\ \mathrm{d}x$$

three parts of the integral are changed, the limits, the function and the infinitesimal $\mathrm{d}x$. So if the substitution is of the form $x = x(u)$ the $u$ limits, $c$ and $d$, are found by solving $a = x(c)$ and $b = x(d)$ and the function is expressed in terms of $u$ as $f(x(u))$




The above diagram shows why the $\mathrm{d}x$ needs to be changed. While the $\delta u$ is the same length for all $u$, the $\delta x$ change as $u$ changes. The rate at which they change is precisely $\dfrac{\mathrm{d}}{\mathrm{d}u}x(u)$. This gives the relation

$$\delta x = \frac{\mathrm{d}x}{\mathrm{d}u}\delta u$$

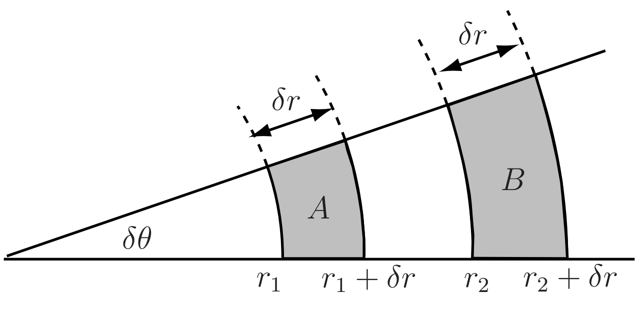
Hence the transformed integral can be written as

$$\int_a^b f(x)\ \mathrm{d}x = \int_c^d f(x(u))\,\frac{\mathrm{d}x}{\mathrm{d}u}\ \mathrm{d}u$$

Here the $\dfrac{\mathrm{d}x}{\mathrm{d}u}$ is playing the part of the Jacobian that we will define.

Another change of coordinates that you have seen is the transformations from cartesian coordinates $(x, y)$ to polar coordinates $(r, \theta)$.
Recall that a double integral in polar coordinates is expressed as

$$\int\!\!\int f(x, y)\ \mathrm{d}x\mathrm{d}y = \int\!\!\int g(r, \theta) r \mathrm{d}r\mathrm{d}\theta$$



We can see from the diagram that the area elements change in size as $r$ increases. The circumference of a circle of radius $r$ is $2\pi r$, so the length of an arc spanned by an angle $\theta$ is $2\pi r \frac{\theta}{2\pi} = r\theta$. Hence the area elements in polar coordinates are approximated by rectangles of width $\delta r$ and length $r\delta\theta$. Thus under the transformation from cartesian to polar coordinates we have the relation

$$\delta x \delta y \to r \delta r \delta \theta$$

that is, $r\delta r\delta\theta$ plays the same role as $\delta x\delta y$. This is why the $r$ term appears in the integrand. Here $r$ is playing the part of the Jacobian.

## 2. The Jacobian

Given an integral of the form

$$\int\int_A f(x,y)\ \mathrm{d}x\mathrm{d}y$$

Assume we have a change of variables of the form $x = x(u,v)$ and $y = y(u,v)$ then the Jacobian of the transformation is defined as

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

**Key Point**

Notice the pattern occurring in the $x$, $y$, $u$ and $v$. Across a row of the determinant the numerators are the same and down a column the denominators are the same.

**Notation** Different textbooks use different notation for the Jacobian. The following are equivalent.

$$J(u,v) = J(x,y;u,v) = J\left(\frac{x,y}{u,v}\right) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

The Jacobian correctly describes how area elements change under such a transformation. The required relationship is

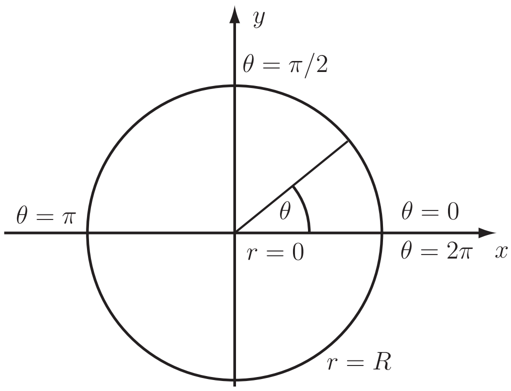
$$\mathrm{d}x\mathrm{d}y \rightarrow |J(u,v)|\,\mathrm{d}u\mathrm{d}v$$

that is, $|J(u,v)|\,\mathrm{d}u\mathrm{d}v$ plays the role of $\mathrm{d}x\mathrm{d}y$.

