Contents 29

vector calculus 2

1. Line integrals involving vectors
2. Surface and volume integrals
3. Integral vector theorems

Learning outcomes

In this Workbook you will learn how to integrate functions involving vectors. You will learn how to evaluate line integrals i.e. where a scalar or a vector is summed along a line or contour. You will be able to evaluate surface and volume integrals where a function involving vectors is summed over a surface or volume. You will learn about some theorems relating to line, surface or volume integrals i.e Stokes' Theorem, Gauss' Theorem and Green's Theorem.

Time allocation

You are expected to spend approximately thirteen hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Line Integrals involving Vectors

29.1

Introduction

The previous section considered the differentiation of scalar and vector fields. The current section considers how to integrate such fields along a contour of integration. Firstly, integrals along a line will be considered in a general (non-vector) context. Subsequently, line integrals involving vectors will be considered. These can integrate either to a scalar or to a vector depending on the form of integral used. Of particular interest are the integrals of conservative vector fields.

Prerequisites

Before starting this Section you should ...

① have a thorough understanding of the basic techniques of integration

② be familiar with the operators div, grad and curl

1. Line Integrals

Workbook 28 was concerned with evaluating an integral over ALL points within a rectangle or other shape (or over a cuboid or other volume). In a related manner, an integral can take place over a line or curve running through a two- (or three-) dimensional shape.

Line Integrals in Two Dimensions

A line integral in two dimensions may be written as

$$\int_C F(x,y)dw$$

There are three main features determining this integral:

$F(x,y)$ This is the function to be integrated e.g. $F(x,y) = x^2 + 4y^2$.

C This is the curve along which integration takes place. e.g. $y = x^2$ or $x = \sin y$ or $x = t - 1$; $y = t^2$. The last case is where x and y are expressed in terms of a parameter t.

dw This states the variable of the integration. Three main cases are dx, dy and ds. Here 's' is arc length and so indicates position along the curve. ds may be written as $ds = \sqrt{(dx)^2 + (dy)^2}$ or $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

A fourth case is when $F(x,y)$ dw has the form: $F_1 dx + F_2 dy$. This is a combination of the cases dx and dy.
In the case where $F(x,y) \geq 0$ and the contour lies in the Oxy plane, the integral $\int_C F(x,y)\, ds$ gives the area under the surface $z = F(x,y)$ and above the curve C. The integrals $\int_C F(x,y)\, dx$ and $\int_C F(x,y)\, dy$ give the areas of the projections of this area onto the planes above the x and y axes (see Figure 1).

Figure 1.

The technique with a line integral is to express all quantities in an integral in terms of a single variable. Often, if the integral is with respect to 'x' or 'y', the curve 'C' and the function 'F' may be expressed in terms of the relevant variable. If the integral is carried out with respect to ds, normally everything is expressed in terms of x. If x and y are given in terms of a parameter t, normally everything is expressed in terms of t.

Example Find $\int_c x\,(1 + 4y)\, dx$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$.

Solution

As this integral concerns only points along C and the integration is carried out with respect to x, y may be replaced by x^2. The limits on x will be 0 to 1. So the integral becomes

$$\begin{aligned} \int_C x(1+4y)\,dx &= \int_{x=0}^{1} x\left(1+4x^2\right)dx = \int_{x=0}^{1}\left(x+4x^3\right)dx \\ &= \left[\frac{x^2}{2}+x^4\right]_0^1 = \left(\frac{1}{2}+1\right)-(0) = \frac{3}{2} \end{aligned}$$

Example Find $\int_c x\,(1+4y)\,dy$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same as the previous Example other than dx being replaced by dy.

Solution

As this integral concerns only points along C and the integration is carried out with respect to y, everything may be expressed in terms of y, i.e. x may be replaced by $y^{1/2}$. The limits on y will be 0 to 1. So the integral becomes

$$\begin{aligned} \int_C x(1+4y)\,dy &= \int_{y=0}^{1} y^{1/2}\,(1+4y)\,dx = \int_{y=0}^{1}\left(y^{1/2}+4y^{3/2}\right)dx \\ &= \left[\frac{2}{3}y^{3/2}+\frac{8}{5}x^{5/2}\right]_0^1 = \left(\frac{2}{3}+\frac{8}{5}\right)-(0) = \frac{34}{15} \end{aligned}$$

Example Find $\int_c x\,(1+4y)\,ds$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. Once again, this is the same as the previous two examples other than the integration being carried out with respect to s, the coordinate along the curve C.

Solution

As this integral is with respect to x, all parts of the integral can be expressed in terms of x, Along $y = x^2$,$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (2x)^2} dx = \sqrt{1 + 4x^2} dx$
So, the integral is

$$\begin{aligned}\int_c x\,(1 + 4y)\,ds &= \int_{x=0}^{1} x\left(1 + 4x^2\right)\sqrt{1 + 4x^2}\,dx \\ &= \int_{x=0}^{1} x\left(1 + 4x^2\right)^{3/2} dx\end{aligned}$$

This can be evaluated using the transformation $U = 1 + 4x^2$ so $dU = 8x dx$ i.e. $dx = \frac{dU}{8}$.
When $x = 0$, $U = 1$ and when $x = 1$, $U = 5$.
The integral therefore equals

$$\begin{aligned}\int_{x=0}^{1} x\left(1 + 4x^2\right)^{3/2} dx &= \frac{1}{8}\int_{U=1}^{5} U^{3/2} dU \\ &= \frac{1}{8}\frac{2}{5}\left[U^{5/2}\right]_1^5 = \frac{1}{20}\left[5^{5/2} - 1\right] \approx 2.745\end{aligned}$$

Example Find $\int_C xy\ dx$ where, on C, x and y are given by $x = 3t^2$, $y = t^3 - 1$ for t starting at $t = 0$ and progressing to $t = 1$.

