Orthogonal Curvilinear Coordinates

28.3

Introduction

The derivatives div, grad and curl from Section 29.2 can be carried out using coordinate systems other than the rectangular cartesian coordinates. This Section shows how to calculate these derivatives in other coordinate systems. Two coordinate systems - cylindrical polar coordinates and spherical polar coordinates - will be illustrated.

Prerequisites

Before starting this Section you should . . .

① be able to find the gradient, divergence and curl of a field in cartesian coordinates.

② be familiar with polar coordinates

Learning Outcomes

After completing this Section you should be able to . . .

✓ be able to find the divergence, gradient or curl of a vector or scalar field expressed in terms of orthogonal curvilinear coordinates.

1. Orthogonal Curvilinear Coordinates

The results shown in Section 29.2 have been given in terms of the familiar cartesian (x, y, z) coordinate system. However, other coordinate systems can be used to better describe some physical situations. A set of coordinates $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$ where the directions at any point indicated by u, v and w are orthogonal (perpendicular) to each other is referred to as a set of *orthogonal curvilinear coordinates*. With each coordinate is associated a *scale factor* h_u, h_v or h_w respectively where $h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$ (with similar expressions for h_v and h_w). The scale factor gives a measure of how a change in the coordinate changes the position of a point.

Two commonly-used sets of orthogonal curvilinear coordinates are *cylindrical polar coordinates* and *spherical polar coordinates*. These are similar to the plane polar coordinates introduced in Section 17.2 but represent extensions to three dimensions.

Cylindrical Polar Coordinates

This corresponds to plane polar (ρ, ϕ) coordinates with an added $z-$coordinate directed out of the xy plane. Normally the variables ρ and ϕ are used instead of r and θ to give the three coordinates ρ, ϕ and z. A cylinder has equation $\rho =$ constant.
The relationship between the coordinate systems is given by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

(i.e. the same z is used by the two coordinate systems). See Figure 1.

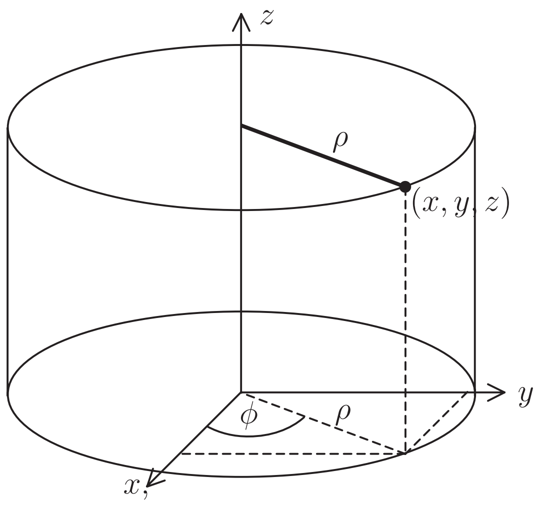

Figure 1

The scale factors h_ρ, h_ϕ and h_z are given as follows

$$h_\rho = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = \sqrt{(\cos\phi)^2 + (\sin\phi)^2 + 0} = 1$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-\rho\sin\phi)^2 + (\rho\cos\phi)^2 + 0} = \rho$$

$$h_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{(0^2 + 0^2 + 1^2)} = 1$$

Spherical Polar Coordinates

In this system a point is referred to by its distance from the origin r and two angles ϕ and θ. The angle θ is the angle between the positive z-axis and the line from the origin to the point. The angle ϕ is the angle from the x-axis to the projection of the point in the xy plane. A useful analogy is of latitude, longitude and height on Earth. The variable r plays the role of height (but height measured above the centre of Earth rather than from the surface). The variable θ plays the role of latitude but is modified so that $\theta = 0$ represents the North Pole, $\theta = 90^\circ = \dfrac{\pi}{2}$ represents the equator and $\theta = 180^\circ = \pi$ represents the South Pole. The variable ϕ plays the role of longitude. A sphere has equation $r =$ constant.
The relationship between the coordinate systems is given by

