

# Integral Vector Theorems

29.3



## Introduction

Various theorems exist equating integrals involving vectors. Often, use of these theorems can make certain vector integrals easier. This section introduces the theorems known as Gauss' Theorem, Stokes' Theorem and Green's Theorem.



## Prerequisites

Before starting this Section you should ...

- ① be able to find the gradient of a scalar field and the divergence and curl of a vector field
- ② be familiar with the integration of vector functions



## Learning Outcomes

After completing this Section you should be able to ...

- ✓ be able to use vector integral theorems to facilitate vector integration.

# 1. Stokes' Theorem

This is a theorem that equates a line integral to a surface integral. For any vector field  $\underline{F}$  and a contour  $C$  which bounds an area  $S$ ,

$$\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Notes

1.  $\underline{dS}$  is a vector perpendicular to the surface and  $\underline{dr}$  is the line element along the contour  $C$ .
2. Both sides of the equation are scalars
3. The theorem is often a useful way of calculating a line integral along a contour composed of several distinct parts (e.g. a square or other figure).
4.  $\nabla \times \underline{F}$  is a vector field representing the curl of the vector field  $\underline{F}$  and may, alternatively, be written as  $\text{curl } \underline{F}$ .

## Justification

Imagine that the surface is divided into a set of infinitesimally small rectangles  $ABCD$  where the axes are adjusted so that  $AB$  and  $CD$  lie parallel to the new  $x$ -axis i.e.  $AB = \delta x$  and  $BC$  and  $AD$  lie parallel to the new  $y$ -axis i.e.  $BC = \delta y$ .

Now,  $\oint_C \underline{F} \cdot \underline{dr}$  is calculated. The contributions along  $AB$ ,  $BC$ ,  $CD$  and  $DA$  are  $\underline{F}(x, y, 0) \cdot \underline{\delta x} = F_x(x, y, z)\delta x$ ,  $\underline{F}(x + \delta x, y, 0) \cdot \underline{\delta y} = F_y(x + \delta x, y, z)\delta y$ ,  $\underline{F}(x, y + \delta y, 0) \cdot (-\underline{\delta x}) = -F_x(x, y + \delta y, z)\delta x$  and  $\underline{F}(x, y, 0) \cdot (-\underline{\delta y}) = -F_y(x, y, z)\delta y$ . Thus,

$$\begin{aligned} \oint_C \underline{F} \cdot \underline{dr} &\approx (F_x(x, y, z) - F_x(x, y + \delta y, z))\delta x + (F_y(x + \delta x, y, z) - F_y(x, y, z))\delta y \\ &\approx -\frac{\partial F_x}{\partial y}\delta x\delta y + \frac{\partial F_y}{\partial x}\delta x\delta y \\ &\approx (\nabla \times \underline{F})_z \delta S \approx (\nabla \times \underline{F}) \cdot \underline{dS} \end{aligned}$$

as  $\delta S$  is perpendicular to the  $x$ - and  $y$ - axes.

Thus, for each small rectangle,

$$\oint_C \underline{F} \cdot \underline{dr} \approx (\nabla \times \underline{F}) \cdot \underline{dS}$$

When the contributions over all the rectangles are summed, the line integrals for the inner parts of the rectangles cancel and all that remains is the line integral around the outside of the shape. The surface integrals sum. Hence, the theorem applies for the area  $S$  bounded by the contour  $C$ .

While the above does not comprise a formal proof of Stokes' Theorem, it gives an appreciation of where the theorem comes from.



### Key Point

$$\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

The closed contour integral of the scalar product of a vector function with the vector along the contour is equal to the integral of the scalar product of the curl of that vector function and the unit normal, over the corresponding surface.

**Example** Verify Stokes' Theorem for the vector function  $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$  and the unit square  $0 \leq x \leq 1, 0 \leq y \leq 1$  for  $z = 0$ .

#### Solution

If  $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$  then  $\nabla \times \underline{F} = (z+1)\underline{i} + (-1-2y)\underline{k} = \underline{i} + (-1-2y)\underline{k}$  (as  $z = 0$ ). Note that  $\underline{dS} = dx\,dy\underline{k}$  so that  $(\nabla \times \underline{F}) \cdot \underline{dS} = (-1-2y)\,dy\,dx$

Thus,

$$\begin{aligned} \int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} &= \int_{x=0}^1 \int_{y=0}^1 (-1-2y)\,dy\,dx \\ &= \int_{x=0}^1 [(-y-y^2)]_{y=0}^1 \,dx = \int_{x=0}^1 (-2)\,dx \\ &= [-2x]_0^1 = -2 + 0 = -2 \end{aligned}$$

To evaluate  $\oint_C \underline{F} \cdot \underline{dr}$ , consider it separately on the four sides.

When  $y = 0$ ,  $\underline{F} = -x\underline{j}$  and  $\underline{dr} = dx\underline{i}$  so  $\underline{F} \cdot \underline{dr} = 0$  so the contribution to the integral is zero.