Solution

Everything can be expressed in terms of t, the parameter. Here $x = 3t^2$ so $dx = 6t\ dt$. The limits on t are $t = 0$ and $t = 1$. The integral becomes

$$\begin{aligned}\int_C xy\ dx &= \int_{t=0}^{1} 3t^2\ (t^3 - 1)\ 6t\ dt = \int_{t=0}^{1} (18t^6 - 18t^3)\ dt \\ &= \left[\frac{18}{7}t^7 - \frac{18}{4}t^4\right]_0^1 = \frac{18}{7} - \frac{9}{2} - 0 = -\frac{27}{14}\end{aligned}$$

Key Point

A line integral is normally evaluated by expressing all variables in terms of one variable.

In general $\displaystyle\int_C f(x,y)\ \mathrm{d}s \neq \int_C f(x,y)\ \mathrm{d}y \neq \int_C f(x,y)\ \mathrm{d}x$.

1. For $F(x,y) = 2x+y^2$, find $\int_C F(x,y)\,dx$, $\int_C F(x,y)\,dy$ and $\int_C F(x,y)\,ds$ where C is the line $y = 2x$ from $(0,0)$ to $(1,2)$.
   (a) Express each integral as a simple integral with respect to a single variable.
   (b) Hence evaluate each integral.
2. For $F(x,y) = 1$, find $\int_C F(x,y)\,dx$, $\int_C F(x,y)\,dy$ and $\int_C F(x,y)\,ds$ where C is the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $(1, \frac{1}{2})$ to $(2, 2 - \frac{1}{4}\ln 2)$.
3. For $F(x,y) = \sin 2x$, find $\int_C F(x,y)\,dx$, $\int_C F(x,y)\,dy$ and $\int_C F(x,y)\,ds$ where C is the curve $y = \sin x$ from $(0,0)$ to $(\frac{\pi}{2}, 1)$.

Your solution

1.)

$\int_{x=0}^{1} (2x + 4x^2)\,dx$, $\frac{7}{3}$, $\int_{y=0}^{2} (y + y^2)\,dy$, $\frac{14}{3}$, $\int_{x=0}^{1} (2x + 4x^2)\sqrt{5}\,dx$, $\frac{7}{3}\sqrt{5}$

Your solution

2.)

1, $\frac{3}{2} - \frac{1}{4}\ln 2$, $\frac{3}{2} - \frac{1}{4}\ln 2$.

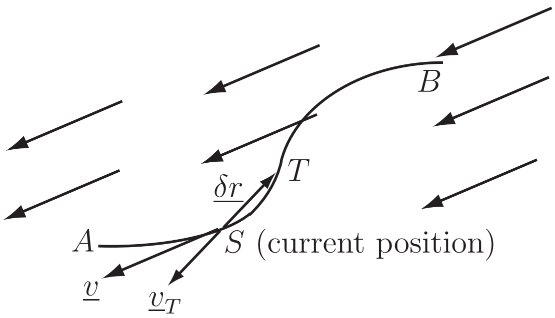
Your solution

3.)

$\frac{2}{3}$, $\frac{2}{3}$, $\frac{2}{3}(2\sqrt{2}-1)$

2. Line Integrals of Scalar Products

Integrals of the form $\int_C \underline{F} \cdot \underline{dr}$, referred to at the end of the previous sub-section, occur in applications such as the following.



Consider a cyclist riding along the road from A to B. Suppose it is necessary to find the total work the cyclist has to do in overcoming the wind.
The work done is proportional to the component of the wind speed in the direction travelled i.e. proportional to $\underline{v} \cdot \delta\underline{r}$
The total work done over AB is approximately $\sum_{\text{all } \delta r} \underline{v} \cdot \delta\underline{r}$, where the summation is carried out over all small elements making up AB.

In the limit $\delta r \to 0$, the total work done over AB is $\lim_{\delta r \to 0} \sum \text{all } \underline{v} \cdot \delta\underline{r} = \int_A^B \underline{v} \cdot \delta\underline{r}$.
This is an example of the integral along a specific line of the scalar product of a vector field and a vector describing the contour. The term *scalar line integral* is often used for integrals of this form but the role of the vector $\underline{v}$ should not be forgotten. The vector $\underline{dr}$ may be considered to be $dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$.
Multiplying out the scalar product, in three dimensions, the 'scalar line integral' of the vector $\underline{F}$ along contour C is given by $\int_C \underline{F} \cdot \underline{dr}$ and equals $\int_C [F_x\,dx + F_y\,dy + F_z\,dz]$ in three dimensions ($\int_C [F_x\,dx + F_y\,dy]$ in two dimensions.)
If the contour C has its start and end points in the same positions i.e. it represents a closed contour, the symbol $\oint_C$ rather than $\int_C$ is used i.e. $\oint_C \underline{F} \cdot \underline{dr}$.
As before, to evaluate the line integral, express the path and the function $\underline{F}$ in terms of either x, y and z, or in terms of a parameter t. Note that in examples t often represents time.

Example Find $\int_C (2xy\ dx - 5x\ dy)$ where C is the curve $y = x^3$ with x varying from $x = 0$ to $x = 1$.