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta. \qquad \text{See Figure 2.}$$

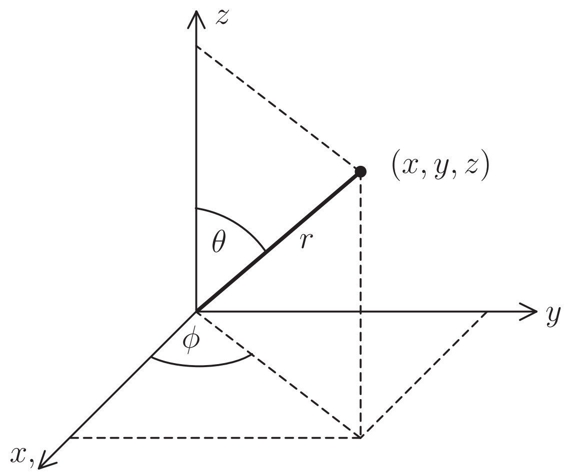

Figure 2

The scale factors h_r, h_θ and h_ϕ are given by

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2} = r$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\sin\phi)^2 + 0} = r\sin\theta$$

2. Vector Derivatives in Orthogonal Coordinates

Given an orthogonal coordinate system u, v, w with unit vectors $\underline{\hat{u}}$, $\underline{\hat{v}}$ and $\underline{\hat{w}}$ and scale factors, h_u, h_v and h_w, it is possible to find the derivatives ∇f, $\nabla \cdot \underline{F}$ and $\nabla \times \underline{F}$.

It is found that

$$\text{grad } f = \nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\underline{\hat{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\underline{\hat{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\underline{\hat{w}}$$

If $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{div } \underline{F} = \nabla \cdot \underline{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right]$$

Also if $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{curl } \underline{F} = \nabla \times \underline{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\underline{\hat{u}} & h_v\underline{\hat{v}} & h_w\underline{\hat{w}} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Key Point

In orthogonal curvilinear coordinates, the vector derivatives ∇f, $\nabla \cdot \underline{F}$ and $\nabla \times \underline{F}$ are influenced by the scale factors h_u, h_v and h_w.

3. Cylindrical Polar Coordinates

In cylindrical polar coordinates (ρ, ϕ, z), the 3 unit vectors are $\underline{\hat{\rho}}$, $\underline{\hat{\phi}}$ and $\underline{\hat{z}}$ with scale factors

$h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$.

The quantities ρ and ϕ are related to x and y by $x = \rho\cos\phi$ and $y = \rho\sin\phi$. The unit vectors are $\underline{\hat{\rho}} = \cos\phi\underline{i} + \sin\phi\underline{j}$ and $\underline{\hat{\phi}} = -\sin\phi\underline{i} + \cos\phi\underline{j}$. In cylindrical polar coordinates, grad $f = \nabla f = \dfrac{\partial f}{\partial \rho}\underline{\hat{\rho}} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\underline{\hat{\phi}} + \dfrac{\partial f}{\partial z}\underline{\hat{z}}$
The scale factor ρ is necessary in the ϕ-component because the derivatives with respect to ϕ are distorted by the distance from the axis $\rho = 0$.

If $\underline{F} = F_\rho\underline{\hat{\rho}} + F_\phi\underline{\hat{\phi}} + F_z\underline{\hat{z}}$ then

$$\text{div } \underline{F} = \nabla \cdot \underline{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right]$$

$$\text{curl}\,\underline{F} = \nabla\times\underline{F} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.$$

Example In cylindrical polar coordinates, find ∇f for

(a) $f = \rho^2 + z^2$

(b) $f = \rho^3\sin\phi$

(c) Show that the result for (b) is consistent with that found working in cartesian coordinates.