When  $x = 1$ ,  $\underline{F} = y^2\underline{i} - \underline{j}$  and  $\underline{dr} = dy\underline{j}$  so  $\underline{F} \cdot \underline{dr} = -dy$  so the contribution to the integral is  $\int_{y=0}^1 (-dy) = [-y]_0^1 = -1$ .

When  $y = 1$ ,  $\underline{F} = \underline{i} - x\underline{j}$  and  $\underline{dr} = -dx\underline{i}$  so  $\underline{F} \cdot \underline{dr} = -dx$  so the contribution to the integral is  $\int_{x=0}^1 (-dx) = [-x]_0^1 = -1$ .

When  $x = 0$ ,  $\underline{F} = y^2\underline{i}$  and  $\underline{dr} = -dy\underline{j}$  so  $\underline{F} \cdot \underline{dr} = 0$  so the contribution to the integral is zero.

The integral  $\oint_C \underline{F} \cdot \underline{dr}$  is the sum of the contributions i.e.  $0 - 1 - 1 + 0 = -2$ .

Thus

$$\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -2$$

**Example** Using cylindrical polar coordinates (effectively plane-polar coordinates as this example just considers the plane  $z = 0$ ), verify Stokes' theorem for the function  $\underline{F} = \rho^2\hat{\phi}$  and the circle  $\rho = a$ .

### Solution

Firstly, find  $\oint_C \underline{F} \cdot \underline{dr}$ . This can be done by integrating along the contour  $\rho = a$  from  $\theta = 0$  to  $\theta = 2\pi$ . Here  $\underline{F} = a^2 \hat{\theta}$  (as  $\rho = a$ ) and  $\underline{dr} = a d\theta \hat{\theta}$  (remembering the scale factor) so  $\underline{F} \cdot \underline{dr} = a^3 d\theta$  and hence

$$\oint_C \underline{F} \cdot \underline{dr} = \int_0^{2\pi} a^3 d\theta = 2\pi a^3$$

As  $\underline{F} = \rho^2 \hat{\phi}$ ,  $\nabla \times \underline{F} = 3\rho \hat{z}$  and  $(\nabla \times \underline{F}) \cdot \underline{dS} = 3\rho$  as  $\underline{dS} = \hat{z}$ .  
Thus

$$\begin{aligned} \int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} &= \int_{\theta=0}^{2\pi} \int_{\rho=0}^a 3\rho \times \rho d\rho d\theta = \int_{\theta=0}^{2\pi} \int_{\rho=0}^a 3\rho^2 d\rho d\theta \\ &= \int_{\theta=0}^{2\pi} [\rho^3]_{\rho=0}^a d\theta = \int_0^{2\pi} a^3 d\theta = 2\pi a^3 \end{aligned}$$

Hence

$$\oint_C \underline{F} \cdot \underline{dr} = \int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = 2\pi a^3$$

**Example** Find the closed line integral  $\oint_C \underline{F} \cdot \underline{dr}$  for the vector field  $\underline{F} = y^2 \underline{i} + (x^2 - z) \underline{j} + 2xy \underline{k}$  and for the contour  $ABCDEFGHA$  in Figure 6.

### Solution

To find the line integral directly would require eight line integrals i.e. along  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FG$ ,  $GH$  and  $HA$ . It is easier to carry out a surface integral to find  $\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS}$  which is equal to the required line integral  $\oint_C \underline{F} \cdot \underline{dr}$  by Stokes' theorem.

$$\text{As } \underline{F} = y^2 \underline{i} + (x^2 - z) \underline{j} + 2xy \underline{k}, \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 - z & 2xy \end{vmatrix} = (2x + 1) \underline{i} - 2y \underline{j} + (2x - 2y) \underline{k}$$

As the contour lies in the  $x - y$  plane, the unit normal is  $\underline{k}$  and  $\underline{dS} = dx dy \underline{k}$

Hence  $(\nabla \times \underline{F}) \cdot \underline{dS} = (2x - 2y) dx dy$ .

To work out  $\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS}$ , it is necessary to divide the area inside the contour into two smaller areas i.e. the rectangle  $ABCDGH$  and the trapezium  $DEFG$ . On the  $ABCDGH$ , the integral is

$$\begin{aligned} \int_{y=0}^4 \int_{x=0}^6 (2x - 2y) dx dy &= \int_{y=0}^4 [x^2 - 2xy]_{x=0}^6 dy = \int_{y=0}^4 (36 - 12y) dy \\ &= [36y - 6y^2]_0^4 = 36 \times 4 - 6 \times 16 - 0 = 48 \end{aligned}$$

On  $DEFG$ , the integral is

$$\begin{aligned} \int_{y=4}^7 \int_{x=1}^{y-2} (2x - 2y) dx dy &= \int_{y=4}^7 [x^2 - 2xy]_{x=1}^{y-2} dy = \int_{y=4}^7 (-y^2 + 2y + 3) dy \\ &= \left[ -\frac{1}{3}y^3 + y^2 + 3y \right]_4^7 = \frac{343}{3} + 49 + 21 + \frac{64}{3} - 16 - 12 = -51 \end{aligned}$$

So the full integral,  $\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = 48 - 51 = -3$ . By Stokes' Theorem,

$$\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -3$$

From Stokes' theorem, it can be seen that surface integrals of the form  $\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS}$  depend only on the contour bounding the surface and not on the internal part of the surface.



