

Numerical integration **31.2**



Introduction

In this Section we will present some methods that can be used to approximate integrals. Attention will be paid to how we ensure that such approximations can be guaranteed to be of a certain level of accuracy.



Prerequisites

Before starting this Section you should ...

- ① review previous material on integrals and integration



Learning Outcomes

After completing this Section you should be able to ...

- ✓ approximate certain integrals
- ✓ be able to ensure that these approximations are of some desired accuracy

1. Numerical integration

The aim in this Section is to describe numerical methods for approximating integrals of the form

$$\int_a^b f(x)dx$$

One motivation for this is in the material on probability that appears in Workbook 38. Normal distributions can be analysed by working out

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

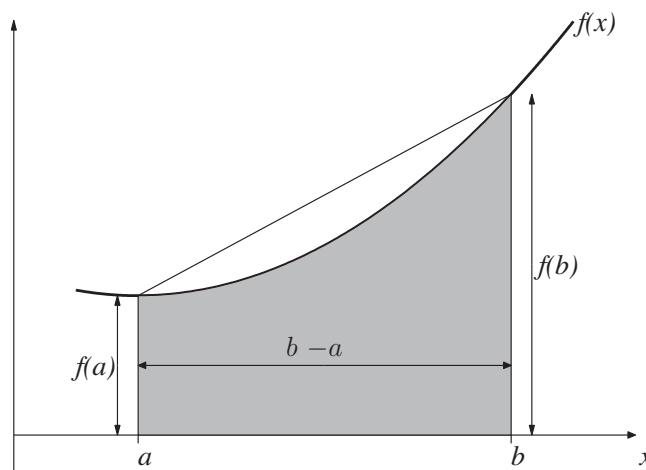
for certain values of a and b . It turns out that it is not possible, using the kinds of functions most engineers would care to know about, to write down a function with derivative equal to $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and values of the integral are approximated instead. Tables of numbers giving the value of this integral for different interval widths appeared at the end of Workbook 39, and it is known that these tables are accurate to the number of decimal places that were given. How can this be known? One aim of this Section is to give a possible answer to that question.

It is clear that, not only do we need a way of approximating integrals, but we also need a way of working out the accuracy of the approximations if we are to be sure that our tables of numbers are to be relied on.

In this Section we will address both of these points. Let us begin with a simple approximation method.

2. The simple trapezium rule

The first approximation we shall look at involves finding the area under a straight line, rather than the area under f . The following diagram shows it best.



We approximate as follows

$$\begin{aligned} \int_a^b f(x)dx &= \text{grey shaded area} \\ &\approx \text{area of the trapezium surrounding the shaded region} \\ &= \text{width of trapezium} \times \text{average height of the two sides} \\ &= \frac{1}{2}(b-a) \left(f(a) + f(b) \right) \end{aligned}$$



Key Point

The simple trapezium rule for approximating $\int_a^b f(x)dx$ is given by approximating the area under the graph of f by the area of a trapezium.

To put it another way,

$$\int_a^b f(x)dx \approx \frac{1}{2}(b-a)(f(a) + f(b))$$

Or, to put it yet another way that may prove helpful a little later on,

$$\int_a^b f(x)dx \approx \frac{1}{2} \times (\text{interval width}) \times \left(f(\text{left-hand end}) + f(\text{right-hand end}) \right)$$

Next we show some instances of implementing this method.

Example Approximate each of these integrals using the simple trapezium rule

$$(a) \int_0^{\pi/4} \sin(x)dx \quad (b) \int_1^2 e^{-x^2/2}dx \quad (c) \int_0^2 \cosh(x)dx$$

Solution

$$(a) \int_0^{\pi/4} \sin(x)dx \approx \frac{1}{2}(b-a)(\sin(a) + \sin(b)) = \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) \left(0 + \frac{1}{\sqrt{2}} \right) = 0.27768,$$

$$(b) \int_1^2 e^{-x^2/2}dx \approx \frac{1}{2}(b-a) \left(e^{-a^2/2} + e^{-b^2/2} \right) = \frac{1}{2} (2-1) \left(e^{-1/2} + e^{-2} \right) = 0.37093,$$

$$(c) \int_0^2 \cosh(x)dx \approx \frac{1}{2}(b-a) (\cosh(a) + \cosh(b)) = \frac{1}{2} (2-0) (1 + \cosh(2)) = 4.76220,$$

where all three answers are given to 5 decimal places.

