Changing Coordinates 27.4

Introduction

We have seen how changing the variable of integration of a single integral or changing the coordinate system for multiple integrals can make integrals easier to evaluate. In this section we introduce the Jacobian. The Jacobian gives a general method for transforming the coordinates of any multiple integral.

Prerequisites

Before starting this Section you should...

1. have a thorough understanding of the various techniques of integration
2. be familiar with the concept of a function of several variables
3. be able to evaluate the determinant of a matrix
4. have completed the previous block

Learning Outcomes

After completing this Section you should be able to...

✓ decide which coordinate transformation simplify an integral
✓ determine the Jacobian for a coordinate transformation
✓ evaluate multiple integrals using a transformation of coordinates
1. You have seen the Jacobian before

When the method of substitution is used to solve an integral of the form

$$\int_b^a f(x) \, dx$$

three parts of the integral are changed, the limits, the function and the infinitesimal $dx$. So if the substitution is of the form $x = x(u)$ the $u$ limits, $c$ and $d$, are found by solving $a = x(c)$ and $b = x(d)$ and the function is expressed in terms of $u$ as $f(x(u))$.

The above diagram shows why the $dx$ needs to be changed. While the $\delta u$ is the same length for all $u$, the $\delta x$ change as $u$ changes. The rate at which they change is precisely $\frac{dx}{du}(u)$. This gives the relation

$$\delta x = \frac{dx}{du}\delta u$$

Hence the transformed integral can be written as

$$\int_a^b f(x) \, dx = \int_c^d f(x(u)) \frac{dx}{du} \, du$$

Here the $\frac{dx}{du}$ is playing the part of the Jacobian that we will define.

Another change of coordinates that you have seen is the transformations from cartesian coordinates $(x, y)$ to polar coordinates $(r, \theta)$.

Recall that a double integral in polar coordinates is expressed as

$$\iint f(x, y) \, dx \, dy = \iint g(r, \theta) \, r \, dr \, d\theta$$
We can see from the diagram that the area elements change in size as $r$ increases. The circumference of a circle of radius $r$ is $2\pi r$, so the length of an arc spanned by an angle $\theta$ is $2\pi r \frac{\theta}{2\pi} = r\theta$. Hence the area elements in polar coordinates are approximated by rectangles of width $\delta r$ and length $r\delta \theta$. Thus under the transformation from cartesian to polar coordinates we have the relation 

$$\delta x \delta y \rightarrow r \delta r \delta \theta$$

that is, $r \delta r \delta \theta$ plays the same role as $\delta x \delta y$. This is why the $r$ term appears in the integrand. Here $r$ is playing the part of the Jacobian.

### 2. The Jacobian

Given an integral of the form 

$$\int \int_A f(x, y) \, dx \, dy$$

Assume we have a change of variables of the form $x = x(u, v)$ and $y = y(u, v)$ then the Jacobian of the transformation is defined as 

$$J(u, v) = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|$$

**Key Point**

Notice the pattern occurring in the $x$, $y$, $u$ and $v$. Across a row of the determinant the numerators are the same and down a column the denominators are the same.

**Notation** Different textbooks use different notation for the Jacobian. The following are equivalent.

$$J(u, v) = J(x, y; u, v) = J\left(\frac{x, y}{u, v}\right) = \left| \frac{\partial (x, y)}{\partial (u, v)} \right|$$
The Jacobian correctly describes how area elements change under such a transformation. The required relationship is

\[ \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta \]

that is, \( |J(u, v)| \, du \, dv \) plays the role of \( dxdy \).

**Key Point**

When transforming area elements employing the Jacobian it is the modulus of the Jacobian that must be used.

**Example** Find the area of the circle of radius \( R \).

\[
\begin{aligned}
\theta &= \pi/2 \\
r &= 0 \\
r &= R \\
\theta &= 0 \\
\theta &= 2\pi
\end{aligned}
\]

**Solution**

Let \( A \) be the region bounded by a circle of radius \( R \) centred at the origin. Then the area of this region is \( \int_A dA \). We will calculate this area by changing to polar coordinates, so consider the usual transformation \( x = r \cos \theta, y = r \sin \theta \) from cartesian to polar coordinates. First we require all the partial derivatives

\[
\begin{aligned}
\frac{\partial x}{\partial r} &= \cos \theta, & \frac{\partial y}{\partial r} &= \sin \theta, & \frac{\partial x}{\partial \theta} &= -r \sin \theta, & \frac{\partial y}{\partial \theta} &= r \cos \theta
\end{aligned}
\]

Thus

\[
J(r, \theta) = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix}
\]

\[
= \cos \theta \times r \cos \theta - (-r \sin \theta) \times \sin \theta
\]

\[
= r \left( \cos^2 \theta + \sin^2 \theta \right) = r
\]
Solution (contd.)