1. Verify Stokes' Theorem for the vector field  $\underline{F} = x^2\underline{i} + 2xy\underline{j} + z\underline{k}$  and the triangle with vertices at  $(0, 0, 0)$ ,  $(3, 0, 0)$  and  $(3, 1, 0)$ .
  - (a) Find the normal vector  $\underline{dS}$
  - (b) Find the vector  $\nabla \times \underline{F}$
  - (c) Evaluate the double integral  $\int_{x=0}^3 \int_{y=0}^{x/3} (\nabla \times \underline{F} \cdot \underline{dS})$
  - (d) Find the integral  $\int \underline{F} \cdot d\underline{r}$  along the 3 sides of the triangle
  - (e) Verify that the two sides of the equation in the theorem are equal.
  
2. Using plane-polar coordinates (or cylindrical polar coordinates with  $z = 0$ ), verify Stokes' Theorem for the vector field  $\underline{F} = \rho\hat{\rho} + \rho \cos \frac{\pi\rho}{2}\hat{\phi}$  and the semi-circle  $\rho \leq 1$ ,  $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ .
  
3. Verify Stokes' theorem for the vector field  $\underline{F} = 2x\underline{i} + (y^2 - z)\underline{j} + xz\underline{k}$  and the contour around the rectangle with vertices at  $(0, -2, 0)$ ,  $(2, -2, 0)$ ,  $(2, 0, 1)$  and  $(0, 0, 1)$ .
  
4. Verify Stokes' Theorem for the vector field  $\underline{F} = -y\underline{i} + x\underline{j} + z\underline{k}$  and for the contour starting from the origin and going to  $(1, 0, 0)$ ,  $(0, 0, 0)$ ,  $(1, 1, 0)$  and  $(1, 1, 1)$  before returning to the origin.
  - (a) Find the surface integral over the triangle  $(0,0,0)$ ,  $(1,0,0)$ ,  $(1,1,0)$
  - (b) Find the surface integral over the triangle  $(1,0,0)$ ,  $(1,1,0)$ ,  $(1,1,1)$
  - (c) Find the line integrals along the four parts of the contour
  - (d) Show that the two sides of the equation of the theorem are equal
  
5. Use Stokes' theorem to evaluate the integral  $\oint_C \underline{F} \cdot d\underline{r}$  where  $\underline{F} = (\sin(\frac{1}{x} + 1) + 5y)\underline{i} + (2x - e^{y^2})\underline{j}$  and  $C$  is the contour starting at  $(0, 0)$  and going to  $(5, 0)$ ,  $(5, 2)$ ,  $(6, 2)$ ,  $(6, 5)$ ,  $(3, 5)$ ,  $(3, 2)$ ,  $(0, 2)$  and returning to  $(0, 0)$ .

**Your solution**

1.)

(a)  $\int dx dy k$ , (b)  $2yzk$ , (c)  $1$ , (d)  $9, 3, -11$ , (e) Both sides are 1.

**Your solution**

2.)

Both sides are 0.

**Your solution**

3.)

Both sides are  $-2$ .

**Your solution**

4.)

(a) 1 (b) 0 (as  $\vec{F}$  is perpendicular to  $\overline{dS}$ ), (c) 0, 1, 1, -1 (d) Both sides are 1.

**Your solution**

5.)

$-57$ .

### Your solution

6.)

Both sides are  $-\bar{i} + 2\bar{j}$ .

## 2. Gauss' Theorem

This is sometimes also known as the divergence theorem and is similar to Stokes' theorem but equates a surface integral to a volume integral. Gauss' Theorem states that for a volume  $V$ , bounded by a closed surface  $S$ , any vector field  $\underline{F}$  satisfies

$$\int \int_S \underline{F} \cdot \underline{dS} = \int \int \int_V \nabla \cdot \underline{F} dV$$

Notes

1.  $\underline{dS}$  is a unit normal pointing outwards.
2. Both sides of the equation are scalars
3. The theorem is often a useful way of calculating a surface integral over a surface composed of several distinct parts (e.g. a cube).
4.  $\nabla \cdot \underline{F}$  is a scalar field representing the divergence of the vector field  $\underline{F}$  and may, alternatively, be written as  $\text{div } \underline{F}$ .
5. Gauss' theorem can be justified in a manner similar to that used for Stokes' theorem.



### Key Point

$$\int \int_S \underline{F} \cdot \underline{dS} = \int \int \int_V \nabla \cdot \underline{F} dV$$

The closed surface integral of the scalar product of a vector function with the unit normal is equal to the integral of the divergence of that vector function over the corresponding volume.



