

Solving quadratic equations

3.2



Introduction

A **quadratic equation** is one which can be written in the form $ax^2 + bx + c = 0$ where a , b and c are numbers and x is the unknown whose value(s) we wish to find. In this section we describe several ways in which quadratic equations can be solved.



Prerequisites

Before starting this Section you should ...

- be able to solve linear equations



Learning Outcomes

After completing this Section you should be able to ...

- ✓ recognise a quadratic equation
- ✓ solve a quadratic equation by factorisation
- ✓ solve a quadratic equation using a standard formula
- ✓ solve a quadratic equation by completing the square
- ✓ interpret the solution of a quadratic graphically

1. Quadratic Equations



Key Point

A **quadratic equation** is one which can be written in the form

$$ax^2 + bx + c = 0 \quad a \neq 0$$

where a , b and c are given numbers and x is the unknown whose value(s) we wish to find.

For example

$$2x^2 + 7x - 3 = 0, \quad x^2 + x + 1 = 0, \quad 0.5x^2 + 3x + 9 = 0$$

are all quadratic equations. To ensure the presence of the x^2 term the number a , in the general expression $ax^2 + bx + c$, cannot be zero. However b and/or c may be zero, so that

$$4x^2 + 3x = 0, \quad 2x^2 - 3 = 0 \quad \text{and} \quad 6x^2 = 0$$

are also quadratic equations. Frequently, quadratic equations occur in non-standard form but where necessary they can be rearranged into standard. For example

$$3x^2 + 5x = 8, \quad \text{can be re-written as} \quad 3x^2 + 5x - 8 = 0$$

$$2x^2 = 8x - 9, \quad \text{can be re-written as} \quad 2x^2 - 8x + 9 = 0$$

$$1 + x = \frac{1}{x}, \quad \text{can be re-written as} \quad x^2 + x - 1 = 0$$

To solve a quadratic equation we must find values of the unknown x which make the left-hand and right-hand sides equal. Such values are known as **solutions** or **roots** of the quadratic equation. We shall now describe three techniques for solving quadratic equations:

- factorisation
- completing the square
- using a formula

Exercises

1. Verify that $x = 2$ and $x = 3$ are both solutions of $x^2 - 5x + 6 = 0$.
2. Verify that $x = -2$ and $x = -3$ are both solutions of $x^2 + 5x + 6 = 0$.

Note the difference between solving quadratic equations in comparison to solving linear equations. A quadratic equation will generally have *two* values of x (solutions) which satisfy it whereas a linear equation only has *one* solution.

2. Solution by factorisation

It may be possible to solve a quadratic equation by factorisation using the method described for factorizing quadratic expressions in Workbook 1 section 5, although you should be aware that not all quadratic equations can be easily factorized.

Example Solve the equation $x^2 + 5x = 0$.

Solution

Factorizing and equating each factor to zero we find

$$x^2 + 5x = 0 \quad \text{is equivalent to} \quad x(x + 5) = 0$$

so that $x = 0$ and $x = -5$ are the two solutions.

Example Solve the quadratic equation $x^2 + x - 6 = 0$.

Solution

Factorizing the left hand side we find $(x^2 + x - 6) = (x + 3)(x - 2)$ so that

$$x^2 + x - 6 = 0 \quad \text{is equivalent to} \quad (x + 3)(x - 2) = 0$$

When the product of two quantities equals zero, at least one of the two must equal zero. In this case either $(x + 3)$ is zero or $(x - 2)$ is zero. It follows that

$$x + 3 = 0, \quad \text{giving} \quad x = -3$$

or

$$x - 2 = 0, \quad \text{giving} \quad x = 2$$

Here there are two solutions, $x = -3$ and $x = 2$. These solutions can be checked quite easily by substitution back into the given equation.

Example Solve the quadratic equation $2x^2 - 7x - 4 = 0$.