It is important to note that, although we have given these integral approximations to 5 decimal places, this does not mean that they are accurate to that many places. We will deal with the accuracy of our approximations later in this Section.

Next are some exercises for you to try.



Approximate these integrals using the simple trapezium method

$$(a) \int_1^5 \sqrt{x} dx \quad (b) \int_1^2 \ln(x) dx$$

Your solution

$$(a) \int_1^5 \sqrt{x} dx \approx \frac{1}{2} (1 + \sqrt{5}) (5 - 1) = 6.47214$$

$$(b) \int_1^2 \ln(x) dx \approx \frac{1}{2} (\ln(1) + \ln(2)) (2 - 1) = 0.34657$$

The answer you obtain for this next exercise can be checked against the table of results in the Workbook concerning normal distributions.



Use the simple trapezium method to approximate $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

Your solution

We find that

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx \frac{1}{2} (1 + e^{-1/2}) \frac{1}{\sqrt{2\pi}} = 0.32046$$

to 5 decimal places.

So we have a means of approximating $\int_a^b f(x) dx$. The question remains whether or not it is a *good* approximation.

How good is it?

We define e_T , the error in the simple trapezium rule to be the difference between the actual value of the integral and our approximation to it, that is

$$e_T = \int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a) + f(b))$$

It is enough for our purposes here to omit some theory and skip straight to the result of interest. In many different text books on the subject it is shown that

$$e_T = -\frac{1}{12}(b-a)^3 f''(c)$$

where c is some number between a and b . (The principal drawback with this expression for e_T is that we do not know what c is, but we will find a way to work around that difficulty later.)

It is worth pausing to ask what meaning we can attach to this expression for e_T . There are two factors which can influence e_T :

1. If $b - a$ is small then, clearly, e_T will also be small. This seems sensible enough - if the integration interval is a small one then there is “less room” to accumulate a large error. (This observation forms part of the motivation for the composite trapezium rule discussed later in this Section.)
2. If f'' is small everywhere in $a < x < b$ then e_T will be small. This fact reflects the fact that we worked out the integral of a straight line function, instead of the integral of f . If f is a long way from being a straight line then f'' will be large and hence so will the error e_T .

We noted above that the expression for e_T is less useful than it might be because it involves the unknown quantity c . We perform a trade-off to get around this problem. The expression above gives an exact value for e_T , but we do not know enough to evaluate it. So we replace the expression with one we can evaluate, but it will not be exact. We replace $f''(c)$ with a worst case value to obtain an **upper bound** on e_T . This worst case value is the largest (positive or negative) value that $f''(x)$ achieves for $a \leq x \leq b$. This leads to

$$|e_T| \leq \max_{a \leq x \leq b} |f''(x)| \frac{(b-a)^3}{12}.$$

We summarise this in a Key Point.



Key Point

The error, $|e_T|$, in the simple trapezium approximation to $\int_a^b f(x)dx$ is bounded above by

$$\max_{a \leq x \leq b} |f''(x)| \frac{(b-a)^3}{12}$$

Example Work out the error bound (to 6 decimal places) for the simple trapezium method approximations to

$$(a) \int_0^{\pi/4} \sin(x) dx \quad (b) \int_0^2 \cosh(x) dx$$

Solution

In each case the trickiest part is working out the maximum value of $f''(x)$.

(a) Here $f(x) = \sin(x)$, therefore $f'(x) = \cos(x)$ and $f''(x) = -\sin(x)$. The function $\sin(x)$ takes values between 0 and $\frac{1}{\sqrt{2}}$ when x varies between 0 and $\pi/4$. Hence

$$e_T < \frac{1}{\sqrt{2}} \times \frac{(\pi/4)^2}{12} = 0.028548 \quad \text{to 6 decimal places.}$$

(b) If $f(x) = \cosh(x)$ then $f''(x) = \cosh(x)$ too. The maximum value of $\cosh(x)$ for x between 0 and 2 will be $\cosh(2) = 3.762196$, to 6 decimal places. Hence, in this case,

$$e_T < (3.762196) \times \frac{(2-0)^3}{12} = 2.508130 \quad \text{to 6 decimal places.}$$

(In this example we used a rounded value of $\cosh(2)$. To be on the safe side, it is best to round this number *up* to make sure that we still have an upper bound on e_T . In this case of course, rounding up is what we would naturally do, because the seventh decimal place was a 6.)