**Example** Verify Gauss' Theorem for the unit cube  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$  and the function  $\underline{F} = x\underline{i} + z\underline{j}$

**Solution**

To find  $\int \int_S \underline{F} \cdot \underline{dS}$ , the integral must be evaluated for all six faces of the cube and the results summed.

On the left face,  $x = 0$ ,  $\underline{F} = z\underline{j}$  and  $\underline{dS} = -\underline{i}dydz$  so  $\underline{F} \cdot \underline{dS} = 0$  and  $\int \int_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0 dydz = 0$

On the right face,  $x = 1$ ,  $\underline{F} = \underline{i} + z\underline{j}$  and  $\underline{dS} = \underline{i}dydz$  so  $\underline{F} \cdot \underline{dS} = 1dydz$  and  $\int \int_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 1 dydz = 1$

On the front face,  $y = 0$ ,  $\underline{F} = x\underline{i} + z\underline{j}$  and  $\underline{dS} = -\underline{j}dxdz$  so  $\underline{F} \cdot \underline{dS} = -z dxdz$  and  $\int \int_S \underline{F} \cdot \underline{dS} = -\int_0^1 \int_0^1 z dxdz = -\frac{1}{2}$

On the back face,  $y = 1$ ,  $\underline{F} = x\underline{i} + z\underline{j}$  and  $\underline{dS} = \underline{j}dxdz$  so  $\underline{F} \cdot \underline{dS} = z dxdz$  and  $\int \int_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 z dxdz = \frac{1}{2}$

On the bottom face,  $z = 0$ ,  $\underline{F} = x\underline{i}$  and  $\underline{dS} = -\underline{k}dydz$  so  $\underline{F} \cdot \underline{dS} = 0dxdy$  and  $\int \int_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0 dxdy = 0$

On the top face,  $z = 1$ ,  $\underline{F} = x\underline{i} + \underline{j}$  and  $\underline{dS} = \underline{k}dydz$  so  $\underline{F} \cdot \underline{dS} = 0dxdy$  and  $\int \int_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0 dxdy = 0$

Thus, summing over all six faces,  $\int \int_S \underline{F} \cdot \underline{dS} = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + 0 = 1$

To find  $\int \int \int_V \nabla \cdot \underline{F} dV$  note that  $\nabla \cdot \underline{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}z = 1 + 0 = 1$ .

So  $\int \int \int_V \nabla \cdot \underline{F} dV = \int_0^1 \int_0^1 \int_0^1 1 dx dy dz = 1$ .

So  $\int \int_S \underline{F} \cdot \underline{dS} = \int \int \int_V \nabla \cdot \underline{F} dV = 1$

Note that in the above example, the volume integral was accomplished by one triple integral while the surface integral required six double integrals. This is often the motivation for using Gauss' Theorem i.e. to carry out one integral rather than several.

**Example** Use Gauss' theorem to evaluate the surface integral  $\int \int_S \underline{F} \cdot \underline{dS}$  where  $\underline{F}$  is the vector field  $x^2y\underline{i} + 2xy\underline{j} + z^3\underline{k}$  and  $S$  is the surface of the unit cube  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ .

### Solution

Note that to carry out the surface integral directly will involve the evaluation of six double integrals (one for each face of the cube). However, by Gauss' theorem, the same result comes from the surface integral  $\int \int \int_V \nabla \cdot \underline{F} \, dV$ . As  $\nabla \cdot \underline{F} = 2xy + 2x + 3z^2$ , the surface integral becomes the triple integral

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 (2xy + 2x + 3z^2) \, dx \, dy \, dz \\ &= \int_0^1 \int_0^1 [x^2y + x^2 + 3xz^2]_{x=0}^1 \, dy \, dz = \int_0^1 \int_0^1 (y + 1 + 3z^2) \, dy \, dz \\ &= \int_0^1 \left[ \frac{1}{2}y^2 + y + 3yz^2 \right]_{y=0}^1 \, dz = \int_0^1 \left( \frac{1}{2} + 1 + 3z^2 \right) \, dz = \int_0^1 \left( \frac{3}{2} + 3z^2 \right) \, dz \\ &= \left[ \frac{3}{2}z + z^3 \right]_0^1 = \frac{11}{6} \end{aligned}$$

The six double integrals also sum to  $\frac{11}{6}$  but this approach requires a greater amount of work.

**Example** Verify Gauss' Theorem for the vector field  $\underline{F} = y^2\underline{j} - xz\underline{k}$  and the triangular prism with vertices at  $(0, 0, 0)$ ,  $(2, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 4, 0)$ ,  $(2, 4, 0)$  and  $(0, 4, 1)$  (see figure 7).

### Solution

As  $\underline{F} = y^2\underline{j} - xz\underline{k}$ ,  $\nabla \cdot \underline{F} = 0 + 2y - x = 2y - x$ .