Solution

Factorizing the left hand side: $(2x^2 - 7x - 4) = (2x + 1)(x - 4)$ so that

$$2x^2 - 7x - 4 = 0 \text{ is equivalent to } (2x + 1)(x - 4) = 0$$

In this case either $(2x + 1)$ is zero or $(x - 4)$ is zero. It follows that

$$2x + 1 = 0, \quad \text{giving} \quad x = -\frac{1}{2}$$

or

$$x - 4 = 0, \quad \text{giving} \quad x = 4$$

There are two solutions, $x = -\frac{1}{2}$ and $x = 4$.

Example Solve the equation $4x^2 + 12x + 9 = 0$.

Solution

Factorizing we find

$$4x^2 + 12x + 9 = (2x + 3)(2x + 3) = (2x + 3)^2$$

This time the factor $(2x + 3)$ occurs twice. The original equation $4x^2 + 12x + 9 = 0$ becomes

$$(2x + 3)^2 = 0$$

so that

$$2x + 3 = 0$$

and we obtain the solution $x = -\frac{3}{2}$. Because the factor $2x + 3$ appears twice in the equation $(2x + 3)^2 = 0$ we say that this root is a **repeated solution** or **double root**.



Solve the quadratic equation $7x^2 - 20x - 3 = 0$.

First factorize the left-hand side

Your solution

$$7x^2 - 20x - 3 =$$

$$(7x - 3)(x + 1)$$

Each factor is then equated to zero to obtain the two solutions

Your solution	
Solution 1: $x =$	Solution 2: $x =$
$\frac{1}{2}$ and $\frac{3}{2}$	

Exercises

Solve the following equations by factorisation:

- | | | |
|------------------------|--------------------------|-------------------------|
| 1. $x^2 - 3x + 2 = 0$ | 2. $x^2 - x - 2 = 0$ | 3. $x^2 + x - 2 = 0$ |
| 4. $x^2 + 3x + 2 = 0$ | 5. $x^2 + 8x + 7 = 0$ | 6. $x^2 - 7x + 12 = 0$ |
| 7. $x^2 - x - 20 = 0$ | 8. $x^2 - 1 = 0$ | 9. $x^2 - 2x + 1 = 0$ |
| 10. $x^2 + 2x + 1 = 0$ | 11. $x^2 + 11x = 0$ | 12. $2x^2 + 2x = 0$ |
| 13. $x^2 - 3x = 0$ | 14. $x^2 + 9x = 0$ | 15. $2x^2 - 5x + 2 = 0$ |
| 16. $6x^2 - x - 1 = 0$ | 17. $-5x^2 + 6x - 1 = 0$ | 18. $-x^2 + 4x - 3 = 0$ |

Answers		
1. 1, 2	3. -2, 1	6. 4, 3
2. -1, -2	4. -1, -2	7. -4, 5
3. 1, -1	5. -7, -1	8. 1, -1
4. 0, 3	6. 1 twice	9. 1 twice
5. 1, 3	7. 2, $\frac{1}{2}$	10. -1 twice
6. $\frac{5}{1}, 1$	8. 1, 3	11. -1, 0
7. $\frac{3}{1}, -\frac{3}{1}$	9. 0, -9	12. 0, -1
8. $\frac{7}{2}, -\frac{3}{2}$	10. 2, $\frac{1}{2}$	13. 0, -1
9. $\frac{1}{2}, -\frac{3}{2}$	11. 2, $\frac{1}{2}$	14. 0, 3
10. $\frac{1}{2}, -\frac{3}{2}$	12. 2, $\frac{1}{2}$	15. 0, 3
11. $\frac{1}{2}, -\frac{3}{2}$	13. 2, $\frac{1}{2}$	16. $\frac{1}{2}, -\frac{3}{2}$
12. $\frac{1}{2}, -\frac{3}{2}$	14. 2, $\frac{1}{2}$	17. $\frac{1}{2}, -\frac{3}{2}$
13. $\frac{1}{2}, -\frac{3}{2}$	15. 2, $\frac{1}{2}$	18. $\frac{1}{2}, -\frac{3}{2}$

3. Completing the square

The technique known as completing the square can be used to solve quadratic equations although it is applicable in many other circumstances as well so it is well worth studying.