**Key Point**

When transforming area elements employing the Jacobian it is the modulus of the Jacobian that must be used.

**Example** Find the area of the circle of radius $R$.




**Solution**

Let $A$ be the region bounded by a circle of radius $R$ centred at the origin. Then the area of this region is $\int_A \mathrm{d}A$. We will calculate this area by changing to polar coordinates, so consider the usual transformation $x = r\cos\theta, y = r\sin\theta$ from cartesian to polar coordinates. First we require all the partial derivatives

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Thus

$$\begin{aligned} J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= \cos\theta \times r\cos\theta - (-r\sin\theta) \times \sin\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r \end{aligned}$$

**Solution (contd.)**

This confirms the previous result for polar coordinates, $\mathrm{d}x\mathrm{d}y \to r\mathrm{d}r\mathrm{d}\theta$. The limits on $r$ are $r = 0$ (centre) to $r = R$ (edge). The limits on $\theta$ are $\theta = 0$ to $\theta = 2\pi$, i.e. starting to the right and going once round anticlockwise.
The required area is

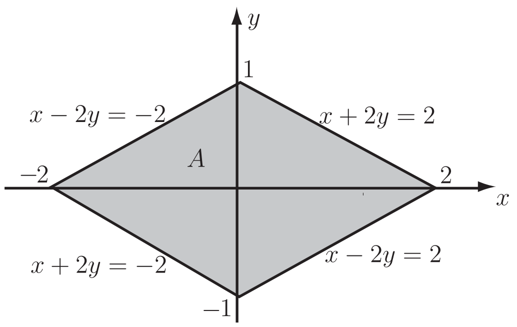
$$\int_A \mathrm{d}A = \int_0^{2\pi}\int_0^R |J(r,\theta)|\,\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^R r\ \mathrm{d}r\mathrm{d}\theta = 2\pi\frac{R^2}{2} = \pi R^2$$

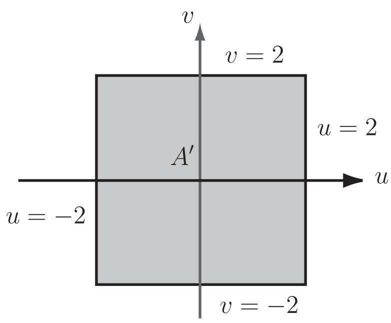
Note that here $r > 0$ so $|J(r,\theta)| = J(r,\theta) = r$.

**Example** The diamond shaped region $A$ in the diagram is bounded by the lines $x+2y = 2$, $x-2y = 2$, $x+2y = -2$ and $x-2y = -2$. We wish to evaluate the integral

$$I = \iint_A (3x+6y)^2\ \mathrm{d}A$$

over this region. Since the region $A$ is neither vertically nor horizontally simple, evaluating $I$ without changing coordinates would require separating the region into two simple triangular regions. So we use a change of coordinates to transform $A$ to a square region and evaluate $I$.




**Solution**

By considering the equations of the boundary lines of region $A$ it is easy to see that the change of coordinates

$$u = x + 2y \qquad (1) \qquad\qquad v = x - 2y \qquad (2)$$

will transform the boundary lines to $u = 2$, $u = -2$, $v = 2$ and $v = -2$. These values of $u$ and $v$ are the new limits of integration. The region $A$ will be transformed to the square region $A'$ shown above. We require the inverse transformations so that we can substitute for $x$ and $y$ in terms of $u$ and $v$. By adding (1) and (2) we obtain $u + v = 2x$ and by subtracting (1) and (2) we obtain $u - v = 4y$, thus the required change of coordinates is

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{4}(u-v)$$

**Solution (contd.)**

Substituting for $x$ and $y$ in the integrand $(3x+6y)^2$ of $I$ gives

$$\left(\frac{3}{2}(u+v)+\frac{6}{4}(u-v)\right)^2 = 9u^2$$

We have the new limits of integration and the new form of the integrand, we now require the Jacobian. The required partial derivatives are

$$\frac{\partial x}{\partial u}=\frac{1}{2} \qquad \frac{\partial x}{\partial v}=\frac{1}{2} \qquad \frac{\partial y}{\partial u}=\frac{1}{4} \qquad \frac{\partial y}{\partial v}=-\frac{1}{4}$$

Then the Jacobian is

$$J(u,v)=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

Then $\mathrm{d}A' = |J(u,v)|\mathrm{d}A = \frac{1}{4}\mathrm{d}A$. Using the new limits, integrand and the Jacobian, the integral can be written

$$I=\int_{-2}^{2}\int_{-2}^{2}\frac{9}{4}u^2\mathrm{d}u\mathrm{d}v.$$

You should evaluate this integral and check that $I = 48$.