Solution

(a) If $f = \rho^2 + z^2$ then $\dfrac{\partial f}{\partial\rho} = 2\rho, \dfrac{\partial f}{\partial\phi} = 0$ and $\dfrac{\partial f}{\partial z} = 2z$ so $\nabla f = 2\rho\underline{\hat{\rho}} + 2z\underline{\hat{z}}$.

(b) If $f = \rho^3\sin\phi$ then $\dfrac{\partial f}{\partial\rho} = 3\rho^2\sin\phi$, $\dfrac{\partial f}{\partial\phi} = \rho^3\cos\phi$ and $\dfrac{\partial f}{\partial z} = 0$ and hence,
$\nabla f = 3\rho^2\sin\phi\underline{\hat{\rho}} + \rho^2\cos\phi\underline{\hat{\phi}}$.

(c) $f = \rho^3\sin\phi = \rho^2\rho\sin\phi = (x^2+y^2)y = x^2y + y^3$ so $\nabla f = 2xy\underline{i} + (x^2+3y^2)\underline{j}$.
Using cylindrical polar coordinates, from (b) we have

$$\begin{aligned}
\nabla f &= 3\rho^2\sin\phi\underline{\hat{\rho}} + \rho^3\cos\phi\underline{\hat{\phi}} \\
&= 3\rho^2\sin\phi(\cos\phi\underline{i} + \sin\phi\underline{j}) + \rho^2\cos\phi(-\sin\phi\underline{i} + \cos\phi\underline{j}) \\
&= \left[3\rho^2\sin\phi\cos\phi - \rho^2\sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2\sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} \\
&= \left[2\rho^2\sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2\sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} \\
&= 2xy\underline{i} + (3y^2 + x^2)\underline{j}
\end{aligned}$$

So the results using cartesian and cylindrical polar coordinates are consistent.

Example Find $\nabla\cdot\underline{F}$ for $\underline{F} = F_\rho\underline{\hat{\rho}} + F_\phi\underline{\hat{\phi}} + F_z\underline{\hat{z}} = \rho^3\underline{\hat{\rho}} + \rho z\underline{\hat{\phi}} + \rho z\sin\phi\underline{\hat{z}}$. Show that the results are consistent with those found using cartesian coordinates.

Solution

Here, $F_\rho = \rho^3$, $F_\phi = \rho z$ and $F_z = \rho z \sin\phi$ so

$$\begin{aligned}
\nabla \cdot \underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right] \\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^4) + \frac{\partial}{\partial \phi}(\rho z) + \frac{\partial}{\partial z}(\rho^2 z \sin\phi)\right] \\
&= \frac{1}{\rho}\left[4\rho^3 + 0 + \rho^2 \sin\phi\right] \\
&= 4\rho^2 + \rho \sin\phi
\end{aligned}$$

Converting to cartesian coordinates,

$$\begin{aligned}
\underline{F} &= F_\rho \underline{\hat{\rho}} + F_\phi \underline{\hat{\phi}} + F_z \underline{\hat{z}} = \rho^3 \underline{\hat{\rho}} + \rho z \underline{\hat{\phi}} + \rho z \sin\phi \underline{\hat{z}} \\
&= \rho^3(\cos\phi \underline{i} + \sin\phi \underline{j}) + \rho z(-\sin\phi \underline{i} + \cos\phi \underline{j}) + \rho z \sin\phi \underline{k} \\
&= (\rho^3 \cos\phi - \rho z \sin\phi)\underline{i} + (\rho^3 \sin\phi + \rho z \cos\phi) + \rho z \underline{k} \\
&= \left[\rho^2(\rho\cos\phi) - \rho\sin\phi z\right]\underline{i} + \left[\rho^2(\rho\sin\phi) + \rho\cos\phi z\right]\underline{j} + \rho\sin\phi z \underline{k} \\
&= \left[(x^2+y^2)x - yz\right]\underline{i} + \left[(x^2+y^2)y + xz\right]\underline{j} + yz\underline{k} \\
&= (x^3 + xy^2 - yz)\underline{i} + (x^2 y + y^3 + xz)\underline{j} + yz\underline{k}
\end{aligned}$$