Thus

$$\begin{aligned}\int \int \int_V \nabla \cdot \underline{F} \, dV &= \int_{x=0}^2 \int_{y=0}^4 \int_{z=0}^{1-x/2} (2y - x) \, dz \, dy \, dx \\ &= \int_{x=0}^2 \int_{y=0}^4 [2yz - xz]_{z=0}^{1-x/2} \, dy \, dx = \int_{x=0}^2 \int_{y=0}^4 (2y - xy - x + \frac{1}{2}x^2) \, dy \, dx \\ &= \int_{x=0}^2 \left[ y^2 - \frac{1}{2}xy^2 - xy + \frac{1}{2}x^2y \right]_{y=0}^4 \, dx = \int_{x=0}^2 (16 - 12x + 2x^2) \, dx \\ &= \left[ 16x - 6x^2 + \frac{2}{3}x^3 \right]_0^2 = \frac{40}{3}\end{aligned}$$

To work out  $\int \int_S \underline{F} \cdot \underline{dS}$ , it is necessary to consider the contributions from the five faces separately.

On the front face,  $y = 0$ ,  $\underline{F} = -xz\underline{k}$  and  $\underline{dS} = -\underline{j}$  thus  $\underline{F} \cdot \underline{dS} = 0$  and the contribution to the integral is zero.

On the back face,  $y = 4$ ,  $\underline{F} = 16\underline{j} - xz\underline{k}$  and  $\underline{dS} = \underline{j}$  thus  $\underline{F} \cdot \underline{dS} = 16$  and the contribution to the integral is  $\int_{x=0}^2 \int_{z=0}^{1-x/2} 16 \, dz \, dx = \int_{x=0}^2 [16z]_{z=0}^{1-x/2} \, dx = \int_{x=0}^2 16(1-x/2) \, dx = [16x - 4x^2]_0^2 = 32 - 16 = 16$

On the left face,  $x = 0$ ,  $\underline{F} = y^2\underline{j}$  and  $\underline{dS} = -\underline{i}$  thus  $\underline{F} \cdot \underline{dS} = 0$  and the contribution to the integral is zero.

On the bottom face,  $z = 0$ ,  $\underline{F} = y^2\underline{j}$  and  $\underline{dS} = -\underline{k}$  thus  $\underline{F} \cdot \underline{dS} = 0$  and the contribution to the integral is zero.

On the top right face,  $z = 1 - x/2$ ,  $\underline{F} = y^2\underline{j} + (\frac{1}{2}x^2 - x)\underline{k}$  and the unit normal  $\hat{n} = \frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}$

Thus  $\underline{dS} = \left[ \frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k} \right] \, dy \, dw$  where  $dw$  measures the distance along the slope for a constant  $y$ . As  $dw = \frac{\sqrt{5}}{2}dx$ ,  $\underline{dS} = \left[ \frac{1}{2}\underline{i} + \underline{k} \right] \, dy \, dx$  thus  $\underline{F} \cdot \underline{dS} = 16$  and the contribution to the integral is  $\int_{x=0}^2 \int_{y=0}^4 (\frac{1}{2}x^2 - x) \, dy \, dx = \int_{x=0}^2 (2x^2 - 4x) \, dx = \left[ \frac{2}{3}x^3 - 2x^2 \right]_0^2 = -\frac{8}{3}$ .

Adding together the contributions,  $\int \int_S \underline{F} \cdot \underline{dS} = 0 + 16 + 0 + 0 - \frac{8}{3} = \frac{40}{3}$

Thus  $\int \int_S \underline{F} \cdot \underline{dS} = \int \int \int_V \nabla \cdot \underline{F} \, dV = \frac{40}{3}$

Gauss' Theorem also applies using orthogonal curvilinear coordinates.



1. Verify Gauss' Theorem for the vector field  $\underline{F} = x\underline{i} - y\underline{j} + z\underline{k}$  and the unit cube  $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ .
  - (a) Find the vector  $\nabla \cdot \underline{F}$
  - (b) Evaluate the triple integral  $\int_{z=0}^1 \int_{y=0}^1 \int_{x=0}^1 \nabla \cdot \underline{F} \, dx dy dz$
  - (c) For each side, evaluate the normal vector  $\underline{dS}$  and the surface integral  $\iint_S \underline{F} \cdot \underline{dS}$
  - (d) Show that the two sides of the equation of the theorem are equal
2. Verify Gauss' Theorem for the vector field  $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$  and the cuboid  $0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 4$ .
3. Verify Gauss' Theorem, using cylindrical polar coordinates, for the vector field  $\underline{F} = \rho^{-2}\hat{\rho}$  over the cylinder  $0 \leq \rho \leq r_0, -1 \leq z \leq 1$  for
  - (a)  $r_0 = 1$
  - (b)  $r_0 = 2$
4. For  $S$  being the surface of the tetrahedron with vertices at  $(0, 0, 0), (1, 0, 0), (0, 1, 0)$  and  $(0, 0, 1)$ , find the surface integral  $\iint_S (x\underline{i} + yz\underline{j}) \cdot \underline{dS}$ 
  - (a) directly
  - (b) by using Gauss' Theorem

Hint :- When evaluating directly, show that the unit normal on the sloping face is  $\frac{1}{\sqrt{3}}(\underline{x} + \underline{y} + \underline{z})$  but that  $\underline{dS} = \underline{x} + \underline{y} + \underline{z}$

### Your solution

1.)

