Surface and Volume Integrals

29.2

Introduction

A vector or scalar field (including one formed from a vector derivative (div, grad or curl)) can be integrated over a surface or volume. This section shows how to carry out such operations.

Prerequisites

Before starting this Section you should ...

① be familiar with vector derivatives

② be familiar with surface and volume integrals

Learning Outcomes

After completing this Section you should be able to ...

✓ be able to carry out operations involving integrations of vector fields.

1. Surface integrals involving vectors

The unit normal

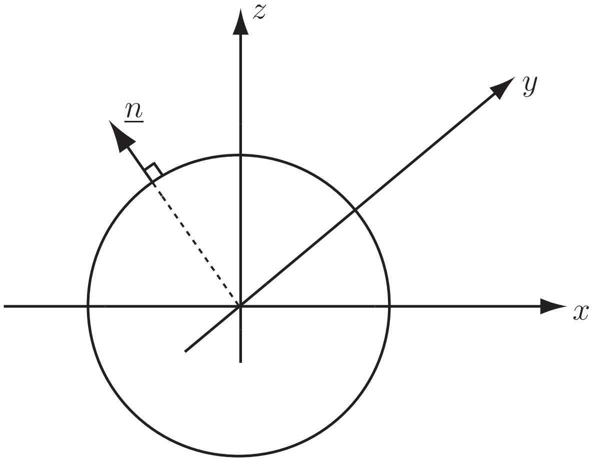
For the surface of any three-dimensional shape, it is possible to find a vector lying perpendicular to the surface and with magnitude 1. The unit vector points outwards from the surface and is usually denoted by $\hat{\underline{n}}$.

Example If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ find the unit normal $\hat{\underline{n}}$

Solution

The unit normal at the point (x, y, z) points away from the centre of the sphere i.e. it lies in the direction of $x\underline{i} + y\underline{j} + z\underline{k}$. To make this a unit vector it must be divided by its magnitude $\sqrt{x^2 + y^2 + z^2}$ i.e. the unit vector

$\hat{\underline{n}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\underline{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\underline{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\underline{k} = \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}$ where $a = \sqrt{x^2 + y^2 + z^2}$



Example For the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, find the unit normal $\hat{\underline{n}}$

Solution

On the face given by $x = 0$, the unit normal points in the negative x-direction. Hence the unit normal is $-\underline{i}$. Similarly :-
On the face $x = 1$ the unit normal is $\underline{i}$
On the face $y = 0$ the unit normal is $-\underline{j}$
On the face $y = 1$ the unit normal is $\underline{j}$
On the face $z = 0$ the unit normal is $-\underline{k}$
On the face $z = 1$ the unit normal is $\underline{k}$

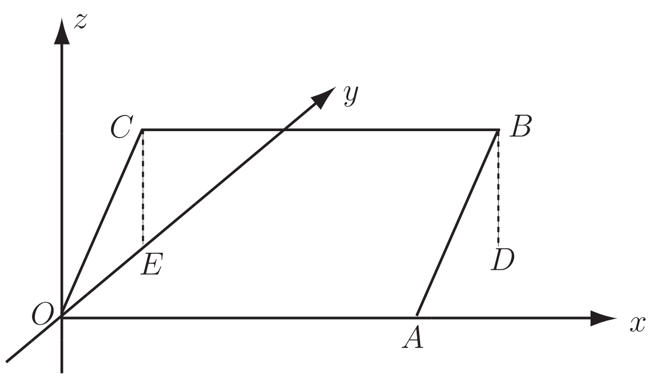
$\underline{dS}$ and the unit normal

The vector $\underline{dS}$ is a vector, an element of the surface with magnitude dudv and direction perpendicular to the surface.

If the plane in question is the Oxy plane, then $\underline{dS} = \underline{\hat{n}}\ du\ dv = \underline{k}\ dx\ dy$.

If the plane in question is not one of the three coordinate planes (Oxy, Oxz, Oyz), appropriate adjustments must be made to express $\underline{dS}$ in terms of dx and dy (or dz and either dx or dy).

Example The rectangle $OABC$ lies in the plane $z = y$. The vertices are $O = (0, 0, 0)$, $A = (1, 0, 0)$,$B = (1, 1, 1)$ and $C = (0, 1, 1)$. Find a unit vector $\underline{\hat{n}}$ normal to the plane and an appropriate vector $\underline{dS}$ expressed in terms of dx and dy.