Solution

It is possible to split this integral into two different integrals and express the first term as a function of x and the second term as a function of y. However, it is also possible to express everything in terms of x. Note that on C, $y = x^3$ so $dy = 3x^2\ dx$ and the integral becomes

$$\begin{aligned}\int_C (2xy\ dx - 5x\ dy) &= \int_{x=0}^{1} \left(2x\ x^3\ dx + 5x\ 3x^2\ dx\right) = \int_0^1 (2x^4 - 15x^3)dx \\ &= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4\right]_0^1 = \frac{2}{5} - \frac{15}{4} - 0 = -\frac{67}{20}\end{aligned}$$

Key Point

An integral of the form $\int_C \underline{F} \cdot \underline{dr}$ may be expressed as $\int_C [F_x\ dx + F_y\ dy + F_z\ dz]$. Knowing the expression for the path C, every term in the integral can be further expressed in terms of one of the variables x, y or z or in terms of a parameter t and hence integrated.

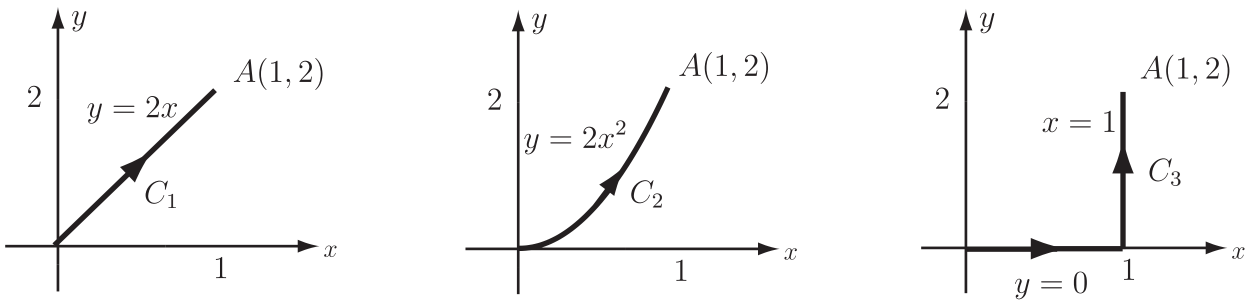
If an integral is two-dimensional there are no terms involving z.

The integral $\int_C \underline{F} \cdot \underline{dr}$ evaluates to a scalar.

Example Three paths from $(0, 0)$ to $(1, 2)$ are defined by

(i) $C_1 :\ y = 2x$
(ii) $C_2 :\ y = 2x^2$
(iii) $C_3 :\ y = 0$ from $(0, 0)$ to $(1, 0)$ and
$x = 1$ from $(1, 0)$ to $(1, 2)$

Sketch each path and find $\displaystyle\int \underline{F} \cdot \underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$, along each path.



Solution

(i) $\int \underline{F} \cdot \underline{dr} = \int \left(y^2 \; \mathrm{d}x + xy \; \mathrm{d}y \right)$. Along $y = 2x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$ so $\mathrm{d}y = 2 \; \mathrm{d}x$. Then

$$\begin{aligned} \int_{C_1} \int \underline{F} \cdot \underline{dr} &= \int_{x=0}^{1} \left((2x)^2 \; \mathrm{d}x + x\,(2x)\,(2 \; \mathrm{d}x) \right) \\ &= \int_0^1 \left(4x^2 + 4x^2 \right) \; \mathrm{d}x = \int_0^1 8x^2 \; \mathrm{d}x = \left[\frac{8}{3} x^2 \right]_0^1 = \frac{8}{3} \end{aligned}$$

Solution

(ii) $\int \underline{F} \cdot \underline{dr} = \int \left(y^2 \; \mathrm{d}x + xy \; \mathrm{d}y \right)$. Along $y = 2x^2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x$ so $\mathrm{d}y = 4x \; \mathrm{d}x$. Then

$$\begin{aligned} \int_{C_2} \underline{F} \cdot \underline{dr} &= \int_{x=0}^{1} \left(\left(2x^2 \right)^2 \; \mathrm{d}x + x \left(2x^2 \right) (4x \; \mathrm{d}x) \right) \\ &= \int_0^1 12x^4 \; \mathrm{d}x = \left[\frac{12}{5} x^5 \right]_0^1 = \frac{12}{5} \end{aligned}$$

Solution

(iii) As the contour C_3, has two distinct parts with different equations, it is necessary to break the full contour OA into the two parts, namely OB and BA where B is the point $(1, 0)$. Hence $\displaystyle\int_{C_3} \underline{F} \cdot \underline{dr} = \int_O^B \underline{F} \cdot \underline{dr} + \int_B^A \underline{F} \cdot \underline{dr}$

Along OB, $y = 0$ so $\mathrm{d}y = 0$. Then

$$\int_O^B \underline{F} \cdot \underline{dr} = \int_{x=0}^1 \left(0^2 \ \mathrm{d}x + x \times 0 \times 0\right) = \int_0^1 0 \ \mathrm{d}x = 0$$

Along AB, $x = 1$ so $\mathrm{d}x = 0$. Then

$$\int_A^B \underline{F} \cdot \underline{dr} = \int_{y=0}^2 \left(y^2 \times 0 + 1 \times y \times \mathrm{d}y\right) = \int_0^2 y \ \mathrm{d}y = \left[\frac{1}{2}y^2\right]_0^2 = 2.$$

Hence $\displaystyle\int_{C_3} \underline{F} \cdot \underline{dr} = 0 + 2 = 2$

Key Point

In general the value of the line integral depends on the path of integration as well as the end points.