(a) 1 (b) 1 (c)  $-\text{dxdy}\bar{k}, 0; \text{dxdy}\bar{j}, 1; -\text{dxdy}\bar{j}, 0; \text{dxdz}\bar{j}, -1; -\text{dydz}\bar{i}, 0; \text{dydz}\bar{i}, 1$  (d) Both sides are 1.

**Your solution**

2.)

Both sides are  $156$ .

**Your solution**

3.)

Both sides equal  $4\pi, 2\pi$ .

### Your solution

4.)

Both sides equal  $\frac{24}{5}$ .

## 3. Green's Theorem

Like Gauss' Theorem, Green's Theorem equates a surface integral to a volume integral. However, Green's Theorem is concerned with two scalar fields  $u(x, y, z)$  and  $w(x, y, z)$ . Two statements of Green's Theorem are as follows

$$\int \int_S (u \nabla w) \cdot \underline{ds} = \int \int \int_V [\nabla u \cdot \nabla w + u \nabla^2 w] dV$$

and

$$\int \int_S [u \nabla w - v \nabla u] \cdot \underline{ds} = \int \int \int_V [u \nabla^2 w - w \nabla^2 u] dV$$

### Proof of Green's Theorem

Green's Theorem can be derived from Gauss' Theorem and a vector derivative identity.

Vector identity (1) from subsection 6 of 37.2 states that  $\nabla \cdot (\phi \underline{A}) = (\nabla \phi) \cdot \underline{A} + \phi(\nabla \cdot \underline{A})$

Letting  $\phi = u$  and  $\underline{A} = \nabla w$ ,

$$\begin{aligned} \nabla \cdot (u \nabla w) &= (\nabla u) \cdot (\nabla w) + u(\nabla \cdot (\nabla w)) \\ &= (\nabla u) \cdot (\nabla w) + u \nabla^2 w \end{aligned}$$

Gauss' Theorem states

$$\int \int_S \underline{F} \cdot \underline{dS} = \int \int \int_V \nabla \cdot \underline{F} \, dV$$

Now, letting  $\underline{F} = u\nabla w$ ,

$$\begin{aligned} \int \int_S (u\nabla w) \cdot \underline{dS} &= \int \int \int_V \nabla \cdot (u\nabla w) \, dV \\ &= \int \int \int_V [(\nabla u) \cdot (\nabla w) + u\nabla^2 w] \, dV \end{aligned}$$

This is the first statement of Green's Theorem. Reversing the roles of  $u$  and  $w$ ,

$$\int \int_S (w\nabla u) \cdot \underline{dS} = \int \int \int_V [(\nabla w) \cdot (\nabla u) + w\nabla^2 u] \, dV$$

Subtracting the last two equations yields the second statement of Green's Theorem.



### Key Point

1.

$$\int \int_S (u \nabla w) \cdot \underline{dS} = \int \int \int_V [\nabla u \cdot \nabla w + u \nabla^2 w] \, dV$$

2.

$$\int \int_S [u \nabla w - v \nabla u] \cdot \underline{dS} = \int \int \int_V [u \nabla^2 w - w \nabla^2 u] \, dV$$

**Example** Verify Green's Theorem (first statement) for  $u = (x - x^2)y$ ,  $w = xy + z^2$  and the unit cube,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ .

## Solution

As  $w = xy + z^2$ ,  $\nabla w = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ . Thus  $u\nabla w = (xy - x^2y)(y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k})$  and the surface integral is of this quantity (scalar product with  $d\mathbf{S}$  integrated over the surface of the unit cube. On the three faces  $x = 0$ ,  $x = 1$ ,  $y = 0$ , the vector  $u\nabla w = \mathbf{0}$  and so the contribution to the surface integral is zero.

On the face  $y = 1$ ,  $u\nabla w = (x - x^2)(\mathbf{i} + x\mathbf{j} + 2z\mathbf{k})$  and  $d\mathbf{S} = \mathbf{j}$  so  $(u\nabla w) \cdot d\mathbf{S} = x^2 - x^3$  and the contribution to the integral is  $\int_{x=0}^1 \int_{z=0}^1 (x^2 - x^3) dz dx = \int_0^1 (x^2 - x^3) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$ .

On the face  $z = 0$ ,  $u\nabla w = (x - x^2)y(y\mathbf{i} + x\mathbf{j})$  and  $d\mathbf{S} = -\mathbf{k}$  so  $(u\nabla w) \cdot d\mathbf{S} = 0$  and the contribution to the integral is zero.

On the face  $z = 1$ ,  $u\nabla w = (x - x^2)y(y\mathbf{i} + x\mathbf{j} + 2\mathbf{k})$  and  $d\mathbf{S} = \mathbf{k}$  so  $(u\nabla w) \cdot d\mathbf{S} = 2y(x - x^2)$  and the contribution to the integral is  $\int_{x=0}^1 \int_{y=0}^1 2y(x - x^2) dy dx = \int_0^1 (x^2 - x^3) dx = \int_{x=0}^1 [y^2(x - x^2)]_{y=0}^1 dx = \int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$ .