Work out the error bound (to 5 significant figures) for the simple trapezium method approximations to

$$(a) \int_1^5 \sqrt{x} dx \quad (b) \int_1^2 \ln(x) dx$$

Your solution

(a)

$$e^x > \frac{1}{3} \times \frac{1}{2} = 1.33333$$

(a) If $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. The negative power here means that f'' takes its biggest value at the left hand end and we see that $\max_{1 \leq x \leq 5} |f''(x)| = \frac{1}{4}$. Therefore

Your solution

(b)

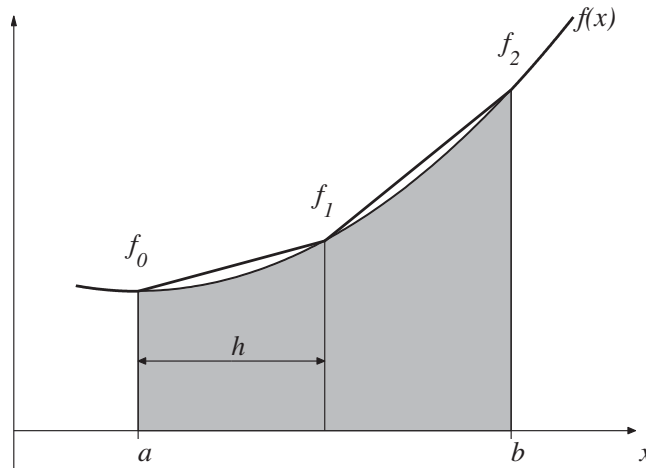
$$e^x > \frac{1}{3} \times 1 = 0.083333$$

(b) Here $f(x) = \ln(x)$ hence $f'(x) = 1/x$ and $f''(x) = -1/x^2$. It follows then that $\max_{1 \leq x \leq 2} |f''(x)| = 1$ and we conclude that

One deficiency in the simple trapezium rule is that there is nothing we can do to improve it. Having computed an error bound to measure the quality of the approximation we have no way to go back and work out a better approximation to the integral. It would be preferable if there were a parameter we could alter to tune the accuracy of the method. The following approach uses the simple trapezium method in a way that allows us to adjust the accuracy of the answer we obtain.

3. The composite trapezium rule

The general idea here is to split the interval $[a, b]$ into a sequence of N smaller subintervals of equal width $h = (b-a)/N$. Then we apply the simple trapezium rule to each of the subintervals. The first diagram below shows the case where $N = 2$ (and $\therefore h = \frac{1}{2}(b-a)$). To simplify notation later on we let $f_0 = f(a)$, $f_1 = f(a+h)$ and $f_2 = f(a+2h) = f(b)$.

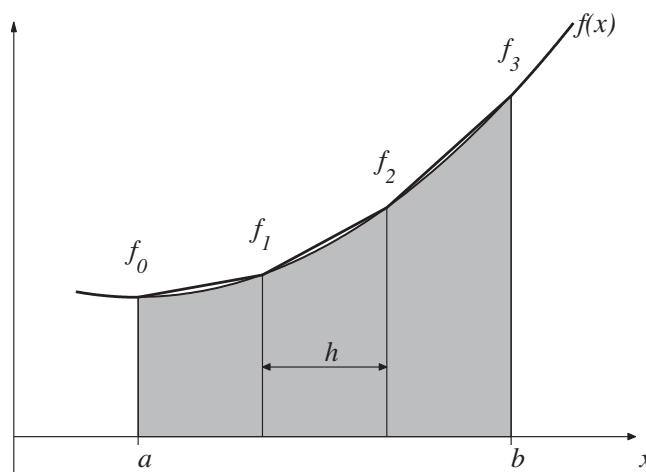


Applying the simple rule to each subinterval we get

$$\begin{aligned}\int_a^b f(x)dx &\approx (\text{area of first trapezium}) + (\text{area of second trapezium}) \\ &= \frac{1}{2}h(f_0 + f_1) + \frac{1}{2}h(f_1 + f_2) = \frac{1}{2}h(f_0 + 2f_1 + f_2)\end{aligned}$$

where we remember that the width of each of the subintervals is h , rather than the $b-a$ we had in the simple trapezium rule.