Example Find $\displaystyle\int_A^O \underline{F} \cdot \underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$ (as in the previous example) and the path from A to O is the straight line from $(1, 2)$ to $(0, 0)$, that is the reverse of C_1 in (i) above.

Deduce $\displaystyle\oint_C \underline{F} \cdot \underline{dr}$, the integral around the closed path C formed by the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$ and the line $y = 2x$ from $(1, 2)$ to $(0, 0)$.

Solution

Reversing the path swaps the limits of integration, this results in a change of sign for the value of the integral.

$$\int_A^O \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_O^A \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\frac{8}{3}$$

The integral along the parabola (calculated in (iii) above) evaluates to $\dfrac{12}{5}$, then

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \oint_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15} \approx -0.267$$

Example Consider the two vector fields

$$\begin{aligned} \underline{F} &= y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k} \\ \text{and} \quad \underline{G} &= x\underline{i} + (4x-y)\underline{j} \end{aligned}$$

Let C_1 and C_2 be the curves from $O = (0,0,0)$ to $A = (1,1,1)$, given by

$$\begin{aligned} C_1 &: x = t, \quad y = t, \quad z = t \quad (0 \le t \le 1) \\ C_2 &: x = t, \quad y = t^2, \quad z = t^2 \quad (0 \le t \le 1) \end{aligned}$$

(a) Evaluate the scalar integral of each vector field along each path.

(b) Find the value of $\oint_C \underline{F}\cdot\underline{dr}$ and $\oint_C \underline{F}\cdot\underline{dr}$ where C is the closed path along C_1 from O to A and back along C_2 to O.

Solution

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = 1 \text{ and } \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\underline{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\underline{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

Now by substituting for $x = y = z = t$ in $\underline{F}$ we have

$$\underline{F} = t^5\underline{i} + 2t^5\underline{j} + 3t^5\underline{k}$$

Hence $\underline{F} \cdot \dfrac{d\underline{r}}{\mathrm{d}t} = t^5 + 2t^5 + 3t^5 = 6t^5$. The values of $t = 0$ and $t = 1$ correspond to the start and end point of C_1 and so these are the required limits of integration. Now

$$\int_{C_1} \underline{F} \cdot \underline{dr} = \int_0^1 \underline{F} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\ \mathrm{d}t = \int_0^1 6t^5\ \mathrm{d}t = \left[t^6\right]_0^1 = 1$$

Substituting for $x = y = z = t$ in $\underline{G}$ we have

$$\underline{G} = t\underline{i} + 3t\underline{j} \text{ and } \underline{G} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = t + 3t = 4t$$

The limits of integration are again $t = 0$ and $t = 1$, then

$$\int_{C_1} \underline{G} \cdot \underline{dr} = \int_0^1 \underline{G} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\ \mathrm{d}t = \int_0^1 4t\ \mathrm{d}t = \left[2t^2\right]_0^1 = 2$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\dfrac{\mathrm{d}\underline{r}}{\mathrm{d}t} = 2t\underline{i} + \underline{j} + 2t\underline{k}$. Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{F}$ we have

$$\underline{F} = t^8\underline{i} + 2t^9\underline{j} + 3t^8\underline{k} \text{ and } \underline{F} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = 2t^9 + 2t^9 + 6t^9 = 10t^9$$

$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_0^1 10t^9\ \mathrm{d}t = \left[t^{10}\right]_0^1 = 1$$

Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{G}$ we have

$$\underline{G} = t^2\underline{i} + \left(4t^2 - t\right)\underline{j} \text{ and } \underline{G} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = 2t^3 + 4t^2 - t$$

$$\int_{C_2} \underline{G} \cdot \underline{dr} = \int_0^1 \left(2t^3 + 4t^2 - t\right) \mathrm{d}t = \left[\frac{1}{2}t^4 + \frac{4}{3}t^3 - \frac{1}{2}t^2\right]_0^1 = \frac{4}{3}$$

(b) For the closed path C

$$\begin{aligned} \oint_C \underline{F} \cdot \underline{dr} &= \int_{C_1} \underline{F} \cdot \underline{dr} - \int_{C_2} \underline{F \cdot \mathrm{d}r} = 1 - 1 = 0 \\ \oint_C \underline{G} \cdot \underline{dr} &= \int_{C_1} \underline{G} \cdot \underline{dr} - \int_{C_2} \underline{G} \cdot \underline{dr} = 2 - \frac{4}{3} = \frac{2}{3} \end{aligned}$$

Summary

Vector Field	Path	Line Integral
$\underline{F}$	C_1	1
$\underline{F}$	C_2	1
$\underline{F}$	closed	0
$\underline{G}$	C_1	2
$\underline{G}$	C_2	4/3
$\underline{G}$	closed	2/3

Note that the line integral of $\mathbf{F}$ is 1 for both paths. This would hold for any path from $(0, 0, 0)$ to $(1, 1, 1)$. The field $\mathbf{F}$ is an example of a *conservative vector field*; these are discussed in detail in the next subsection.

In $\int_C \underline{F} \cdot \underline{dr}$, the vector field $\underline{F}$ may be the divergence of a scalar field or the curl of a vector field.