Solution

Note two vectors in the rectangle are $\overrightarrow{OA} = \underline{i}$ and $\overrightarrow{OC} = \underline{j} + \underline{k}$. A vector perpendicular to the plane is $\underline{i} \times (\underline{j} + \underline{k}) = -\underline{j} + \underline{k}$. However, this vector is of magnitude $\sqrt{2}$ so the unit normal vector is $\underline{\hat{n}} = \frac{1}{\sqrt{2}}(-\underline{j} + \underline{k}) = -\frac{1}{\sqrt{2}}\underline{j} + \frac{1}{\sqrt{2}}\underline{k}$.
The vector $\underline{dS}$ is therefore $(-\frac{1}{\sqrt{2}}\underline{j} + \frac{1}{\sqrt{2}}\underline{k})\ du\ dv$ where du and dv are increments in the plane of the rectangle $OABC$. Now, one increment, say du, may point in the $x - direction$ while dv will point in a direction up the plane, parallel to OC. Thus $du = dx$ and (by Pythagoras) $dv = \sqrt{(dy)^2 + (dz)^2}$. However, as $z = y$, $dz = dy$ and hence $dv = \sqrt{2}dy$.
Thus, $\underline{dS} = (-\frac{1}{\sqrt{2}}\underline{j} + \frac{1}{\sqrt{2}}\underline{k})\ dx\sqrt{2}\ dy = (-\underline{j} + \underline{j})\ dx\ dy$.
Note :- the factor of $\sqrt{2}$ could also have been found by comparing the area of rectangle $OABC$, i.e. 1, with the area of its projection in the Oxy plane i.e. $OADE$ or area $\frac{1}{\sqrt{2}}$.

Integrating a scalar field

A function can be integrated over a surface in a manner similar to that shown in sections 29.1 and 29.2. Often, such integrals can be carried out with respect to an element containing the unit normal.

Example Evaluate the integral

$$\int_A \frac{1}{1+x^2}\underline{dS}$$

where $\underline{S}$ is the unit normal over the area A and A is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

In this integral, $\underline{S}$ becomes $\underline{k}\ dx\ dy$ i.e. the unit normal times the surface element. Thus the integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{x=0}^{1}\frac{\underline{k}}{1+x^2}\,dx\,dy &= \int_{y=0}^{1}\left[\underline{k}\tan^{-1}x\right]_0^1\,dy \\ &= \int_{y=0}^{1}\left[\underline{k}(\frac{\pi}{4}-0)\right]_0^1\,dy = \frac{\pi}{4}\underline{k}\int_{y=0}^{1}dy \\ &= \frac{\pi}{4}\underline{k}\end{aligned}$$

Example Find $\int\int_S u\underline{dS}$ where $u = r^2 = x^2 + y^2 + z^2$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

The unit cube has six faces and the normal vector $\underline{\hat{n}}$ points in a different direction on each face. The surface integral must be evaluated for each face separately and the results summed.
On the face $x = 0$, the normal $\underline{\hat{n}} = -\underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(0^2+y^2+z^2)(-\underline{i})\,dz\,dy &= -\underline{i}\int_{y=0}^{1}\left[y^2z+\frac{1}{3}z^3\right]_{z=0}^{1}dy \\ &= -\underline{i}\int_{y=0}^{1}\left[y^2+\frac{1}{3}\right]dy = -\underline{i}\left[\frac{1}{3}y^3+\frac{1}{3}y\right]_0^1 = -\frac{2}{3}\underline{i}\end{aligned}$$

On the face $x = 1$, the normal $\underline{\hat{n}} = \underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(1^2+y^2+z^2)(\underline{i})\,dz\,dy &= \underline{i}\int_{y=0}^{1}\left[z+y^2z+\frac{1}{3}z^3\right]_{z=0}^{1}dy \\ &= \underline{i}\int_{y=0}^{1}\left[y^2+\frac{4}{3}\right]dy = \underline{i}\left[\frac{1}{3}y^3+\frac{4}{3}y\right]_0^1 = \frac{5}{3}\underline{i}\end{aligned}$$