The next improvement will come from taking $N = 3$ subintervals. Here $h = \frac{1}{3}(b-a)$ is smaller than in the diagram above and we denote $f_0 = f(a)$, $f_1 = f(a+h)$, $f_2 = f(a+2h)$ and $f_3 = f(a+3h) = f(b)$. (Notice that f_1 and f_2 mean something different than they did in the $N = 2$ case.)

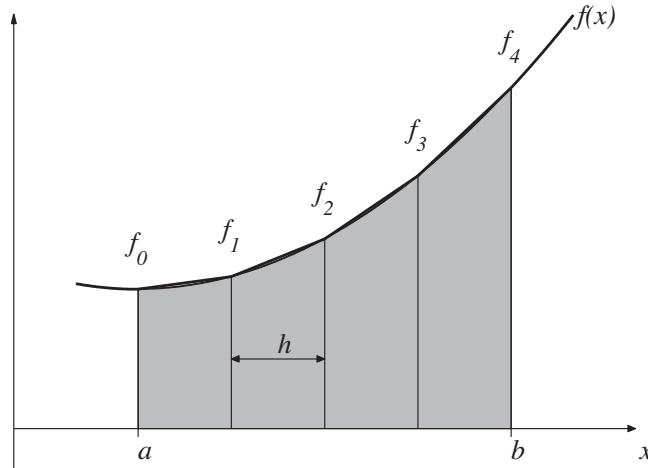


As the diagram shows, these approximations are getting closer and closer to the grey shaded

area and in this case we have

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{2}h(f_0 + f_1) + \frac{1}{2}h(f_1 + f_2) + \frac{1}{2}h(f_2 + f_3) \\ &= \frac{1}{2}h \left(f_0 + 2[f_1 + f_2] + f_3 \right).\end{aligned}$$

The pattern is probably becoming clear by now, but here is one more improvement. In the diagram below $N = 4$, $h = \frac{1}{4}(b - a)$ and we denote $f_0 = f(a)$, $f_1 = f(a + h)$, $f_2 = f(a + 2h)$, $f_3 = f(a + 3h)$ and $f_4 = f(a + 4h) = f(b)$.



This leads to

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{2}h(f_0 + f_1) + \frac{1}{2}h(f_1 + f_2) + \frac{1}{2}h(f_2 + f_3) + \frac{1}{2}h(f_3 + f_4) \\ &= \frac{1}{2}h \left(f_0 + 2[f_1 + f_2 + f_3] + f_4 \right).\end{aligned}$$

We generalise this idea into the following Key Point.



Key Point

The composite trapezium rule for approximating $\int_a^b f(x)dx$ is carried out as follows:

1. Choose N , the number of subintervals,

2. $\int_a^b f(x)dx \approx \frac{1}{2}h \left(f_0 + 2[f_1 + f_2 + \cdots + f_{N-1}] + f_N \right),$

where

$$h = \frac{b-a}{N}, \quad f_0 = f(a), \quad f_1 = f(a+h), \dots, f_n = f(a+nh), \dots,$$

and $f_N = f(a+Nh) = f(b).$

Example Using 4 subintervals in the composite trapezium rule, and working to 6 decimal places, approximate

$$\int_0^2 \cosh(x) dx$$

Solution

In this case $h = (2 - 0)/4 = 0.5$.

We require $\cosh(x)$ evaluated at 5 x -values and the results are tabulated below

x_n	$f_n = \cosh(x_n)$
0	1.000000
0.5	1.127626
1	1.543081
1.5	2.352410
2	3.762196

to 6 decimal places. It follows that

$$\begin{aligned} \int_0^2 \cosh(x) dx &\approx \frac{1}{2}h (f_0 + f_4 + 2[f_1 + f_2 + f_3]) \\ &= \frac{1}{2}(0.5) (1 + 3.762196 + 2[1.127626 + 1.543081 + 2.35241]) \\ &= 3.452107 \end{aligned}$$



Using 4 subintervals in the composite trapezium rule approximate

$$\int_1^2 \ln(x) dx$$

Your solution

$$\int_2^1 \ln(x) dx \approx \frac{2}{1} h (f_0 + f_4 + 2[f_1 + f_2 + f_3]) = \frac{2}{1} (0.25) (0 + 0.693147 + 2[0.223144 + 0.405465 + 0.559616]) = 0.383700$$

to 6 decimal places. It follows that

x_n	1	0.000000
$f_n = \ln(x_n)$	1.25	0.223144
	1.5	0.405465
	1.75	0.559616
	2	0.693147

In this case $h = (2 - 1)/4 = 0.25$. We require $\ln(x)$ evaluated at 5 x -values and the results are tabulated below

How good is it?