Thus,  $\int \int_S (u \nabla w) \cdot d\mathbf{s} = 0 + 0 + 0 + \frac{1}{12} + 0 + \frac{1}{6} = \frac{1}{4}$ .

Now evaluate  $\int \int \int_V [\nabla u \cdot \nabla w + u \nabla^2 w] dV$ .

Note that  $\nabla u = (1 - 2x)y\mathbf{i} + (x - x^2)\mathbf{j}$  and  $\nabla^2 w = 2$  so

$$\begin{aligned} \nabla u \cdot \nabla w + u \nabla^2 w &= (1 - 2x)y^2 + (x - x^2)x + 2(x - x^2)y \\ &= x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2 \end{aligned}$$

and the integral

$$\begin{aligned} \int \int \int_V [\nabla u \cdot \nabla w + u \nabla^2 w] dV &= \int_{z=0}^1 \int_{y=0}^1 \int_{x=0}^1 (x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2) dx dy dz \\ &= \int_{z=0}^1 \int_{y=0}^1 \left[ \frac{x^3}{3} - \frac{x^4}{4} + x^2y - \frac{2}{3}x^3y + xy^2 - x^2y^2 \right]_{x=0}^1 dy dz \\ &= \int_{z=0}^1 \int_{y=0}^1 \left( \frac{1}{12} + \frac{y}{3} \right) dy dz = \int_{z=0}^1 \left[ \frac{y}{12} + \frac{y^2}{6} \right]_{y=0}^1 dz \\ &= \int_{z=0}^1 \left( \frac{1}{4} \right) dz = \left[ \frac{z}{4} \right]_{z=0}^1 = \frac{1}{4} \end{aligned}$$

Hence

$$\int \int_S (u \nabla w) \cdot d\mathbf{s} = \int \int \int_V [\nabla u \cdot \nabla w + u \nabla^2 w] dV = \frac{1}{4}$$

## Green's Theorem in the Plane

This states that

$$\oint_C (P dx + Q dy) = \int \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

## Justification



Green's Theorem in the plane can be derived from Stoke's Theorem.

$$\int \int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Now let  $\underline{F}$  be the vector field  $P(x, y)\underline{i} + Q(x, y)\underline{j}$  i.e. there is no dependence on  $z$  and there are no components in the  $z$ - direction. Now

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underline{k}$$

and  $\underline{dS} = dx dy \underline{k}$  giving  $(\nabla \times \underline{F}) \cdot \underline{dS} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$ .

Thus Stokes' Theorem becomes

$$\int \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C \underline{F} \cdot \underline{dr}$$

and Green's Theorem in the plane Follows



### Key Point

$$\oint_C (P dx + Q dy) = \int \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

**Example** Evaluate the line integral  $\oint_C [(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy]$  around the rectangle  $0 \leq x \leq 3, 0 \leq y \leq 1$ .

### Solution

The integral could be accomplished by four line integrals but it is easier to note that  $[(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy]$  is of the form  $P dx + Q dy$  with  $P = 4x^2 + y - 3$  and  $Q = 3x^2 + 4y^2 - 2$ . It is thus of a suitable form for Green's Theorem in the plane.

Note that  $\frac{\partial Q}{\partial x} = 6x$  and  $\frac{\partial P}{\partial y} = 1$ .

Green's Theorem in the plane becomes

$$\begin{aligned}\oint_C ((4x^2 + y - 3) dx + (3x^2 + 4y^2 - 2) dy) &= \int_{y=0}^1 \int_{x=0}^3 (6x - 1) dx dy \\ &= \int_{y=0}^1 [3x^2 - x]_{x=0}^3 dy = \int_{y=0}^1 24 dy = 24\end{aligned}$$

The same result could be gained by evaluating four line integrals.

**Example** Verify Green's Theorem in the plane for the integral  $\oint_C [4z dy + (y^2 - 2)]$  and the contour starting at the origin  $O = (0, 0, 0)$  and going to  $A = (0, 2, 0)$  and  $B = (0, 0, 1)$  before returning to the origin.

### Solution

The whole of the contour is in the plane  $x = 0$  and Green's Theorem in the plane becomes

$$\oint_C (P dy + Q dz) = \int \int_S \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z} \right) dy dz$$

Firstly evaluate  $\oint_C [4z dy + (y^2 - 2)]$ .

On  $OA$ ,  $z = 0$  and  $dz = 0$ . As the integrand is zero, the integral will also be zero.

On  $AB$ ,  $z = (1 - \frac{y}{2})$  and  $dz = 0$ . The integral is

$$\int_{y=2}^0 [(4 - 2y)dy - \frac{1}{2}(y^2 - 2)dy] = \int_2^0 (5 - 2y - \frac{1}{2}y^2)dy = [5y - y^2 - \frac{1}{6}y^3]_2^0 = -\frac{14}{3}$$

On  $BO$ ,  $y = 0$  and  $dy = 0$ . The integral is  $\int_1^0 (-2)dz = [-2z]_1^0 = 2$ .