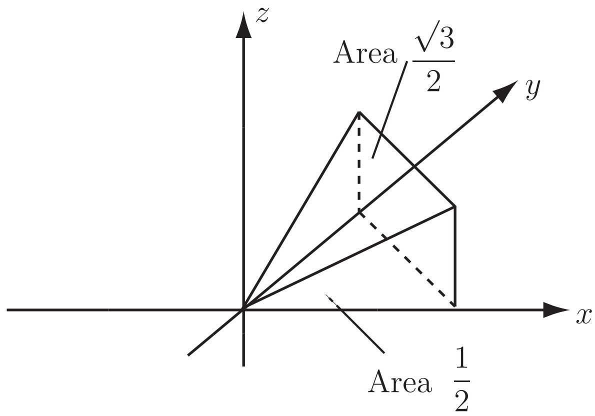
The net contribution from the faces $x = 0$ and $x = 1$ is $-\frac{2}{3}\underline{i} + \frac{5}{3}\underline{i} = \underline{i}$.
Due to the symmetry of the scalar field u and the unit cube, the net contribution from the faces $y = 0$ and $y = 1$ is $\underline{j}$ while the net contribution from the faces $z = 0$ and $z = 1$ is $\underline{k}$.
The sum i.e. the surface integral $\int\int_S u\ \underline{dS} = \underline{i} + \underline{j} + \underline{k}$

Key Point

A scalar function integrated with respect to a unit normal gives a vector quantity

When the surface does not lie in one of the planes (Oxy plane, Oxz plane, Oyz plane), extra care must be taken when finding $\underline{dS}$.

Example Find $\int\int_S f\underline{dS}$ where f is the function $2x$ and S is the surface of the triangle bounded by $(0,0,0)$, $(0,1,1)$ and $(1,0,1)$.



Solution

The unit vector $\underline{n}$ is perpendicular to two vectors in the plane e.g. $(\underline{j}+\underline{k})$ and $(\underline{i}+\underline{k})$. The vector $(\underline{j}+\underline{k})\times(\underline{i}+\underline{k}) = \underline{i}+\underline{j}+\underline{k}$ and has magnitude $\sqrt{3}$. Hence the normal vector $\underline{n} = \frac{1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}-\frac{1}{\sqrt{3}}\underline{k}$. As the area of the triangle is $\frac{\sqrt{3}}{2}$ and the area of its projection in the Oxy plane is $\frac{1}{2}$, the vector $\underline{dS} = \frac{\sqrt{3}/2}{1/2}\underline{n}\ dy\ dx = (\underline{i}+\underline{j}+\underline{k})\ dy\ dx$.
Thus

$$\begin{aligned}
\int\int_S f\underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{1-x} 2x\ dy\ dx\ (\underline{i}+\underline{j}+\underline{k}) \\
&= \int_{x=0}^{1} [2xy]_{y=0}^{1-x}\ dx\ (\underline{i}+\underline{j}+\underline{k}) \\
&= \int_{x=0}^{1} (2x-2x^2)\ dx\ (\underline{i}+\underline{j}+\underline{k}) \\
&= \left[x^2-\frac{2}{3}x^3\right]_0^1 (\underline{i}+\underline{j}+\underline{k}) = \frac{1}{3}(\underline{i}+\underline{j}+\underline{k})
\end{aligned}$$

The scalar function being integrated may be the divergence of a suitable vector function.

Example Find $\int\int_S (\nabla \cdot \underline{F}) \underline{dS}$ where $\underline{F} = 2x\underline{i} + yz\underline{j} + xy\underline{k}$ and S is the surface of the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

Solution

Note that $\nabla \cdot \underline{F} = 2 + z = 2$ as $z = 0$ everywhere along S. As the triangle lies in the Oxy plane, the normal vector $\underline{n} = \underline{k}$ and $\underline{dS} = \underline{k}\, dy\, dx$.
Thus,

$$\begin{aligned} \int\int_S (\nabla \cdot \underline{F})\underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{x} 2\, dy\, dx \\ &= \int_0^1 [2y]_0^x\, dx = \int_0^1 2x\, dx = \left[x^2\right]_0^1 = 1 \end{aligned}$$

1. Evaluate the integral $\int\int_S 4x\underline{dS}$ where S represents the trapezium with vertices at $(0,0)$, $(3,0)$, $(2,1)$ and $(0,1)$.

2. Evaluate the integral $\int\int_S xy\underline{dS}$ where S is the triangle with vertices at $(0,0,4)$, $(0,2,0)$ and $(1,0,0)$.