- Example**
- (a) Show that $(x + 3)^2 = x^2 + 6x + 9$
- (b) Hence show that $x^2 + 6x$ can be written as $(x + 3)^2 - 9$.

<p>Solution</p> <p>(a) Removing the brackets we find</p> $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$ <p>Thus</p> $(x + 3)^2 = x^2 + 6x + 9$ <p>(b) By subtracting 9 from both sides of the previous equation it follows that</p> $(x + 3)^2 - 9 = x^2 + 6x$

Example

- (a) Show that $(x - 4)^2 = x^2 - 8x + 16$
(b) Hence show that $x^2 - 8x$ can be written as $(x - 4)^2 - 16$.

Solution

- (a) Removing the brackets we find

$$(x - 4)^2 = (x - 4)(x - 4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

- (b) Subtracting 16 from both sides we can write

$$(x - 4)^2 - 16 = x^2 - 8x$$

We shall now generalise the results of the previous two examples. Noting that

$$(x + k)^2 = x^2 + 2kx + k^2$$

we can write

$$x^2 + 2kx = (x + k)^2 - k^2$$

Note that the constant term in the brackets on the right hand side is always half the coefficient of x on the left. This process is called **completing the square**.

**Key Point**

Completing the square

The expression $x^2 + 2kx$ is equivalent to $(x + k)^2 - k^2$

Example Complete the square for the expression $x^2 + 16x$.

Solution

Comparing $x^2 + 16x$ with the general form $x^2 + 2kx$ we see that $k = 8$. Hence

$$x^2 + 16x = (x + 8)^2 - 8^2 = (x + 8)^2 - 64$$

Note that the constant term in the brackets on the right, that is 8, is half the coefficient of x on the left, which is 16.

Example Complete the square for the expression $5x^2 + 4x$.

Solution

Consider $5x^2 + 4x$. First of all the coefficient 5 is removed outside a bracket as follows

$$5x^2 + 4x = 5\left(x^2 + \frac{4}{5}x\right)$$

We can now complete the square for the quadratic expression in the brackets:

$$x^2 + \frac{4}{5}x = \left(x + \frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2 = \left(x + \frac{2}{5}\right)^2 - \frac{4}{25}$$

Finally, multiplying both sides by 5 we find

$$5x^2 + 4x = 5\left(\left(x + \frac{2}{5}\right)^2 - \frac{4}{25}\right)$$

Completing the square can be used to solve quadratic equations as shown in the following examples.

Example Solve the equation $x^2 + 6x + 2 = 0$ by completing the square.

Solution

First of all just consider $x^2 + 6x$, and note that we can write this as

$$x^2 + 6x = (x + 3)^2 - 9$$

Then the quadratic equation can be written as

$$x^2 + 6x + 2 = (x + 3)^2 - 9 + 2 = 0$$

that is

$$(x + 3)^2 = 7$$

Taking the square root of both sides gives

$$\begin{aligned}x + 3 &= \pm\sqrt{7} \\x &= -3 \pm \sqrt{7}\end{aligned}$$

The two solutions are $x = -3 + \sqrt{7} = -0.3542$ and $x = -3 - \sqrt{7} = -5.6458$.

Example Solve the equation $x^2 - 8x + 5 = 0$

Solution

First consider $x^2 - 8x$ which we can write as

$$x^2 - 8x = (x - 4)^2 - 16$$

so that the equation becomes

$$x^2 - 8x + 5 = (x - 4)^2 - 16 + 5 = 0$$

$$(x - 4)^2 = 11$$

$$x - 4 = \pm\sqrt{11}$$

$$x = 4 \pm \sqrt{11}$$

So $x = 7.3166$ or $x = 0.6834$ (to 4d.p.) depending on whether we take the plus or minus sign.