Example Find $\int_C [\nabla(x^2y)] \cdot \underline{dr}$ where C is the contour $y = 2x - x^2$ from $(0, 0)$ to $(2, 0)$.

Solution

Note that $\nabla(x^2y) = 2xy\underline{i} + x^2\underline{j}$ so the integral is $\int_C [2xy\ dx + x^2\ dy]$.
On $y = 2x - x^2$, $dy = (2 - 2x)\ dx$ so the integral becomes

$$\begin{aligned}\int_C \left[2xy\ dx + x^2\ dy\right] &= \int_{x=0}^{2} \left[2x(2x - x^2)\ dx + x^2(2 - 2x)\ dx\right] \\ &= \int_0^2 (6x^2 - 4x^3)\ dx = \left[2x^3 - x^4\right]_0^2 = 0\end{aligned}$$

1. Evaluate $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$ along each of the following paths from $(1, 1)$ to $(2, 4)$.

   (a) C_1 : the straight line $y = 3x - 2$.

   (b) C_2 : the parabola $y = x^2$.

   (c) C_3 : the straight line $x = 1$ from $(1, 1)$ to $(1, 4)$ followed by the straight line $y = 4$ from $(1, 4)$ to $(2, 4)$.

2. For the function $\underline{F}$ and paths in question 1, deduce $\oint \underline{F} \cdot \underline{dr}$ for the closed paths

   (a) C_1 followed by the reverse of C_2.

   (b) C_2 followed by the reverse of C_3.

   (c) C_3 followed by the reverse of C_1.

3. Consider $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. Find the value of the line integral along each of the paths from $(0, 0)$ to $(1, 4)$.

   (a) $y = 4x$

   (b) $y = 4x^2$

   (c) $y = 4x^{1/2}$

   (d) $y = 4x^3$

4. Consider the two vector fields $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$ and $\underline{G} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$ and the two curves between $(0, 0, 0)$ and $(1, -1, 2)$
   $C_1 : x = t^2,\ y = -t,\ z = 2t$ for $0 \le t \le 1$.
   $C_2 : x = t - 1,\ y = 1 - t,\ z = 2t - 2$ for $1 \le t \le 2$.

   (a) Find $\int_{C_1} \underline{F} \cdot \underline{dr}$, $\int_{C_2} \underline{F} \cdot \underline{dr}$, $\int_{C_1} \underline{G} \cdot \underline{dr}$, $\int_{C_2} \underline{G} \cdot \underline{dr}$

   (b) Find $\oint_C \underline{F} \cdot \underline{dr}$ and $\oint_C \underline{G} \cdot \underline{dr}$ where C is the closed path from $(0, 0, 0)$ to $(1, -1, 2)$ along C_1 and back to $(0, 0, 0)$ along C_2.

5. Find $\int_C \underline{F} \cdot \underline{dt}$ along $y = 2x$ from $(0, 0)$ to $(2, 4)$ for

   (a) $\underline{F} = \nabla(x^2y)$

   (b) $\underline{F} = \nabla \times (\frac{1}{2}x^2y^2\underline{k})$

Your solution

1.)

$11, \frac{34}{3}, 8$

Your solution

2.)

$-\frac{1}{3}, \frac{10}{3}, -3.$

Your solution

3.)

All are 12.

Your solution

4.)

$2, \frac{5}{3}, 0, 0, \frac{1}{3}, 0.$

Your solution

5.)

$16, -16.$

3. Conservative vector fields

For certain of the line integrals in the previous section, the integral depended only on the vector field $\underline{F}$ and the start and end points of the contour but not on the actual path of the contour between the start and end points. However, for other line integrals, the result depended on the actual details of the path of the contour.

Vector fields are classified according to whether the line integrals are path dependent or path independent. Those vector fields for which **all** line integrals between **all** pairs of points are path independent are called **conservative vector fields**.
There are five properties of conservative vector field. Since it is impossible to check the value of every line integral over every path, it is possible to use these five properties (and in particular property $P3$ below to determine whether a vector field is conservative. They are also used to simplify calculations with conservative vector fields.

P1 The line integral $\displaystyle\int_A^B \underline{F}\cdot \underline{dr}$ depends only on the end points A and B and is independent of the actual path taken.

P2 The line integral around any closed curve is zero. That is $\oint_C \underline{F} \cdot \underline{dr} = 0$ for all C.

P3 The curl of a conservative vector field $\underline{F}$ is zero i.e. $\nabla \times \underline{F} = \underline{0}$

P4 For any conservative vector field $\underline{F}$, it is possible to find a scalar field ϕ such that $\nabla\phi = \underline{F}$. Then, $\oint_C \underline{F} \cdot \underline{dr} = \phi(B) - \phi(A)$ where A and B are the start and end points of contour C .

P5 All gradient fields are conservative. That is $\underline{F} = \nabla\phi$ is a conservative vector field for any scalar field ϕ.

Example The following vector fields were considered in the examples of the previous subsection.

1. $\underline{F}_1 = y^2\underline{i} + xy\underline{j}$
2. $\underline{F}_2 = 2x\underline{i} + 2y\underline{j}$
3. $\underline{F}_3 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$
4. $\underline{F}_4 = x\underline{i} + (4x - y)\,\underline{j}$

Determine which of these vector fields are conservative e.g. by referring to the answers given in the solution. For those that are conservative find a scalar field ϕ such that $\underline{F} = \nabla\phi$ and use P4 to verify the line integrals found.