So

$$\begin{aligned}
\nabla \cdot \underline{F} &= \frac{\partial}{\partial x}(x^3 + xy^2 - yz) + \frac{\partial}{\partial y}(x^2 y + y^3 + xz) + \frac{\partial}{\partial z}(yz) \\
&= (3x^2 + y^2) + (x^2 + 3y^2) + y = 4x^2 + 4y^2 + y \\
&= 4(x^2 + y^2) + y \\
&= 4\rho^2 + \rho\sin\phi
\end{aligned}$$

So $\nabla \cdot \underline{F}$ is the same in both coordinate systems

Example Find $\nabla \times \underline{F}$ for $\underline{F} = \rho^2 \underline{\hat{\rho}} + z\sin\phi \underline{\hat{\phi}} + 2z\cos\phi \underline{\hat{z}}$

Solution

$$\nabla \times \underline{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho^2 & \rho z\sin\phi & 2z\cos\phi \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\underline{\rho}}\left[\frac{\partial}{\partial \phi}2z\cos\phi - \frac{\partial}{\partial z}\rho z\sin\phi\right] + \rho\hat{\underline{\phi}}\left[\frac{\partial}{\partial z}\rho^2 - \frac{\partial}{\partial \rho}2z\cos\phi\right] + \hat{\underline{z}}\left[\frac{\partial}{\partial \rho}\rho z\sin\phi - \frac{\partial}{\partial \phi}\rho^2\right]\right]$$

$$= \frac{1}{\rho}\left[\hat{\underline{\rho}}(-2z\sin\phi - \rho\sin\phi) + \rho\hat{\underline{\phi}}(0) + \hat{\underline{z}}(z\sin\phi)\right]$$

$$= -\frac{(2z\sin\phi)}{\rho}\hat{\underline{\rho}} + \frac{z\sin\phi}{\rho}\hat{\underline{z}}$$

Example A magnetic field $\underline{B}$ is given by $\underline{B} = \rho^{-2}\hat{\underline{\phi}} + k\underline{z}$. Find $\nabla \cdot \underline{B}$ and $\nabla \times \underline{B}$.

Solution

$$\nabla \cdot \underline{B} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}0 + \frac{\partial}{\partial \phi}\rho^{-2} + \frac{\partial}{\partial z}k\rho\right] = \frac{1}{\rho}[0+0+0] = 0$$

$$\nabla \times \underline{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & B_\phi & B_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho^{-2} & 0 & k \end{vmatrix}$$

$$= \underline{0}$$

All magnetic fields satisfy $\nabla \cdot \underline{B} = 0$ i.e. an absence of magnetic monopoles. There is a class of magnetic fields known as potential fields that satisfy $\nabla \times \underline{B} = \underline{O}$

(a) Using cylindrical polar coordinates, find ∇f for $f = r^2 z\sin\theta$

(b) Using cylindrical polar coordinates, find ∇f for $f = z\sin 2\theta$

Your solution

(a)

$2\rho z\sin\theta\underline{\hat{\rho}} + \rho z\cos\theta\underline{\hat{\phi}} + \rho^2\sin\theta\underline{\hat{z}}$

Your solution

(b)

$\frac{2}{\rho}z\cos 2\theta\underline{\hat{\phi}} + \sin 2\theta\underline{\hat{z}}$

Find $\nabla\cdot\underline{F}$ for $\underline{F} = \rho\cos\phi\underline{\hat{\rho}} - \rho\sin\phi\underline{\hat{\rho}} + \rho z\underline{\hat{z}}$

(a) Find the derivatives $\dfrac{\partial}{\partial\rho}[\rho F_\rho],\ \dfrac{\partial}{\partial\phi}[F_\phi],\ \dfrac{\partial}{\partial z}[\rho F_z]$