3. Find the integral $\int\int_S xyz\ \underline{dS}$ where S is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

4. Evaluate the integral $\int\int_S \left[\nabla \cdot (x^2\underline{i} + yz\underline{j} + x^2y\underline{k})\right] \underline{dS}$ where S is the rectangle with vertices at $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ and $(1,0,1)$.

Your solution

1.) (a) Find the vector $\underline{dS}$ (b) write the surface integral as a double integral (c) evaluate this double integral

(a) $\underline{k}$ (b) $\int_{y=0}^{1}\int_{x=0}^{3-y} 4x\, dx dy\underline{k}$ (c) $\dfrac{38}{3}\underline{k}$

Your solution

2.)

$\frac{8}{3}\underline{i} + \frac{4}{3}\underline{j} + \frac{2}{3}\underline{k}$

Your solution

3.)

$\frac{1}{4}(\underline{x} + \underline{y} + \underline{z})$

Your solution

4.)

$\frac{5}{2}\underline{i}$

Integrating a vector field

In a similar manner, a vector field may be integrated over a surface. Two common integrals are $\int_S \underline{F}(\underline{r})\cdot \underline{dS}$ and $\int_S \underline{F}(\underline{r})\times \underline{dS}$ which integrate to a scalar and a vector respectively. Again, when the unit normal $\underline{dS}$ is expressed appropriately, the expression will reduce to a double integral.

Example Evaluate the integral

$$\int_A (x^2 y\underline{i} + z\underline{j} + (2x+y)\underline{k})\cdot \underline{dS}$$

where $\underline{S}$ is the unit normal over the area A and A is the square $0 \le x \le 1$, $0 \le y \le 1$, $z = 0$.

Solution

On A, the unit normal is $dx\ dy\ \underline{k}$ so the integral becomes

$$\begin{aligned} & \int_A (x^2 y\underline{i} + z\underline{j} + (2x+y)\underline{k})\cdot(\underline{k}\ dx\ dy) \\ = & \int_{y=0}^{1}\int_{x=0}^{1}(2x+y)\ dx\ dy = \int_{y=0}^{1}\left[x^2+xy\right]_{x=0}^{1}\ dy \\ = & \int_{y=0}^{1}(1+y)\ dy = \left[y+\frac{1}{2}y^2\right]_0^1 = \frac{3}{2} \end{aligned}$$

Example Evaluate $\int_A \underline{r}\cdot\underline{dS}$ where the surface A represents the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and $\underline{r}$ represents the vector $x\underline{i} + y\underline{j} + z\underline{k}$.

Solution

The unit normal $\underline{dS}$ will be a constant vector on each face but will be different for each face. On the face $x = 0$ (left), $\underline{dS} = -dy\ dz\ \underline{i}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{y=0}^{1}(0\underline{i} + y\underline{j} + z\underline{k}) \cdot (-\ dy\ dz\ \underline{i}) = \int_{z=0}^{1}\int_{y=0}^{1} 0\ dy\ dz = 0$$

Similarly on the face $y = 0$ (front), $\underline{dS} = -dx\ dz\ \underline{j}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{x=0}^{1}(x\underline{i} + 0\underline{j} + z\underline{k}) \cdot (-\ dx\ dz\ \underline{j}) = \int_{z=0}^{1}\int_{x=0}^{1} 0\ dx\ dz = 0$$

Furthermore on the face $z = 0$ (bottom), $\underline{dS} = -dx\ dy\ \underline{k}$ and the integral on this face is

$$\int_{x=0}^{1}\int_{y=0}^{1}(x\underline{i} + y\underline{j} + 0\underline{k}) \cdot (-\ dx\ dy\ \underline{k}) = \int_{x=0}^{1}\int_{y=0}^{1} 0\ dx\ dy = 0$$

On these three faces, the contribution to the integral is zero. However, on the face $x = 1$ (right), $\underline{dS} = +dy\ dz\ \underline{i}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{y=0}^{1}(1\underline{i} + y\underline{j} + z\underline{k}) \cdot (+\ dy\ dz\ \underline{i}) = \int_{z=0}^{1}\int_{y=0}^{1} 1\ dy\ dz = 1$$

(using the techniques of double integration from Workbook 27).