Solution

1. Two different values were obtained for line integrals over the paths C_1 and C_2. Hence, by P1, $\underline{F}_1$ is not conservative. [It is also possible to reach this conclusion by finding that $\nabla \times \underline{F} = -y\underline{k} \neq \underline{0}$].

Solution

2 Both line integrals from $(0,0)$ to $(4,2)$ had the same value i.e. 20 and for the closed path the line integral was 0. This alone does not mean that $\underline{F}_2$ is conservative as there could be other untried paths giving different values. So by using P3

$$\nabla \times \underline{F}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-0) = \underline{0}$$

As $\nabla \times \underline{F}_2 = \underline{0}$, P3 gives that $\underline{F}_2$ is a conservative vector field.
Now, find a ϕ such that $\underline{F}_2 = \nabla\phi$. Then
$\dfrac{\partial\phi}{\partial x}\underline{i} + \dfrac{\partial\phi}{\partial y}\underline{j} = 2x\underline{i} + 2y\underline{j}$ Thus

$$\left.\begin{array}{l} \dfrac{\partial\phi}{\partial x} = 2x \quad \Rightarrow \quad \phi = x^2 + f(y) \\[2ex] \dfrac{\partial\phi}{\partial y} = 2y \quad \Rightarrow \quad \phi = y^2 + g(x) \end{array}\right\} \Rightarrow \phi = x^2 + y^2 (+ \text{ constant})$$

Using P4: $\displaystyle\int_{(0,0)}^{(4,2)} \underline{F}_2 \cdot \underline{dr} = \int_{(0,0)}^{(4,2)} (\nabla\phi) \cdot \underline{dr} = \phi(4,2) - \phi(0,0) = (4^2 + 2^2) - (0^2 + 0^2) = 20.$

3 The fact that line integrals by two different paths between the same start and end points is consistent with $\underline{F}_3$ being a conservative field. So too is the fact that the integral around a closed path is zero. However, neither fact can be used to *conclude* that $\underline{F}_3$ is a conservative field. This can be done by showing that $\nabla \times \underline{F}_3 = \underline{0}$.

Now, $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\underline{i} - (3y^2z^2 - 3y^2z^2)\underline{j} + (2yz^3 - 2yz^3)\underline{k} = \underline{0}.$

As $\nabla \times \underline{F}_3 = \underline{0}$, P3 gives that $\underline{F}_3$ is a conservative field.
To find ϕ that satisfies $\nabla\phi = \underline{F}_3$, it is necessary to satisfy

$$\left.\begin{array}{l} \dfrac{\partial\phi}{\partial x} = y^2z^3 \quad \rightarrow \quad \phi = xy^2z^3 + f(y,z) \\[2ex] \dfrac{\partial\phi}{\partial y} = 2xyz^3 \quad \rightarrow \quad \phi = xy^2z^3 + g(x,z) \\[2ex] \dfrac{\partial\phi}{\partial z} = 3xy^2z^2 \quad \rightarrow \quad \phi = xy^2z^3 + h(x,y) \end{array}\right\} \rightarrow \phi = xy^2z^3$$

Using P4: $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \underline{F}_3 \cdot \underline{dr} = \phi(1,1,1) - \phi(0,0,0) = 1 - 0 = 1.$

Solution

4 As the integral along C_1 is 2 and the integral along C_2 (same start and end points but different intermediate points) is $\frac{4}{3}$, F_4 is NOT a conservative field.
Note that $\nabla \times \underline{F}_4 = 4\underline{k} \neq \underline{0}$ so this is an independent conclusion that $\underline{F}_4$ is NOT conservative.

Example

1. Show that $I = \int_{(0,0)}^{(2,1)} [(2xy+1)dx + (x^2 - 2y)dy]$ is independent of the path taken
2. Find I using property P1.
3. Find I using property P4.
4. Find $I = \oint_C [(2xy+1)dx + (x^2 - 2y)dy]$ where C is
   (a) the circle $x^2 + y^2 = 1$
   (b) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Solution

1. The integral $I = \int_{(0,0)}^{(2,1)} [(2xy+1)dx + (x^2-2y)dy]$ may be re-written $\int_C \underline{F} \cdot \underline{dr}$ where $\underline{F} = (2xy+1)\underline{i} + (x^2-2y)\underline{j}$.

   Now $\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+1 & x^2-2y & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$

   As $\nabla \times \underline{F} = 0$, F is a conservative field and I is independent of the path taken between $(0,0)$ and $(2,1)$.

2. As I is independent of the path taken from $(0,0)$ to $(2,1)$, it can be evaluated along ANY such path. One possibility is the straight line $y = \frac{1}{2}x$. On this line, $dy = \frac{1}{2}dx$. The integral I becomes

$$\begin{aligned} I &= \int_{(0,0)}^{(2,1)} \big[(2xy+1)dx + (x^2-2y)dy\big] = \int_{x=0}^{2} \left[(2x \times \frac{1}{2}x + 1)dx + (x^2 - 4x)\frac{1}{2}dx\right] \\ &= \int_0^2 (\frac{3}{2}x^2 - \frac{1}{2}x + 1)dx = \left[\frac{1}{2}x^3 - \frac{1}{4}x^2 + x\right]_0^2 = 4 - 1 + 2 - 0 = 5 \end{aligned}$$

3. If $\underline{F} = \nabla\phi$ then

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= 2xy+1 \quad \rightarrow \quad \phi = x^2y + x + f(y) \\ \frac{\partial\phi}{\partial y} &= x^2-2y \quad \rightarrow \quad \phi = x^2y - y^2 + g(x) \end{aligned}\right\} \rightarrow \phi = xy^2z^3.$$