This confirms the previous result for polar coordinates, \( dxdy \rightarrow rdrd\theta \). The limits on \( r \) are \( r = 0 \) (centre) to \( r = R \) (edge). The limits on \( \theta \) are \( \theta = 0 \) to \( \theta = 2\pi \), i.e. starting to the right and going once round anticlockwise.

The required area is

\[
\int_A dA = \int_0^{2\pi} \int_0^R |J(r, \theta)| \, dr \, d\theta = \int_0^{2\pi} \int_0^R r \, dr \, d\theta = 2\pi \frac{R^2}{2} = \pi R^2
\]

Note that here \( r > 0 \) so \( |J(r, \theta)| = J(r, \theta) = r \).

Example

The diamond shaped region \( A \) in the diagram is bounded by the lines \( x + 2y = 2, x - 2y = 2, x + 2y = -2 \) and \( x - 2y = -2 \). We wish to evaluate the integral

\[
I = \iint_A (3x + 6y)^2 \, dA
\]

over this region. Since the region \( A \) is neither vertically nor horizontally simple, evaluating \( I \) without changing coordinates would require separating the region into two simple triangular regions. So we use a change of coordinates to transform \( A \) to a square region and evaluate \( I \).

Solution

By considering the equations of the boundary lines of region \( A \) it is easy to see that the change of coordinates

\[
u = x + 2y \quad (1) \quad \quad \quad v = x - 2y \quad (2)
\]

will transform the boundary lines to \( u = 2, \ u = -2, \ v = 2 \) and \( v = -2 \). These values of \( u \) and \( v \) are the new limits of integration. The region \( A \) will be transformed to the square region \( A' \) shown above. We require the inverse transformations so that we can substitute for \( x \) and \( y \) in terms of \( u \) and \( v \). By adding (1) and (2) we obtain \( u + v = 2x \) and by subtracting (1) and (2) we obtain \( u - v = 4y \), thus the required change of coordinates is

\[
x = \frac{1}{2} (u + v) \quad y = \frac{1}{4} (u - v)
\]
Solution (contd.)

Substituting for $x$ and $y$ in the integrand $(3x + 6y)^2$ of $I$ gives

$$\left(\frac{3}{2}(u + v) + \frac{6}{4}(u - v)\right)^2 = 9u^2$$

We have the new limits of integration and the new form of the integrand, we now require the Jacobian. The required partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{4}, \quad \frac{\partial y}{\partial v} = -\frac{1}{4}$$

Then the Jacobian is

$$J(u, v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

Then $dA' = |J(u, v)|dA = \frac{1}{4}dA$. Using the new limits, integrand and the Jacobian, the integral can be written

$$I = \int_{-2}^{2} \int_{-2}^{2} \frac{9}{4} u^2 du dv.$$  

You should evaluate this integral and check that $I = 48$.

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1. Given the transformations $u = x + y, \ v = x - y$ express $x$ and $y$ in terms of $u$ and $v$ to find the inverse transformations.

2. Find the Jacobian $J(u, v)$ for the transformation in part 1.

3. Express the integral $I = \int \int (x^2 + y^2) \ dx \ dy$ in terms of $u$ and $v$, using the transformations in part 1.

4. Find the limits on $u$ and $v$ for the rectangle with vertices $(x, y) = (0, 0), \ (2, 2), \ (-1, 5), \ (-3, 3)$ and hence evaluate $I$.

Your solution

1.)

$$(a - n)^\frac{2}{3} = \sqrt[3]{(a + n)^2} = x$$
2. \[ \frac{\partial f}{\partial n} = \frac{f}{n} \]

3. \[ \frac{\partial f}{\partial n} = \frac{f}{n} \int \frac{v}{1} \int \frac{u}{1} \]

4. \[ \frac{\partial f}{\partial n} = \frac{f}{n} \int \frac{v}{1} \int \frac{u}{1} \]

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3. The Jacobian in 3 dimensions

When changing the coordinate system of a triple integral

\[ I = \iiint_V f(x, y, z) \, dV \]

we need to extend the above definition of the Jacobian to 3 dimensions. For given transformations \( x = x(u, v, w), y = y(u, v, w) \) and \( z = z(u, v, w) \) the Jacobian is

\[
J(u, v, w) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]

---

Key Point

The same pattern persists as in the 2 dimensional case. Across a row of the determinant the numerators are the same and down a column the denominators are the same.
The volume element \( dV = dx dy dz \) becomes \( dV = |J(u, v, w)| du dv dw \). As before the limits and integrand must also be transformed.