Solve the equation $x^2 - 4x + 1 = 0$ by completing the square.

First examine the two left-most terms in the equation: $x^2 - 4x$. Complete the square for these terms:

Your solution

$$x^2 - 4x =$$

$$(x - 2)^2 - 4$$

The equation $x^2 - 4x + 1 = 0$ can then be written

Your solution

$$x^2 - 4x + 1 =$$

$$(x - 2)^2 - 4 + 1 = 0$$

from which you can now obtain the roots

Your solution

$$(x - 2)^2 = 3, \text{ so } x - 2 = \pm\sqrt{3}. \text{ Therefore } x = 2 \pm \sqrt{3} = 3.7321 \text{ or } 0.2679 \text{ to 4d.p.}$$

Exercises

1. Solve the quadratic equations at the end of the previous section by completing the square.

4. Solution by formula

When it is difficult to factorize a quadratic equation, it may be possible to solve it using a formula which is used to calculate the roots. The formula is obtained by completing the square in the general quadratic $ax^2 + bx + c$. We proceed by removing the coefficient of a :

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

Thus the solution of $ax^2 + bx + c = 0$ is the same as the solution to

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

So, solving:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \text{which leads to} \quad x = -\frac{b}{2a} \pm \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}$$

Simplifying this expression further we obtain the important result:



Key Point

If $ax^2 + bx + c = 0$ then the two solutions (roots) are

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

To apply the formula to a specific quadratic equation it is necessary to identify carefully the values of a , b and c , paying particular attention to the signs of these numbers. Substitution of these values into the formula then gives the desired solutions.

Note that if the quantity $b^2 - 4ac$ is a positive number we can take its square root and the formula will produce two solutions known as **distinct real roots**. If $b^2 - 4ac = 0$ there will be a single root known as a **repeated root**. The value of this root is $x = -\frac{b}{2a}$. Finally if $b^2 - 4ac$ is negative we say the equation possesses **complex roots**. These require special treatment and are described in Workbook 10.



Key Point

When finding roots of the quadratic equation $ax^2 + bx + c = 0$ first calculate the quantity

$$b^2 - 4ac$$

- If $b^2 - 4ac > 0$ the quadratic has two real distinct roots
- If $b^2 - 4ac = 0$ the quadratic has real and equal roots
- If $b^2 - 4ac < 0$ the quadratic has no real roots: they are complex

Example Compare the given equation with the standard form $ax^2 + bx + c = 0$ and identify a , b and c . Calculate $b^2 - 4ac$ in each case and use this information to state the nature of the roots.

- a) $3x^2 + 2x - 7 = 0$ b) $3x^2 + 2x + 7 = 0$ c) $3x^2 - 2x + 7 = 0$
d) $x^2 + x + 2 = 0$ e) $-x^2 + 3x - \frac{1}{2} = 0$ f) $5x^2 - 3 = 0$
g) $x^2 - 2x + 1 = 0$

Solution

- (a) $a = 3$, $b = 2$ and $c = -7$. So $b^2 - 4ac = (2)^2 - 4(3)(-7) = 88$.
The roots are real and distinct.
- (b) $a = 3$, $b = 2$ and $c = 7$. So $b^2 - 4ac = (2)^2 - 4(3)(7) = -80$.
The roots are complex.
- (c) $a = 3$, $b = -2$ and $c = 7$. So $b^2 - 4ac = (-2)^2 - 4(3)(7) = -80$.
Again the roots are complex.
- (d) $a = 1$, $b = 1$ and $c = 2$. So $b^2 - 4ac = 1^2 - 4(1)(2) = -7$.
The roots are complex.
- (e) $a = -1$, $b = 3$ and $c = -\frac{1}{2}$. So $b^2 - 4ac = 3^2 - 4(-1)(-\frac{1}{2}) = 7$.
The roots are real and distinct.
- (f) $a = 5$, $b = 0$ and $c = -3$. So $b^2 - 4ac = 0 - 4(5)(-3) = 60$.
The roots are real and distinct.
- (g) $a = 1$, $b = -2$ and $c = 1$. So $b^2 - 4ac = (-2)^2 - 4(1)(1) = 0$.
There is a single repeated root.