1. Given the transformations $u = x+y$, $v = x-y$ express $x$ and $y$ in terms of $u$ and $v$ to find the inverse transformations.
2. Find the Jacobian $J(u,v)$ for the transformation in part 1.
3. Express the integral $I = \iint (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$ in terms of $u$ and $v$, using the transformations in part 1.
4. Find the limits on $u$ and $v$ for the rectangle with vertices $(x,y) = (0,0),\ (2,2),\ (-1,5),\ (-3,3)$ and hence evaluate $I$.

**Your solution**

1.)

$x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v)$

**Your solution**

2.

$J(u,v) = -\dfrac{1}{2}$

**Your solution**

3.

$\dfrac{1}{4}\displaystyle\iint \left(u^2+v^2\right)\,\mathrm{d}u\mathrm{d}v$

**Your solution**

4.

$\frac{1}{4}\int_{u=-6}^{0}\int_{v=0}^{4}(u^2+v^2)\mathrm{d}u\mathrm{d}v = 104$

## 3. The Jacobian in 3 dimensions

When changing the coordinate system of a triple integral

$$I = \iiint_V f(x,y,z)\mathrm{d}V$$

we need to extend the above definition of the Jacobian to 3 dimensions. For given transformations $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ the Jacobian is

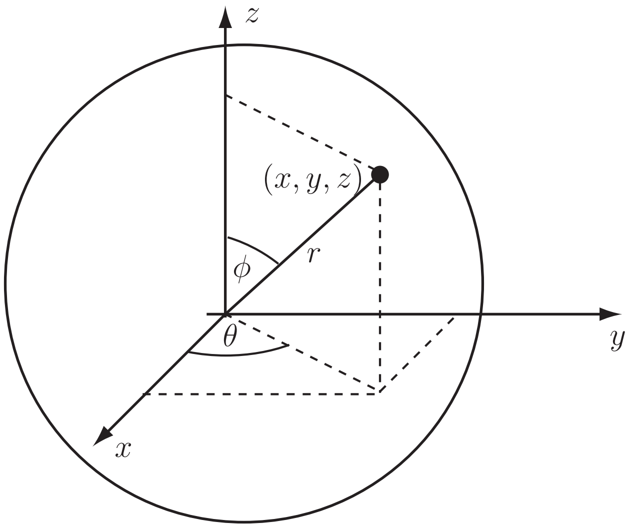
$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

**Key Point**

The same pattern persists as in the 2 dimensional case. Across a row of the determinant the numerators are the same and down a column the denominators are the same.

The volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ becomes $\mathrm{d}V = |J(u, v, w)|\mathrm{d}u\mathrm{d}v\mathrm{d}w$. As before the limits and integrand must also be transformed.

**Example** Use spherical coordinates to find the volume of a sphere of radius $R$.




**Solution**

The change of coordinates from Cartesian to spherical polar coordinates is given by the transformation equations

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi$$

We now need the nine partial derivatives

$$\begin{array}{lll}
\dfrac{\partial x}{\partial r} = \cos\theta\sin\phi & \dfrac{\partial x}{\partial \theta} = -r\sin\theta\sin\phi & \dfrac{\partial x}{\partial \phi} = r\cos\theta\cos\phi \\
\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi & \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi & \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \\
\dfrac{\partial z}{\partial r} = \cos\phi & \dfrac{\partial z}{\partial \theta} = 0 & \dfrac{\partial z}{\partial \phi} = r\sin\phi
\end{array}$$

Hence we have

$$J(r, \theta, \phi) = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

It is easiest to expand this determinant along the bottom row.

$$J(r, \theta, \phi) = \cos\phi \begin{vmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + 0 - r\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi \end{vmatrix}$$