**Example** Use spherical coordinates to find the volume of a sphere of radius \( R \).

![Diagram of spherical coordinates](image)

**Solution**

The change of coordinates from Cartesian to spherical polar coordinates is given by the transformation equations

\[
\begin{align*}
x &= r \cos \theta \sin \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \phi
\end{align*}
\]

We now need the nine partial derivatives

\[
\begin{align*}
\frac{\partial x}{\partial r} &= \cos \theta \sin \phi \\
\frac{\partial y}{\partial r} &= \sin \theta \sin \phi \\
\frac{\partial z}{\partial r} &= \cos \phi \\
\frac{\partial x}{\partial \theta} &= -r \sin \theta \sin \phi \\
\frac{\partial y}{\partial \theta} &= r \cos \theta \sin \phi \\
\frac{\partial z}{\partial \theta} &= 0 \\
\frac{\partial x}{\partial \phi} &= r \cos \theta \cos \phi \\
\frac{\partial y}{\partial \phi} &= r \sin \theta \sin \phi \\
\frac{\partial z}{\partial \phi} &= -r \sin \phi
\end{align*}
\]

Hence we have

\[
J(r, \theta, \phi) = \begin{vmatrix} 
\cos \theta \sin \phi & -r \sin \theta \sin \phi & r \cos \theta \cos \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\cos \phi & 0 & -r \sin \phi 
\end{vmatrix}
\]

It is easiest to expand this determinant along the bottom row.

\[
J(r, \theta, \phi) = \cos \phi \begin{vmatrix} 
-r \sin \theta \sin \phi & r \cos \theta \cos \phi \\
r \cos \theta \sin \phi & r \sin \theta \cos \phi 
\end{vmatrix} + 0 - r \sin \phi
\]

Check that this gives \( J(r, \theta, \phi) = -r^2 \sin \phi \). Notice that \( J(r, \theta, \phi) \leq 0 \) for \( 0 \leq \phi \leq \pi \), so \( |J(r, \theta, \phi)| = r^2 \sin \phi \). The limits are found as follows. The variable \( \phi \) is related to ‘latitude’ with \( \phi = 0 \) representing the ‘North Pole’ with \( \phi = \pi/2 \) representing the equator and \( \phi = \pi \) representing the ‘South Pole’.
Solution (contd.)

The variable $\theta$ is related to ‘longitude’ with values of 0 to $2\pi$ covering every point for each value of $\phi$. Thus limits on $\phi$ are 0 to $\pi$ and limits on $\theta$ are 0 to $2\pi$. The limits on $r$ are $r = 0$ (centre) to $r = R$ (surface).

To find the volume of the sphere we then integrate the volume element $dV = r^2 \sin \phi \, dr \, d\theta \, d\phi$ between these limits.

$$
Volume = \int_0^\pi \int_0^{2\pi} \int_0^R r^2 \sin \phi \, dr \, d\theta \, d\phi = \int_0^\pi \int_0^{2\pi} \frac{1}{3} R^3 \sin \phi \, d\theta \, d\phi = \int_0^\pi 2\pi R^3 \sin \phi \, d\phi = \frac{4}{3} \pi R^3
$$

Example

Find the volume integral of the function $f(x, y, z) = x - y$ over the parallelepiped with the vertices of the base at $(x, y, z) = (0, 0, 0), (2, 0, 0), (3, 1, 0)$ and $(1, 1, 0)$ and the vertices of the upper face at $(x, y, z) = (0, 1, 2), (2, 1, 2), (3, 2, 2)$ and $(1, 2, 2)$

![Diagram of parallelepiped](image)

Solution

This will be a difficult integral to derive limits for in terms of $x, y$ and $z$. However, it can be noted that the base is described by $z = 0$ while the upper face is described by $z = 2$. Similarly, the front face is described by $2y - z = 0$ with the back face being described by $2y - z = 2$. Finally the left face satisfies $2x - 2y + z = 0$ while the right face satisfies $2x - 2y + z = 4$.