Example Solve the quadratic equation $2x^2 + 3x - 6 = 0$ using the formula.

Solution

We compare the given equation with the standard form $ax^2 + bx + c = 0$ in order to identify a , b and c . We see that here $a = 2$, $b = 3$ and $c = -6$. Note particularly the sign of c . Substituting these values into the formula we find

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{(2)(2)} \\ &= \frac{-3 \pm \sqrt{9 + 48}}{4} \\ &= \frac{-3 \pm \sqrt{57}}{4} \\ &= \frac{-3 \pm 7.5498}{4} \end{aligned}$$

Hence the two roots are $x = 1.1375$, if the positive sign is taken and $x = -2.6375$ if the negative sign is taken. However, it is often sufficient to leave the solution in the so-called **surd form** $x = \frac{-3 \pm \sqrt{57}}{4}$.



Solve the equation $3x^2 - x - 6 = 0$.

First identify a , b and c .

Your solution

$a =$ $b =$ $c =$

$9 = 2$ $1 = 4$ $6 = 2$

Substitute these values into the formula

Your solution

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ so $x =$

$$\frac{9}{2 \pm 1} = \frac{(9)(2)}{(9)(2) - 2(1) - 4(6)}$$

Finally calculate the values of x to 4d.p.:

Your solution

$x =$ _____ or $x =$ _____

1.5907, -1.2573

Exercises

Solve the following quadratic equations by using the formula. Give answers exactly (where possible) or to 4d.p.:

- | | | |
|------------------------|-------------------------|------------------------|
| 1. $x^2 + 8x + 1 = 0$ | 2. $x^2 + 7x - 2 = 0$ | 3. $x^2 + 6x - 2 = 0$ |
| 4. $-x^2 + 3x + 1 = 0$ | 5. $-2x^2 - 3x + 1 = 0$ | 6. $2x^2 + 5x - 3 = 0$ |

- | | | | | | |
|---------------------|--------------------|--------------------|--------------------|--------------------|----------------------|
| 1. -0.1270, -7.8730 | 2. -7.2749, 0.2749 | 3. 0.3166, -6.3166 | 4. 3.3028, -0.3028 | 5. -1.7808, 0.2808 | 6. $\frac{7}{4}, -3$ |
|---------------------|--------------------|--------------------|--------------------|--------------------|----------------------|

5. Geometrical description of quadratics

We can plot a graph of the function $y = ax^2 + bx + c$ (given values of a, b and c). If the graph crosses the horizontal axis it will do so when $y = 0$, and so the x coordinates at such points are solutions of $ax^2 + bx + c = 0$. Depending on the sign of a and of the nature of the solutions there are essentially just six different types of graph that can occur. These are displayed in figure 1.