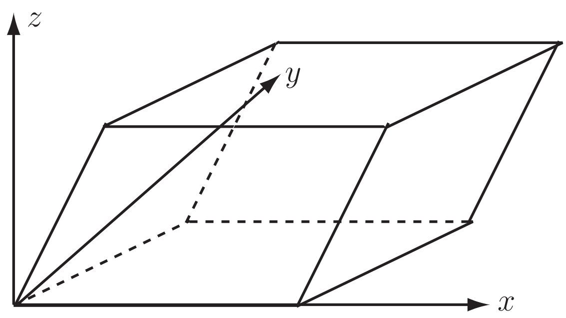
Check that this gives $J(r, \theta, \phi) = -r^2\sin\phi$. Notice that $J(r, \theta, \phi) \leq 0$ for $0 \leq \phi \leq \pi$, so $|J(r, \theta, \phi)| = r^2\sin\phi$. The limits are found as follows. The variable $\phi$ is related to 'latitude' with $\phi = 0$ representing the 'North Pole' with $\phi = \pi/2$ representing the equator and $\phi = \pi$ representing the 'South Pole'.

**Solution (contd.)**

The variable $\theta$ is related to 'longitude' with values of 0 to $2\pi$ covering every point for each value of $\phi$. Thus limits on $\phi$ are 0 to $\pi$ and limits on $\theta$ are 0 to $2\pi$. The limits on $r$ are $r = 0$ (centre) to $r = R$ (surface).
To find the volume of the sphere we then integrate the volume element $\mathrm{d}V = r^2 \sin\phi \; \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$ between these limits.

$$\text{Volume} = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\phi \; \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}R^3 \sin\phi \; \mathrm{d}\theta\mathrm{d}\phi = \int_0^{\pi} \frac{2\pi}{3}R^3 \sin\phi \; \mathrm{d}\phi = \frac{4}{3}\pi R^3$$

**Example** Find the volume integral of the function $f(x,y,z) = x - y$ over the parallelepipid with the vertices of the base at $(x,y,z) = (0,0,0), (2,0,0), (3,1,0)$ and $(1,1,0)$ and the vertices of the upper face at $(x,y,z) = (0,1,2)$, $(2,1,2)$, $(3,2,2)$ and $(1,2,2)$




**Solution**

This will be a difficult integral to derive limits for in terms of $x$, $y$ and $z$. However, it can be noted that the base is described by $z = 0$ while the upper face is described by $z = 2$. Similarly, the front face is described by $2y - z = 0$ with the back face being described by $2y - z = 2$. Finally the left face satisfies $2x - 2y + z = 0$ while the right face satisfies $2x - 2y + z = 4$.
The above suggests a change of variable with the new variables satisfying $u = 2x - 2y + z$, $v = 2y - z$ and $w = z$ and the limits on $u$ being 0 to 4, the limits on $v$ being 0 to 2 and the limits on $w$ being 0 to 2.
Inverting the relationship between $u$, $v$, $w$ and $x$, $y$ and $z$, gives

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(v+w) \qquad z = w$$

**Solution (contd.)**

The Jacobian is given by

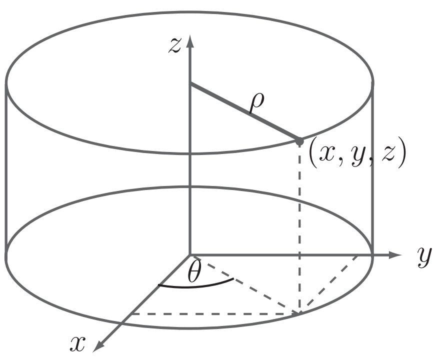
$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{4}$$

Note that the function $f(x,y,z) = x - y$ equals $\frac{1}{2}(u+v) - \frac{1}{2}(v+w) = \frac{1}{2}(u-w)$. Thus the integral is

$$\begin{aligned} & \int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{2}(u-w)\frac{1}{4}\ du\ dv\ dw \\ = \ & \int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{8}(u-w)\ du\ dv\ dw = \int_{w=0}^{2}\int_{v=0}^{2} \left[\frac{1}{16}u^2 - \frac{1}{8}uw\right]_0^4\ dv\ dw \\ = \ & \int_{w=0}^{2}\int_{v=0}^{2} \left(1 - \frac{1}{2}w\right)\ dv\ dw = \int_{w=0}^{2} \left[v - \frac{vw}{2}\right]_0^2\ dw = \int_{w=0}^{2} (2-w)\ dw \\ = \ & \left[2w - \frac{1}{2}w^2\right]_0^2 = 4 - \frac{4}{2} - 0 \\ = \ & 2 \end{aligned}$$

Find the Jacobian for the following transformations.