(b) Combine these to find $\nabla\cdot\underline{F}$

Your solution

(a) $2\rho\cos\phi, \quad -\rho\cos\phi, \quad \rho^2$

(b)

$$\begin{aligned} \nabla \cdot \underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right] \\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^2\cos\phi) + \frac{\partial}{\partial \phi}(-\rho\sin\phi) + \frac{\partial}{\partial z}(\rho^2 z)\right] \\ &= \frac{1}{\rho}\left[2\rho\cos\phi - \rho\cos\phi + \rho^2\right] \\ &= \rho\cos\phi + \rho^2 \end{aligned}$$

Find $\nabla \times \underline{F}$ for $\underline{F} = F_\rho\underline{\hat{\rho}} + F_\phi\underline{\hat{\phi}} + F_z\underline{\hat{z}} = \rho^3\underline{\hat{\rho}} + \rho z\underline{\hat{\phi}} + \rho z\sin\phi\underline{\hat{z}}$. Show that the results are consistent with those found using cartesian coordinates.

(a) Find the curl $\quad \nabla \times \underline{F}$

(b) Find $\underline{F}$ in cartesian coordinates.

(c) Hence find $\nabla \times \underline{F}$ in cartesian coordinates.

(d) Using $\underline{\hat{r}} = \cos\phi\underline{i} + \sin\phi\underline{j}$ and $\underline{\hat{\theta}} = -\sin\phi\underline{i} + \cos\phi\underline{j}$ show that the solution to part (a) is equal to the solution for part (c).

Your solution

(a) $(z\cos\phi - \rho)\underline{\hat{\rho}} - 2\sin\phi\underline{\hat{\phi}} + 2z\underline{\hat{z}}$

(b) $(x^3 + xy^2 - yz)\underline{i} + (x^2y + y^3 + xz)\underline{j} + yz\underline{k}$

(c) $(z - x)\underline{i} - y\underline{j} + 2z\underline{k}$

(a) For $\underline{F} = \rho\underline{\hat{\rho}} + (\rho\sin\theta + z)\underline{\hat{\phi}} + \rho z\underline{\hat{z}}$, find $\nabla\cdot\underline{F}$ and $\nabla\times\underline{F}$.

(b) For $f = r^2 z^2 cos2\phi$, find $\nabla\times(\nabla f)$.

Your solution

(a)

$1 + \cos\theta + \rho, -\underline{\hat{\rho}} - z\cos\theta\underline{\hat{\phi}} + (2\rho\sin\phi + z)\underline{\hat{z}}$

Your solution

(b)

$\underline{0}$

4. Spherical Polar Coordinates

In spherical polar coordinates (r, θ, ϕ), the 3 unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$ and $\underline{\hat{\phi}}$ with scale factors $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. The quantities r, θ and ϕ are related to x, y and z by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. In spherical polar coordinates,

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\underline{\hat{\theta}} + \frac{1}{r^2\sin\theta}\frac{\partial f}{\partial\phi}\underline{\hat{\phi}}$$

If $\underline{F} = F_r\underline{\hat{r}} + F_\theta\underline{\hat{\theta}} + F_\phi\underline{\hat{\phi}}$

then

$$\text{div } \underline{F} = \nabla\cdot\underline{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial\theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right]$$

$$\text{curl}\ \underline{F} = \nabla \times \underline{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r^2 \sin\theta \hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r^2 \sin\theta F_\phi \end{vmatrix}$$

Example In spherical polar coordinates, find ∇f for

(a) $f = r$

(b) $f = \dfrac{1}{r}$

(c) $f = r^2 \sin(\phi + \theta)$

Solution

(a)

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial f}{\partial \phi}\hat{\underline{\phi}} \\ &= \frac{\partial(r)}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial(r)}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial(r)}{\partial \phi}\hat{\underline{\phi}} \\ &= 1\hat{\underline{r}} = \hat{\underline{r}} \end{aligned}$$

(b)