Similarly, on the face $y = 1$ (back), $\underline{dS} = +dx\ dz\ \underline{j}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{x=0}^{1}(x\underline{i} + 1\underline{j} + z\underline{k}) \cdot (+\ dx\ dz\ \underline{j}) = \int_{z=0}^{1}\int_{x=0}^{1} 1\ dx\ dz = 1$$

and finally,on the face $z = 1$ (top), $\underline{dS} = +dx\ dy\ \underline{k}$ and the integral on this face is

$$\int_{y=0}^{1}\int_{x=0}^{1}(x\underline{i} + y\underline{j} + 1\underline{k}) \cdot (+\ dx\ dy\ \underline{k}) = \int_{y=0}^{1}\int_{x=0}^{1} 1\ dx\ dy = 1$$

Adding together the contributions from the various faces gives $\int_A \underline{r} \cdot \underline{dS} = 0+0+0+1+1+1 = 3$

Example If $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$, evaluate $\int\int_S \underline{F} \times \underline{dS}$ where S is the part of the plane $z = 0$ bounded by $x = \pm 1$, $y = \pm 1$.

Solution

Here $\underline{dS} = \; dx\, dy\underline{k}$ and hence

$$\underline{F} \times \underline{dS} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & y^2 & z^2 \\ 0 & 0 & dx\, dy \end{vmatrix} = y^2\, dx\, dy\underline{i} - x^2\, dx\, dy\underline{j} \text{ and}$$

$\int\int_S \underline{F} \times \underline{dS} = \int_{y=-1}^{1}\int_{x=-1}^{1} y^2\, dx\, dy\; \underline{i} - \int_{y=-1}^{1}\int_{x=-1}^{1} x^2\, dx\, dy\; \underline{j}$

The integral

$$\begin{aligned} \int_{y=-1}^{1}\int_{x=-1}^{1} y^2\, dx\, dy &= \int_{y=-1}^{1} \left[y^2 x\right]_{x=-1}^{1}\, dy \\ &= \int_{y=-1}^{1} 2y^2\, dy = \left[\frac{2}{3}y^3\right]_{-1}^{1} = \frac{4}{3} \end{aligned}$$

Similarly $\int_{y=-1}^{1}\int_{x=-1}^{1} x^2\, dx\, dy = \frac{4}{3}$.

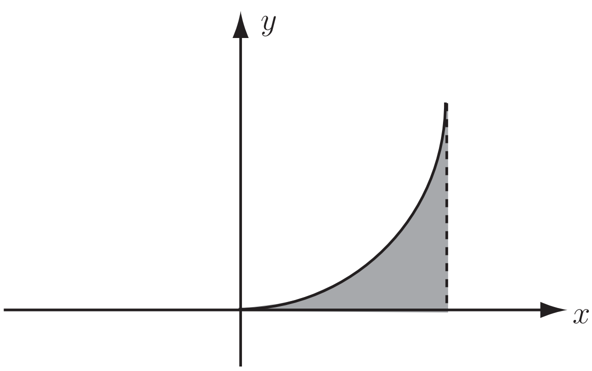
Thus $\int\int_S \underline{F} \times \underline{dS} = \frac{4}{3}\underline{i} - \frac{4}{3}\underline{j}$

Key Point

1. An integral of the form $\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ evaluates to a scalar.
2. An integral of the form $\int_S \underline{F}(\underline{r}) \times \underline{dS}$ evaluates to a vector.

The vector function involved may be the gradient of a scalar or the curl of a vector.

Example Integrate $\int\int_S (\nabla\phi).\underline{dS}$ where $\phi = x^2 + 2yz$ and S is the area between $y = 0$ and $y = x^2$ for $0 \leq x \leq 1$ and $z = 0$.



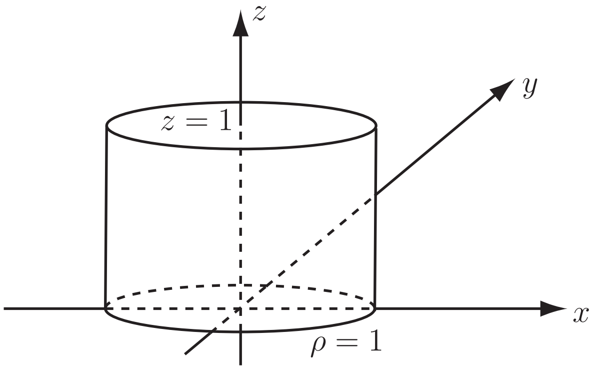
Solution

Here $\nabla\phi = 2x\underline{i} + 2z\underline{j} + 2y\underline{k}$ and $\underline{dS} = \underline{k}\; dy\; dx$. Thus $(\nabla\phi).\underline{dS} = 2y\; dy\; dx$ and

$$\begin{aligned}\int\int_S (\nabla\phi).\underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{x^2} 2y\; dy\; dx \\ &= \int_{x=0}^{1} \left[y^2\right]_{y=0}^{x^2}\; dx = \int_{x=0}^{1} x^4\; dx \\ &= \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}\end{aligned}$$