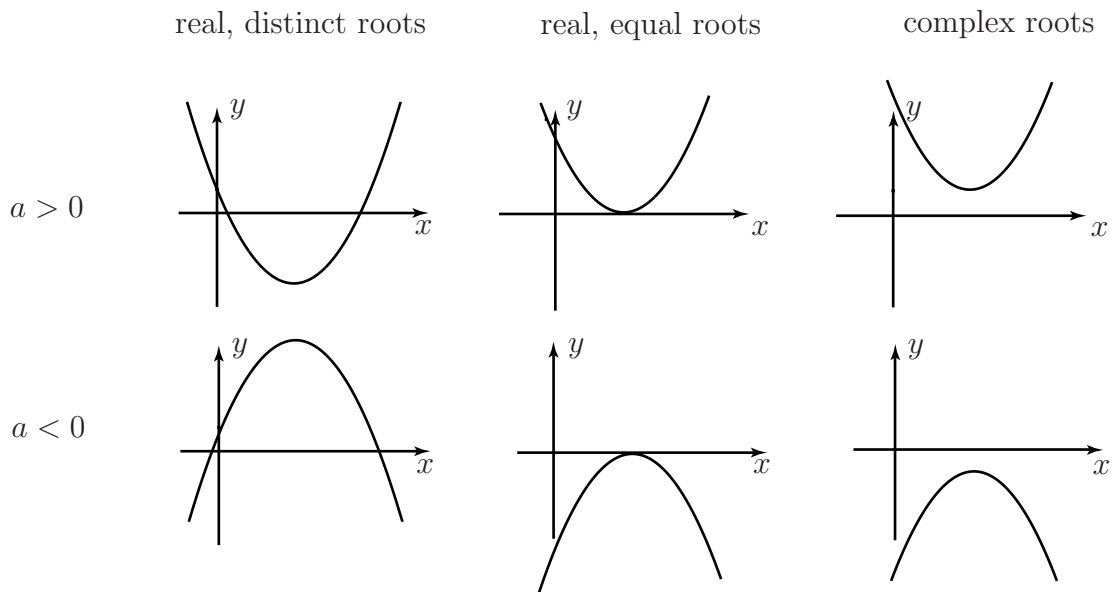


Figure 1. The possible graphs of a quadratic

Sometimes a graph of the quadratic is used to locate the solutions; however, this approach is generally inaccurate. This is illustrated in the following example.

Example Solve the equation $x^2 - 4x + 1 = 0$ by plotting a graph of the function:

$$y = x^2 - 4x + 1$$

Solution

By constructing a table of function values we can plot the graph as shown in Figure 2.

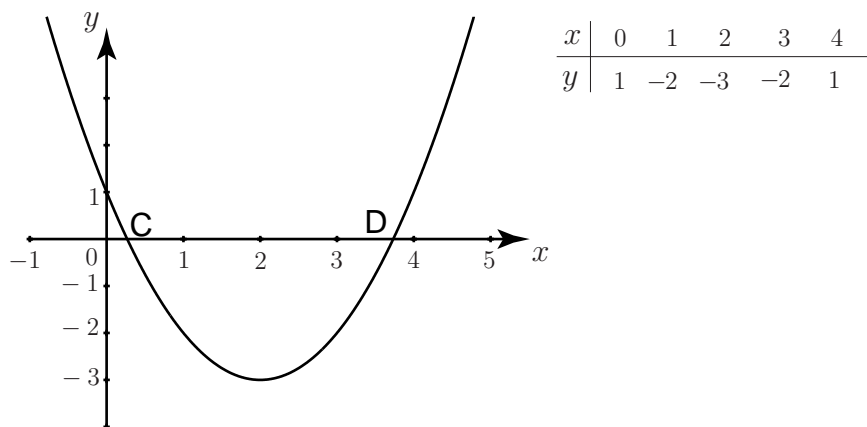


Figure 2. The graph of $y = x^2 - 4x + 1$ cuts the x axis at C and D .

The solutions of the equation $x^2 - 4x + 1 = 0$ are found by looking for points where the graph crosses the horizontal axis. The two points are approximately $x = 0.3$ and $x = 3.7$ marked C and D on the Figure.

Exercises

1. Solve the following quadratic equations:

(a) $x^2 - 9 = 0$ (b) $s^2 - 25 = 0$, (c) $3x^2 - 12 = 0$.

2. Solve the equation $x^2 - 5x + 6 = 0$.

3. Solve the equation $6s^2 + s - 15 = 0$.

4. Solve the equation $x^2 + 7x = 0$.

5. Solve the equation $2x^2 - 3x - 7 = 0$.

Answers 1(a) $x = 3, -3$, (b) $s = 5, -5$, (c) $x = 2, -2, 2, 3, 2$,
 3. $s = 3/2, -5/3$, 4. $x = 0, -7$, 5. $-2.7656, 1.2656$ (4d.p.).