   These are consistent if $\phi = x^2y + x - y^2$ (plus a constant which may be set equal to zero). So $I = \phi(2,1) - \phi(0,0) = (4+2-1) - 0 = 5$

4. As F is a conservative field, all integrals around a closed contour are zero.

1. Determine whether the following vector fields are conservative

   (a) $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$

   (b) $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$

   (c) $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$

   (d) $\underline{F} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$

2. Consider the integral $\int_C \underline{F} \cdot \underline{dr}$ with $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. Noting that $\underline{F}$ is a conservative vector field, find a scalar field ϕ so that $\nabla\phi = \underline{F}$. Hence evaluate the integral $\int_C \underline{F} \cdot \underline{dr}$ where C is an integral with start-point $(0, 0)$ and end point $(1, 4)$.

3. For the following conservative vector fields $\underline{F}$, find a scalar field ϕ such that $\nabla\phi = \underline{F}$ and hence evaluate the $I = \int_C \underline{F} \cdot \underline{dr}$ for the contours C indicated.

   (a) $\underline{F} = (4x^3y - 2x)\underline{i} + (x^4 - 2y)\underline{j}$; any path from $(0, 0)$ to $(2, 1)$

   (b) $\underline{F} = (e^x + y^3)\underline{i} + (3xy^2)\underline{j}$; circle $x^2 + y^2 = 1$.

   (c) $\underline{F} = (y^2 + \sin z)\underline{i} + 2xy\underline{j} + x\cos z\underline{k}$; any path from $(1, 1, 0)$ to $(2, 0, \pi)$

   (d) $\underline{F} = \frac{1}{x}\underline{i} + 4y^3z^2\underline{j} + 2y^4z\underline{k}$; any path from $(1, 1, 1)$ to $(1, 2, 3)$

Your solution

1.)

No, Yes, No, Yes

Your solution

2.)

$x^3y^2 - y$, 12

Your solution

3.)

$x^4y - x^2 - y^2$,11;$e^x + xy^3$,0;$xy^2 + x\sin z$,1; $\ln x + y^4z^2$,143

4. Vector Line Integrals

It is also possible to form the integrals $\int_C f(x,y,z)\ \underline{dr}$ and $\int_C \underline{F}(x,y,z) \times \underline{dr}$. Each of these integrals evaluates to a vector.

Remembering that $\underline{dr} = dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}$, an integral of the form $\int_C f(x,y,z)\ \underline{dr}$ becomes $\int_C f(x,y,z)dx\ \ \underline{i} + \int_C f(x,y,z)dy\ \ \underline{j} + \int_C f(x,y,z)dz\ \ \underline{k}$. The first term can be evaluated by expressing y and z in terms of x. Similarly the second and third terms can be evaluated by expressing all terms as functions of y and z respectively. Alternatively, all variables can be expressed in terms of a parameter t. If an integral is two-dimensional, the term in z will be absent.

Example Evaluate the integral $\int_C xy^2\underline{dr}$ where C represents the contour $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution

This is a two-dimensional integral so the term in z will be absent.

$$\begin{aligned} I &= \int_C xy^2\underline{dr} = \int_C xy^2(dx\underline{i} + dy\underline{j}) = \int_C xy^2 dx\ \underline{i} + \int_C xy^2 dy\ \underline{j} \\ &= \int_{x=0}^{1} x(x^2)^2 dx\ \underline{i} + \int_{y=0}^{1} y^{1/2}y^2 dy\ \underline{j} = \int_0^1 x^5 dx\ \underline{i} + \int_0^1 y^{5/2} dy\ \underline{j} \\ &= \left[\frac{1}{6}x^6\right]_0^1\ \underline{i} + \left[\frac{2}{7}x^{7/2}\right]_0^1\ \underline{j} = \frac{1}{6}\underline{i} + \frac{2}{7}\underline{j} \end{aligned}$$

Example Find $I = \int_C xz\underline{dr}$ for the contour C given parametrically by $x = \cos t$, $y = \sin t$, $z = t - \pi$ starting at $t = 0$ and going to $t = 2\pi$ (i.e. the contour starts at $(1, 0, -\pi)$ and finishes at $(1, 0, \pi)$).

Solution

The integral becomes $\int_C x(dx\underline{i} + dy\underline{j} + dz\underline{k})$.
Now, $x = \cos t$, $y = \sin t$, $z = t - \pi$ so $dx = -\sin t\ dt$, $dy = \cos t\ dt$ and $dz = dt$. So

$$
\begin{aligned}
I &= \int_0^{2\pi} \cos t(-\sin t\ dt\underline{i} + \cos t\ dt\underline{j} + dt\underline{k}) \\
&= -\int_0^{2\pi} \cos t \sin t\ dt\ \underline{i} + \int_0^{2\pi} \cos^2 t\ dt\ \underline{j} + \int_0^{2\pi} \cos t\ dt\ \underline{k} \\
&= -\frac{1}{2}\int_0^{2\pi} \sin 2t\ dt\ \underline{i} + \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\ dt\ \underline{j} + [\sin t]_0^{2\pi}\ \underline{k} \\
&= \frac{1}{4}[\cos 2t]_0^{2\pi}\ \underline{i} + \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi}\ \underline{j} + 0\underline{k} \\
&= 0\underline{i} + \pi\ \underline{j} = \pi\underline{j}
\end{aligned}
$$

Integrals of the form $\int_C \underline{F} \times \underline{dr}$ can be evaluated as follows. The vector field $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ and $\underline{dr} = dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}$ so

$$
\begin{aligned}
\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix} &= (F_2\ dz - F_3\ dy)\underline{i} + (F_3\ dx - F_1\ dz)\underline{j} + (F_1\ dy - F_2\ dx)\underline{k} \\
&= (F_3\underline{j} - F_2\underline{k})\underline{i} + (F_1\underline{k} - F_3\underline{i})\underline{j} + (F_2\underline{i} - F_1\underline{j})dz
\end{aligned}
$$

There are a maximum of six terms involved in one such integral; the exact details may dictate which form to use.