Summing,  $\oint_C [4z dy + (y^2 - 2)] = -\frac{8}{3}$

In this example,  $P = 4z$  and  $Q = y^2 - 2$ . Thus  $\frac{\partial P}{\partial z} = 4$  and  $\frac{\partial Q}{\partial y} = 2y$ . Hence,

$$\begin{aligned} \int \int_S \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z} \right) dy dz &= \int_{y=0}^2 \int_{z=0}^{1-y/2} (2y - 4) dy dz \\ &= \int_{y=0}^2 [2yz - 4z]_{z=0}^{1-y/2} dz = \int_{y=0}^2 (-y^2 + 4y - 4) dz \\ &= \left[ -\frac{1}{3}y^3 + 2y^2 - 4y \right]_0^2 = -\frac{8}{3} \end{aligned}$$

Hence

$$\oint_C (P dy + Q dz) = \int \int_S \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z} \right) dy dz = -\frac{8}{3}$$

One very useful, special case of Green's Theorem in the plane is when  $Q = x$  and  $P = -y$ . The theorem becomes

$$\oint_C (-y dx + x dy) = \int \int_S (1 - (-1)) dx dy$$

The right hand side becomes  $\int \int_S 2 dx dy$  i.e.  $2A$  where  $A$  is the area inside the contour  $C$ . Hence

$$A = \frac{1}{2} \oint_C (x dy - y dx)$$

This result is known as the area theorem.

**Example** Verify the area theorem  $A = \frac{1}{2} \oint_C (x dy - y dx)$  for the segment of the circle  $x^2 + y^2 = 4$  lying above the line  $y = 1$ .

### Solution

Firstly, the area of the segment  $ADBC$  can be found by subtracting the area of the triangle  $OADB$  from the area of the sector  $OACB$ . The triangle has area  $\frac{1}{2} \times 2\sqrt{3} \times 1 = \sqrt{3}$ . The sector has area  $\frac{\pi}{3} \times 2^2 = \frac{4}{3}\pi$ . Thus segment  $ADBC$  has area  $\frac{4}{3}\pi - \sqrt{3}$ .

Now, evaluate the integral  $\oint_C (x dy - y dx)$  around the segment. Along the line,  $y = 1$ ,  $dy = 0$  so the integral  $\int_C (x dy - y dx)$  becomes  $\int_{-\sqrt{3}}^{\sqrt{3}} (x \times 0 - 1 dx) = \int_{-\sqrt{3}}^{\sqrt{3}} (-dx) = -2\sqrt{3}$ .

Along the arc of the circle,  $y = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$  so  $dy = -x(4 - x^2)^{-1/2} dx$ . The integral  $\int_C (x dy - y dx)$  becomes

$$\begin{aligned} \int_{\sqrt{3}}^{-\sqrt{3}} [-x^2(4 - x^2)^{-1/2} - (4 - x^2)^{1/2}] dx &= \int_{-\sqrt{3}}^{\sqrt{3}} -\sqrt{3}^{\sqrt{3}} \frac{4}{\sqrt{4 - x^2}} dx \\ &= \int_{-\pi/3}^{\pi/3} 4 \frac{1}{2 \cos \theta} 2 \cos \theta d\theta \\ &= \int_{-\pi/3}^{\pi/3} 4 d\theta = \frac{8}{3}\pi \end{aligned}$$

So,  $\frac{1}{2} \oint_C (x dy - y dx) = \frac{1}{2} [\frac{8}{3}\pi - 2\sqrt{3}] = \frac{4}{3}\pi - \sqrt{3}$  Hence both sides of the theorem equal  $\frac{4}{3}\pi - \sqrt{3}$



- Verify the first statement of Green's Theorem for the functions  $u = xyz$ ,  $w = y^2$  and the unit cube  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ .
  - Find the function  $(u \nabla w)$
  - Evaluate the integral  $\iint_S (u \nabla w) \cdot \underline{dS}$  for the six faces of the cube
  - Find the function  $\nabla u \cdot \nabla w + u \nabla^2 w$
  - Evaluate the integral  $\iiint_V [\nabla u \cdot \nabla w + u \nabla^2 w] dV$
  - Show that the two sides in the equation of the theorem are equal.
- Verify the area theorem for
  - The area above  $y = 0$  but below  $y = 1 - x^2$ .
  - The segment of the circle  $x^2 + y^2 = 1$ , to the upper left of the line  $y = 1 - x$ .

**Your solution**

1.)

(a)  $2xy^2z\bar{j}$  (b)  $0, 0, 0, \frac{1}{2}, 0, 0$  (c)  $4xyz$  (d) Both sides are  $\frac{2}{3}$ .

**Your solution**

2.)

4.

**Your solution**

3.)

$\frac{4}{3}, \frac{4}{\pi} - \frac{1}{2}$ .