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial f}{\partial \phi}\hat{\underline{\phi}} \\ &= \frac{\partial(\frac{1}{r})}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial(\frac{1}{r})}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial(\frac{1}{r})}{\partial \phi}\hat{\underline{\phi}} \\ &= -\frac{1}{r^2}\hat{\underline{r}} \end{aligned}$$

(c)

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial f}{\partial \phi}\hat{\underline{\phi}} \\ &= \frac{\partial(r^2 \sin(\phi+\theta)}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial(r^2 \sin(\phi+\theta)}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}\frac{\partial(r^2 \sin(\phi+\theta)}{\partial \phi}\hat{\underline{\phi}} \\ &= 2r\sin(\phi+\theta)\hat{\underline{r}} + \frac{1}{r}r^2\cos(\phi+\theta)\hat{\underline{\theta}} + \frac{1}{r^2 \sin\theta}r^2\cos(\phi+\theta)\hat{\underline{\phi}} \\ &= 2r\sin(\phi+\theta)\hat{\underline{r}} + r\cos(\phi+\theta)\hat{\underline{\theta}} + \frac{\cos(\phi+\theta)}{\sin\theta}\hat{\underline{\phi}} \end{aligned}$$

Example Using spherical polar coordinates, find $\nabla\cdot\underline{F}$ for the following vector functions.

(a) $\underline{F} = r\underline{\hat{r}}$ (b) $\underline{F} = r^2\sin\theta\underline{\hat{r}}$ (c) $\underline{F} = r\sin\theta\,\underline{\hat{r}} + r^2\sin\phi\,\underline{\hat{\theta}} + r\cos\theta\,\underline{\hat{\phi}}$

Solution

(a)

$$\begin{aligned}\nabla\cdot\underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial\theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\times r) + \frac{\partial}{\partial\theta}(r\sin\theta\times 0) + \frac{\partial}{\partial\phi}(r\times 0)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^3\sin\theta) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\phi}(0)\right] = \frac{1}{r^2\sin\theta}\left[3r^2\sin\theta + 0 + 0\right] = 3\end{aligned}$$

Note :- in cartesian coordinates, the corresponding vector is $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ with $\nabla\cdot\underline{F} = 1+1+1 = 3$ (hence consistency).

(b)

$$\begin{aligned}\nabla\cdot\underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ F_r) + \frac{\partial}{\partial\theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ r^2\sin\theta) + \frac{\partial}{\partial\theta}(r\sin\theta\times 0) + \frac{\partial}{\partial\phi}(r\times 0)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^4\sin^2\theta) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\phi}(0)\right] \\ &= \frac{1}{r^2\sin\theta}\left[4r^3\sin^2\theta + 0 + 0\right] = 4r\sin\theta\end{aligned}$$

(c)

$$\begin{aligned}\nabla\cdot\underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ F_r) + \frac{\partial}{\partial\theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ r\sin\theta) + \frac{\partial}{\partial\theta}(r\sin\theta\times r^2\sin\phi) + \frac{\partial}{\partial\phi}(r\times\ r\cos\theta)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^3\sin^2\theta) + \frac{\partial}{\partial\theta}(r^3\sin\theta\sin\phi) + \frac{\partial}{\partial\phi}(r^2\cos\phi)\right] \\ &= \frac{1}{r^2\sin\theta}\left[3r^2\sin^2\theta + r^3\cos\theta\ \sin\phi + 0\right] = 3\sin\theta + r\cot\theta\ \sin\phi\end{aligned}$$

Example Find $\nabla\times\underline{F}$ for the following vector fields $\underline{F}$.