Example Evaluate the integral $\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr}$ where C represents the curve $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

Solution

Note that the z component of F and dz are both zero.

So $\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & 3xy & 0 \\ dx & dy & 0 \end{vmatrix} = (x^2dy - 3xydx)\underline{k}$

and $\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} = \int_C (x^2dy - 3xydx)\underline{k}$
Now, on C, $y = 2x^2$ so $dy = 4xdx$ and

$$\begin{aligned} \int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} &= \int_C (x^2dy - 3xydx)\underline{k} \\ &= \int_{x=0}^{1} \left[x^2 \times 4xdx - 3x \times 2x^2dx\right]\underline{k} = \int_0^1 (-2x^3)dx\underline{k} \\ &= \left[\frac{1}{2}x^4\right]_0^1 \underline{k} = \frac{1}{2}\underline{k} \end{aligned}$$

A scalar or vector involved in a vector line integral may itself be a vector derivative.

Example Find the vector line integral $\int_C (\nabla\cdot\underline{F})\ \underline{dr}$ where $\underline{F}$ is the vector $x^2\underline{i}+2xy\underline{j}+2xz\underline{k}$ and C is the curve $y = x^2$, $z = x^3$ from $x = 0$ to $x = 1$ i.e. from $(0, 0, 0)$ to $(1, 1, 1)$.

Solution

As $\underline{F} = x^2\underline{i} + 2xy\underline{j} + 2xz\underline{k}$, $\nabla \cdot \underline{F} = 2x + 2x + 2x = 6x$.
The integral $\int_C (\nabla \cdot \underline{F})\ \underline{dr} = \int_C 6x(dx\underline{i} + dy\underline{j} + dz\underline{k} = \int_C 6x\ dx\ \underline{i} + \int_C 6x\ dy\ \underline{j} + \int_C 6x\ dz\ \underline{k}$.
The first term is

$$\int_C 6x\ dx\ \underline{i} = \int_{x=0}^{1} 6x\ dx\ \underline{i} = \left[3x^2\right]_0^1 \underline{i} = 3\underline{i}$$

In the second term, as $y = x^2$ on C, dy may be replaced by $2x\ dx$ so

$$\int_C 6x\ dy\ \underline{j} = \int_{x=0}^{1} 6x \times 2x\ dx\ \underline{j} = \int_0^1 12x^2\ dx\ \underline{j} = \left[4x^3\right]_0^1 \underline{j} = 4\underline{j}$$

In the third term, as $z = x^3$ on C, dz may be replaced by $3x^2\ dx$ so

$$\int_C 6x\ dz\ \underline{k} = \int_{x=0}^{1} 6x \times 3x^2\ dx\ \underline{k} = \int_0^1 18x^3\ dx\ \underline{k} = \left[\frac{9}{2}x^4\right]_0^1 \underline{k} = \frac{9}{2}\underline{k}$$

On summing, $\int_C (\nabla \cdot \underline{F})\ \underline{dr} = 3\underline{i} + 4\underline{j} + \frac{9}{2}\underline{k}$.

1. Find the vector line integral $\int_C f\underline{dr}$ where $f = x^2$ and C is

   (a) the curve $y = x^{1/2}$ from $(0,0)$ to $(9,3)$.

   (b) the line $y = x/3$ from $(0,0)$ to $(9,3)$.

2. When C represents the contour $y = 4 - 4x$, $z = 2 - 2x$ from $(0,4,2)$ to $(1,0,0)$ and $\underline{F}$ is the vector field $(x - z)\underline{j}$, evaluate the vector line integral $\int_C \underline{F} \times \underline{dr}$.

3. Evaluate the vector line integral $\int_C (\nabla \cdot \underline{F})\ \underline{dr}$ in the case where $\underline{F} = x\underline{i} + xy\underline{j} + xy^2\underline{k}$ and C is the contour described by $x = 2t$, $y = t^2$, $z = 1 - t$ for t starting at $t = 0$ and going to $t = 1$.

4. When C is the contour $y = x^3$, $z = 0$, from $(0,0,0)$ to $(1,1,0)$, evaluate the vector line integrals

   (a) $\int_C [\nabla(xy)] \times \underline{dr}$

   (b) $\int_C [\nabla \times (x^2\underline{i} + y^2\underline{k})] \times \underline{dr}$

Your solution

1.)

$243\underline{i} + \frac{9}{2}\underline{j}, 243\underline{i} + \frac{243}{2}\underline{j}$.

Your solution

2.)

$\underline{i} + \frac{1}{2}\underline{j}$

Your solution

3.)

$4\underline{i} + \frac{7}{3}\underline{j} - 2\underline{k}$

Your solution

4.)

$\underline{0}$, $\underline{k}$