1. $x = 2u + 3v - w$, $y = v - 5w$, $z = u + 4w$
2. $x = u^2 + vw$, $y = 2v + u^2w$, $z = uvw$
3. Cylindrical polar coordinates. $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$



**Your solution**

1.

$-6$

**Your solution**

2.

$4u^2v - 2u^4w + u^2vw^2 - 2v^2w$

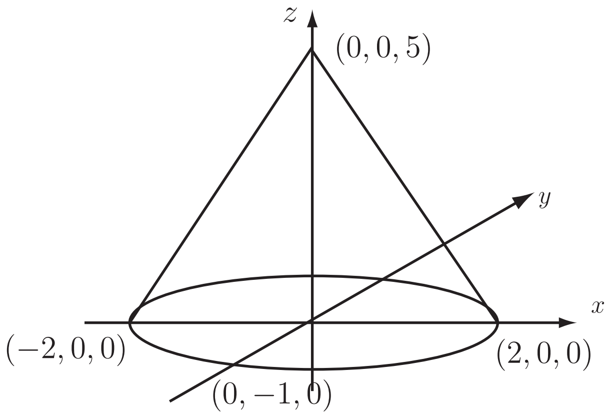
**Your solution**

3.

$p$

## Exercises

1. The function $f = x^2 + y^2$ is to be integrated over an elliptical cone with base being the ellipse, $x^2/4 + y^2 = 1, z = 0$ and apex (point) at the point $(0, 0, 5)$. This integral has relevance in topics such as moments of inertia. The integral can be made simpler by means of the change of variables $x = 2(1 - \frac{w}{5})\tau \cos\theta$, $y = (1 - \frac{w}{5})\tau \sin\theta$, $z = w$.




   (a) Find the limits on the variables $\tau$, $\theta$ and $w$.

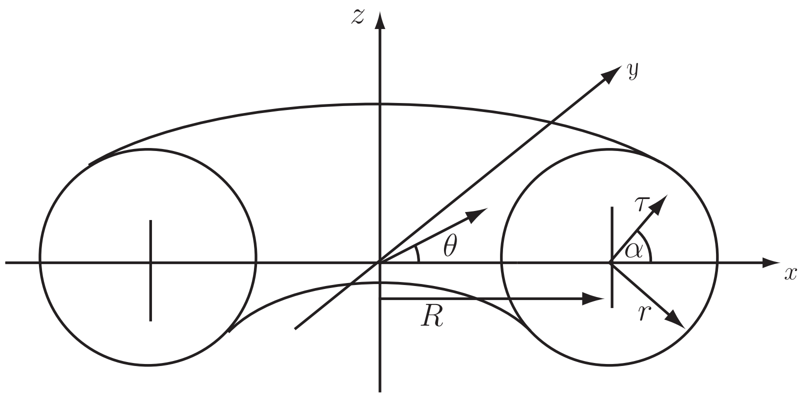
   (b) Find the Jacobian $J(\tau, \theta, w)$ for this transformation.

   (c) Express the integral $\int\int\int (x^2 + y^2)\, dz\, dy\, dz$ in terms of $\tau$, $\theta$ and $w$.

   (d) Evaluate this integral. [Hint :- it may be worth noting that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$].

2. Using cylindrical polar coordinates, integrate the function $f = z\sqrt{x^2 + y^2}$ over the volume between the surfaces $z = 0$ and $z = 1 + x^2 + y^2$ for $0 \leq x^2 + y^2 \leq 1$.

3. A torus (doughnut) has major radius $R$ and minor radius $r$. Using the transformation $x = (R + \tau\cos\alpha)\cos\theta$, $y = (R + \tau\cos\alpha)\sin\theta, z = \tau\sin\alpha$, find the volume of the torus. [Hints :- limits on $\alpha$ and $\theta$ are 0 to $2\pi$, limits on $\tau$ are 0 to $r$. Show that Jacobian is $\tau(R + \tau\cos\alpha)$].



1. a) $\tau : 0$ to $1$, $\theta : 0$ to $2\pi$, $w : 0$ to $5$,

   b) $2(1 - \frac{w}{5})^2\tau$,

   c) $2\int_{\tau=0}^{1}\int_{\theta=0}^{2\pi}\int_{w=0}^{5}(1 - \frac{w}{5})^4\tau^3(4\cos^2\theta + \sin^2\theta)\; dw\; d\theta\; d\tau$,

   d) $\frac{5}{2}\pi$

2. $\frac{92}{105}\pi$

3. $2\pi^2 Rr^2$