The above suggests a change of variable with the new variables satisfying $u = 2x - 2y + z$, $v = 2y - z$ and $w = z$ and the limits on $u$ being 0 to 4, the limits on $v$ being 0 to 2 and the limits on $w$ being 0 to 2.

Inverting the relationship between $u, v, w$ and $x, y$ and $z$, gives

$$
x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(v + w) \quad z = w
$$
Solution (contd.)

The Jacobian is given by

\[
J(u, v, w) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix} = \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 1
\end{vmatrix} = \frac{1}{4}
\]

Note that the function \( f(x, y, z) = x - y \) equals \( \frac{1}{2}(u + v) - \frac{1}{2}(v + w) = \frac{1}{2}(u - w) \). Thus the integral is

\[
\int_{w=0}^{2} \int_{v=0}^{2} \int_{u=0}^{4} \frac{1}{2}(u - w) \frac{1}{4} du \, dv \, dw
\]

\[
= \int_{w=0}^{2} \int_{v=0}^{2} \int_{u=0}^{4} \frac{1}{8}(u - w) \, du \, dv \, dw = \int_{w=0}^{2} \int_{v=0}^{2} \left[ \frac{1}{16} u^2 - \frac{1}{8} uw \right]_0^4 \, dv \, dw
\]

\[
= \int_{w=0}^{2} \int_{v=0}^{2} \left( 1 - \frac{1}{2} w \right) \, dv \, dw = \int_{w=0}^{2} \left[ v - \frac{v w}{2} \right]_0^2 \, dw = \int_{w=0}^{2} (2 - w) \, dw
\]

\[
= \left[ 2w - \frac{1}{2} w^2 \right]_0^2 = 4 - \frac{4}{2} - 0
\]

\[
= 2
\]

Find the Jacobian for the following transformations.

1. \( x = 2u + 3v - w, \ y = v - 5w, \ z = u + 4w \)

2. \( x = u^2 + vw, \ y = 2v + u^2 w, \ z = uvw \)

3. Cylindrical polar coordinates. \( x = \rho \cos \theta, \ y = \rho \sin \theta, \ z = z \)
Your solution

1.

Your solution

2.

\[ m_{c}a_{\zeta} - \zeta m_{c}n + n_{v}n_{\zeta} - a_{c}n_{\zeta} \]

Your solution

3.

\[ \delta \]
Exercises

1. The function \( f = x^2 + y^2 \) is to be integrated over an elliptical cone with base being the ellipse, \( x^2/4 + y^2 = 1, z = 0 \) and apex (point) at the point \( (0,0,5) \). This integral has relevance in topics such as moments of inertia. The integral can be made simpler by means of the change of variables \( x = 2(1 - \omega^5) \tau \cos \theta, y = (1 - \omega^5) \tau \sin \theta, z = w \).

   (a) Find the limits on the variables \( \tau, \theta \) and \( w \).
   (b) Find the Jacobian \( J(\tau, \theta, w) \) for this transformation.
   (c) Express the integral \( \int \int \int (x^2 + y^2) \, dz \, dy \, dz \) in terms of \( \tau, \theta \) and \( w \).
   (d) Evaluate this integral. [Hint :- it may be worth noting that \( \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \)].

2. Using cylindrical polar coordinates, integrate the function \( f = z \sqrt{x^2 + y^2} \) over the volume between the surfaces \( z = 0 \) and \( z = 1 + x^2 + y^2 \) for \( 0 \leq x^2 + y^2 \leq 1 \).

3. A torus (doughnut) has major radius \( R \) and minor radius \( r \). Using the transformation \( x = (R + \tau \cos \alpha) \cos \theta, y = (R + \tau \cos \alpha) \sin \theta, z = \tau \sin \alpha \), find the volume of the torus. [Hints :- limits on \( \alpha \) and \( \theta \) are 0 to \( 2\pi \), limits on \( \tau \) are 0 to \( r \). Show that Jacobian is \( \tau(R + \tau \cos \alpha) \)].
1.a) \( \tau:0 \to 1, \quad \theta:0 \to 2\pi, \quad w:0 \to 5 \),

b) \( 2(1 - w^5)^2 \tau \),

c) \( 2 \int_1^{\pi} \int_0^{2\pi} \int_0^5 (1 - w^5)^4 \tau^3 (4\cos^2 \theta + \sin^2 \theta) \, dw \, d\theta \, d\tau \),

d) \( \frac{\tau}{2} (p \theta p \theta p \theta) (1 - \mu) \int_0^1 \int_0^2 \int_0^\pi \tau^2 \theta^2 \),