For integrals of the form $\int\int_S \underline{F}\cdot\underline{dS}$, non-cartesian geometry e.g. cylindrical or spherical polar coordinates may be used. Once again, it is necessary to include any scale factors along with the unit normal.

Example Using cylindrical polar coordinates, (see Section 28.3), find the integral $\int_S \underline{F}(\underline{r})\cdot\underline{dS}$ for $\underline{F} = \rho z\underline{\hat{\rho}} + z\sin^2\phi\underline{\hat{z}}$ and S being the complete surface (including ends) of the cylinder $\rho \le a$, $0 \le z \le 1$.



Solution

The integral $\int_S \underline{F}(\underline{r})\cdot\underline{dS}$ must be evaluated separately for the curved surface and the ends. For the curved surface, $\underline{dS} = \underline{\hat{\rho}}\; a\; d\phi\; dz$ (with the a coming from ρ the scale factor for ϕ and the fact that $\rho = a$ on the curved surface). Thus, $\underline{F}\cdot\underline{dS} = a^2\; z\; d\phi\; dz$ and

$$\begin{aligned}\int\int_S \underline{F}(\underline{r})\cdot\underline{dS} &= \int_{z=0}^{a}\int_{\phi=0}^{2\pi} a^2 z\; d\phi\; dz \\ &= 2\pi a^2\int_{z=0}^{a} z\; dz = 2\pi a^2\left[\frac{1}{2}z^2\right]_0^a = \pi a^4\end{aligned}$$

Solution

On the bottom, $z = 0$ so $\underline{F} = \underline{0}$ and the contribution to the integral is zero.
On the top, $z = 1$ and $\underline{dS} = \underline{\hat{z}}\ r\ dr\ d\phi$ and $\underline{F} \cdot \underline{dS} = \rho\ z\ \sin^2\phi\ d\phi\ d\rho = \rho\ h\ \sin^2\phi\ d\phi\ d\rho$ and

$$\begin{aligned}\int\int_S \underline{F}(\underline{r}) \cdot \underline{dS} &= \int_{rho=0}^{a}\int_{\phi=0}^{2\pi} \rho\ h\ \sin^2\phi\ d\phi\ d\rho \\ &= h\pi \int_{rho=0}^{a} \rho\ d\rho = \frac{1}{2}h\pi a^2\end{aligned}$$

So $\int\int_S \underline{F}(\underline{r}) \cdot \underline{dS} = \pi a^4 + \frac{1}{2}h\pi a^2 = \pi a^2(a^2 + \frac{h}{2})$

1. For $\underline{F} = (x^2+y^2)\underline{i}+(x^2+z^2)\underline{j}+2xz\underline{k}$ and S being the rectangle bounded by $(1,0,1)$, $(1,0,-1)$, $(-1,0,-1)$ and $(-1,0,1)$ find the integral $\int_S \underline{F}\cdot\underline{dS}$
2. For $\underline{F} = (x^2+y^2)\underline{i}+(x^2+z^2)\underline{j}+2xz\underline{k}$ and S being the rectangle bounded by $(1,0,1)$, $(1,0,-1)$, $(-1,0,-1)$ and $(-1,0,1)$ (i.e. the same $\underline{F}$ and S as in question 1), find the integral $\int_S \underline{F} \times \underline{dS}$
3. Evaluate the integral $\int\int_S \nabla\phi \cdot \underline{dS}$ for $\phi = x^2 z \sin y$ and S being the rectangle bounded by $(0,0,0)$, $(1,0,1)$, $(1,\pi,1)$ and $(0,\pi,0)$.
4. Evaluate the integral $\int\int_S (\nabla \times \underline{F}) \times \underline{dS}$ where $\underline{F} = xe^y\underline{i} + ze^y\underline{j}$ and S represents the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.
5. Using spherical polar coordinates, evaluate the integral $\int\int_S \underline{F}\cdot\underline{dS}$ where $\underline{F} = r\cos\theta\ \underline{\hat{r}}$ and S is the curved surface of the top half of the sphere $r = a$.