(a) $\underline{F} = r^k\underline{\hat{r}}$, where k is a constant

(b) $\underline{F} = r^2\cos\theta\,\underline{\hat{r}} + \sin\theta\,\underline{\hat{\theta}} + \sin^2\theta\,\underline{\hat{\phi}}$

Solution

(a)

$$\begin{aligned}
\nabla\times\underline{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r^2\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r^2\sin\theta F_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r^2\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^k & r\times 0 & r^2\sin\theta\times 0 \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(0)-\frac{\partial}{\partial\phi}(0)\right)\underline{\hat{r}}+\left(\frac{\partial}{\partial\phi}(r^k)-\frac{\partial}{\partial r}(0)\right)r\underline{\hat{\theta}}\right. \\
&\quad \left.+\left(\frac{\partial}{\partial r}(0)-\frac{\partial}{\partial\theta}(r^k)\right)r^2\sin\theta\,\underline{\hat{\phi}}\right] \\
&= 0\,\underline{\hat{r}}+0\,\underline{\hat{\theta}}+0\,\underline{\hat{\phi}}=\underline{0}
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla\times\underline{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r^2\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r^2\sin\theta F_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r^2\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^2\cos\theta & r\times\sin\theta & r^2\sin\theta\times\sin^2\theta \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(r^2\sin^3\theta)-\frac{\partial}{\partial\phi}(r\sin\theta)\right)\underline{\hat{r}}+\left(\frac{\partial}{\partial\phi}(r^2\cos\theta)-\frac{\partial}{\partial r}(r^2\sin^3\theta)\right)r\underline{\hat{\theta}}\right. \\
&\quad \left.+\left(\frac{\partial}{\partial r}(r\sin\theta)-\frac{\partial}{\partial\theta}(r^2\cos\theta)\right)r^2\sin\theta\,\underline{\hat{\phi}}\right] \\
&= \frac{1}{r^2\sin\theta}\left[\left(3r^2\sin^2\theta\cos\theta+0\right)\underline{\hat{r}}+\left(0-2r\sin^3\theta\right)r\underline{\hat{\theta}}+\left(\sin\theta+r^2\sin\theta\right)r^2\sin\theta\,\underline{\hat{\phi}}\right] \\
&= 3\sin\theta\cos\theta\,\underline{\hat{r}}-2\sin^2\underline{\hat{\theta}}+(1+r^2)\sin\theta\,\underline{\hat{\phi}}
\end{aligned}$$

(a) Using spherical polar coordinates, find ∇f for

i. $f = r^4$

ii. $f = \dfrac{r}{r^2+1}$

iii. $f = r^2 \sin 2\theta \cos\phi$

(b) For $\underline{F} = r\sin\theta\underline{\hat{r}} + r\cos\phi\underline{\hat{\theta}} + r\sin\phi\underline{\hat{\phi}}$, find $\nabla\cdot\underline{F}$ and $\nabla\times\underline{F}$.

(c) For $\underline{F} = r^{-4}\cos\theta\underline{\hat{r}} + r^{-4}\sin\theta\underline{\hat{\theta}}$, find $\nabla\cdot\underline{F}$ and $\nabla\times\underline{F}$.

(d) For $\underline{F} = r^2\cos\theta\underline{\hat{r}} + \cos\phi\underline{\hat{\theta}}$ find $\nabla\cdot(\nabla\times\underline{F})$.

Your solution

(a)

(i) $4r^3\underline{\hat{r}}$, (ii) $\dfrac{1-r^2}{(1+r^2)^2}\underline{\hat{r}}$, (iii) $2r\sin 2\theta\cos\phi\underline{\hat{r}} + 2r\cos 2\theta\cos\phi\underline{\hat{\theta}} - 2r\cos\theta\sin\phi\underline{\hat{\phi}}$

Your solution

(b)

$\cos\phi(\cot\theta + \text{cosech}\theta) + 3\sin\theta, \cot\frac{\theta}{2}\sin\phi\underline{\hat{r}} - 2\sin\phi\underline{\hat{\theta}} + (2\cos\phi - \cos\theta)\underline{\hat{\phi}}$

Your solution

(c)

$0, -2r^{-5}\sin\theta\,\underline{\hat{\phi}}$

Your solution

(d)

0