Your solution

1.)

Your solution

2.)

$\frac{4}{3}\underline{k}$

Your solution

3.)

$-\frac{\pi}{3}$

Your solution

4.)

$(e-1)\underline{j}$

Your solution

5.)

πa^3

2. Volume integrals involving vectors

Integrating a scalar function of a vector over a volume is essentially the same procedure as in Section 29.3. The volume element dV may be considered as $dx\ dy\ dz$. However, the scalar function may be the divergence of a vector functions.

Example Integrate $\nabla \cdot \underline{F}$ over the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ where $\underline{F}$ is the vector function $x^2 y\underline{i} + (x - z)\underline{j} + 2xz^2\underline{k}$.

Solution

$\nabla \cdot \underline{F} = \dfrac{\partial}{\partial x}(x^2 y) + \dfrac{\partial}{\partial y}(x - z) + \dfrac{\partial}{\partial z}(2xz^2) = 2xy + 4xz$

The integral is

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2xy + 4xz)\ dz\ dy\ dx = \int_{x=0}^{1}\int_{y=0}^{1} \left[2xyz + 2xz^2\right]_0^1\ dy\ dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[2xy + 2x\right]\ dy\ dx = \int_{x=0}^{1} \left[xy^2 + 2xy\right]_0^1\ dx$$

$$= \int_{x=0}^{1} 3x\ dx = \left[\frac{3}{2}x^2\right]_0^1 = \frac{3}{2}$$

Key Point

The volume integral of a scalar function (including the divergence of a vector) is a scalar

1. Evaluate $\int\int\int_V \nabla \cdot \underline{F} dV$ when $\underline{F}$ is the vector field $yz\underline{i} + xy\underline{j}$ and V is the unit cube $0 \le x \le 1$, $0 \le y \le 1$.

2. For the vector field $\underline{F} = (x^2 y + \sin z)\underline{i} + (xy^2 + e^z)\underline{j} + (z^2 + x^y)\underline{k}$, find the integral $\int\int\int_V \nabla \cdot \underline{F} dV$ where V is the volume inside the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

3. Using spherical polar coordinates and the vector field $\underline{F} = r^2\underline{\hat{r}} + r^2 \sin\theta\underline{\hat{\phi}}$, evaluate the integral $\int\int\int_V \nabla \cdot \underline{F}\ dV$ over the sphere given by $r \le a$.

Your solution

1.)

$\frac{1}{2}$

Your solution

2.)

$\frac{1}{20}$

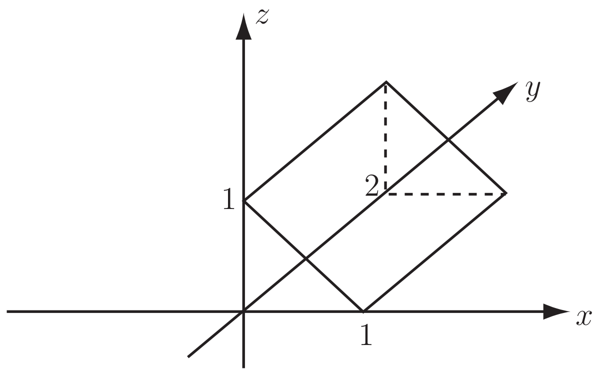
Your solution

3.)

$4\pi a^4$

Integrating a vector function over a volume integral is similar but care should be taken with the various components. It may help to think in terms of a separate volume integral for each component. The vector function may be of the form ∇f or $\nabla \times \underline{F}$.

Example Integrate the function $\underline{F} = x^2\underline{i} + 2\underline{j}$ over the prism given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq (1 - x)$.



Solution

The integral is

$$\begin{aligned}
& \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{1-x} x^2\underline{i} + 2\underline{j}\; dz\; dy\; dx = \int_{x=0}^{1}\int_{y=0}^{2}\left[x^2 z\underline{i} + 2z\underline{j}\right]_{z=0}^{1-x}\; dy\; dx \\
= & \int_{x=0}^{1}\int_{y=0}^{2}\left[x^2(1-x)\underline{i} + 2(1-x)\underline{j}\right]\; dy\; dx = \int_{x=0}^{1}\int_{y=0}^{2}\left[(x^2-x^3)\underline{i} + (2-2x)\underline{j}\right]\; dy\; dx \\
= & \int_{x=0}^{1}\left[(2x^2-2x^3)\underline{i} + (4-4x)\underline{j}\right]\; dx = \left[(\frac{2}{3}x^3 - \frac{1}{2}x^4)\underline{i} + (4x-2x^2)\underline{j}\right]_0^1 \\
= & \frac{1}{6}\underline{i} + 2\underline{j}
\end{aligned}$$

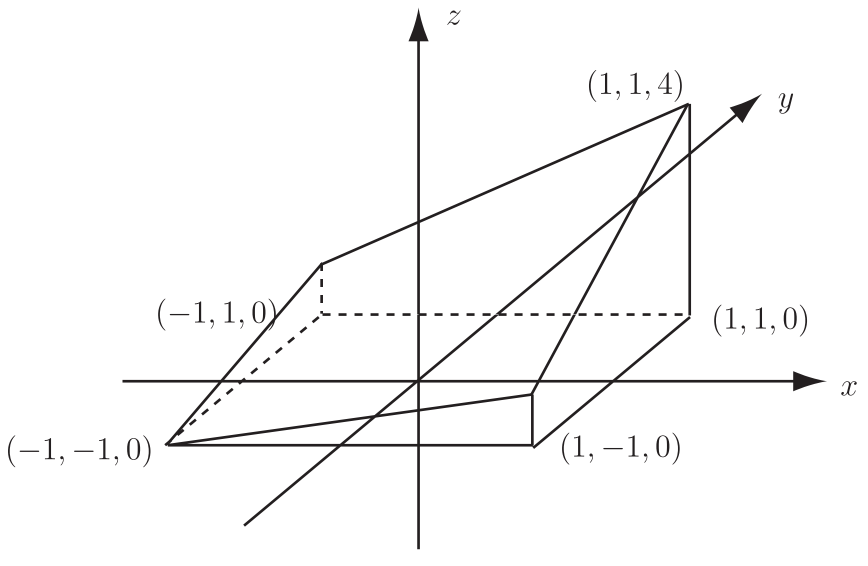
Example For $\underline{F} = x^2y\underline{i} + y^2\underline{j}$ evaluate $\int\int\int_V (\nabla \times \underline{F})dV$ where V is the volume under the plane $z = x + y + 2$ (and above $z = 0$) for $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Solution

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & y^2 & 0 \end{vmatrix} = -x^2\underline{k}$$

so

$$\begin{aligned}
\int\int\int_V (\nabla \times \underline{F})dV &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=0}^{x+y+2}(-x^2)\underline{k}\ dz\ dy\ dx \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1}\left[(-x^2)z\underline{k}\right]_{z=0}^{x+y+2}\ dy\ dx \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1}\left[-x^3 - x^2y - 2x^2\right]\ dy\ dx\ \underline{k} \\
&= \int_{x=-1}^{1}\left[-x^3y - \frac{1}{2}x^2y^2 - 2x^2y\right]_{y=-1}^{1}\ dx\ \underline{k} \\
&= \int_{x=-1}^{1}\left[-2x^3 - 0 - 4x^2\right]\ dx\ \underline{k} = \left[-\frac{1}{2}x^4 - \frac{4}{3}x^3\right]_{-1}^{1}\underline{k} = -\frac{8}{3}\underline{k}
\end{aligned}$$



Key Point

The volume integral of a vector function (including the gradient of a scalar or the curl of a vector) is a vector

1. Evaluate the integral $\int_V \underline{F} dV$ for the case where $\underline{F} = x\underline{i} + y^2\underline{j} + z\underline{k}$ and V being the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

2. For $f = x^2 + yz$, and V being the volume bounded by $y = 0$, $x + y = 1$ and $-x + y = 1$ for $-1 \leq z \leq 1$, find the integral $\int\int\int_V (\nabla f)\, dV$.

3. Evaluate the integral $\int_V (\nabla \times \underline{F}) dV$ for the case where $\underline{F} = xz\underline{i} + (x^3 + y^3)\underline{j} - 4y\underline{k}$ and V being the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

Your solution

1.)

$\frac{8}{3}\underline{j}$

Your solution

2.)

$\frac{2}{3}\underline{k}$

Your solution

3.)

$-32\underline{i} + 8\underline{k}$