We can work out an upper bound on the error incurred by the composite trapezium method. Fortunately, all we have to do here is apply the method for the error in the simple rule over and over again. Let e_T^N denote the error in the composite trapezium rule with N subintervals. Then

$$\begin{aligned} |e_T^N| &\leq \max_{\text{1st subinterval}} \left| f''(x) \right| \frac{h^3}{12} + \max_{\text{2nd subinterval}} \left| f''(x) \right| \frac{h^3}{12} + \dots + \max_{\text{last subinterval}} \left| f''(x) \right| \frac{h^3}{12} \\ &= \frac{h^3}{12} \underbrace{\left(\max_{\text{1st subinterval}} \left| f''(x) \right| + \max_{\text{2nd subinterval}} \left| f''(x) \right| + \dots + \max_{\text{last subinterval}} \left| f''(x) \right| \right)}_{N \text{ terms}} \end{aligned}$$

This is all very well as a piece of theory, but it is awkward to use in practice. The process of working out the maximum value of $|f''|$ separately in each subinterval is very time-consuming. We can obtain a more user-friendly, if less accurate, error bound by replacing each term in the last bracket above with the biggest one. Hence we obtain

$$|e_T^N| \leq \frac{h^3}{12} \left(N \max_{a \leq x \leq b} \left| f''(x) \right| \right)$$

This upper bound can be rewritten on recalling that $Nh = b - a$, and we now summarise the result in a key point.



Key Point

The error, $|e_T^N|$, in the N -subinterval composite trapezium approximation to $\int_a^b f(x)dx$ is bounded above by

$$\max_{a \leq x \leq b} |f''(x)| \frac{(b-a)h^2}{12}$$

This formula can be used to decide how many sub-intervals to use to guarantee a specific accuracy.

Example The function f is known to have a second derivative with the property that

$$|f''(x)| < 12$$

for x between 0 and 4. Using the error bound given earlier in this Section determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_0^4 f(x)dx$$

can be guaranteed to be accurate to 3 decimal places. (In other words we require it to have an error in it that is less than $\frac{1}{2} \times 10^{-3}$.)

Solution

We require that

$$12 \times \frac{(b-a)h^2}{12} < 0.0005$$

that is

$$4h^2 < 0.0005.$$

This implies that $h^2 < 0.000125$ and therefore $h < 0.0111803$.

Now $N = (b-a)/h = 4/h$ and it follows that

$$N > 357.7708$$

Clearly, N must be a whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.0005 is $N = 358$.

It is worth remembering that the error bound we are using here is a pessimistic one. We effectively use the same (worst case) value for $f''(x)$ all the way through the integration interval. Odds are that fewer subintervals will give the required accuracy, but the value for N we found here will guarantee a good enough approximation.

Next are two exercises for you to try.



The function f is known to have a second derivative with the property that

$$|f''(x)| < 14$$

for x between -1 and 4 . Using the error bound given earlier in this Section determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_{-1}^4 f(x)dx$$

can be guaranteed to have an error in it that is less than 0.00001 .

Your solution

Clearly, N must be a whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.00001 is $N = 1208$.

$$N > 1207.6147$$

This implies that $h^2 < 0.00001714$ and therefore $h < 0.0041404$.
Now $N = (b - a)/h = 5/h$ and it follows that

$$70h^2 \frac{12}{12} > 0.00001$$

that is

$$14 \times \frac{(b - a)h^2}{12} > 0.00001$$

We require that



It is given that the function $e^{-x^2/2}$ has a second derivative that is never greater than 1 in absolute value.

- (a) Use this fact to determine how many subintervals are required for the composite trapezium method to deliver an approximation to

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

that is guaranteed accurate to 2 decimal places.

- (b) Find an approximation to the integral that is accurate to 2 decimal places.

Your solution

(a) We require that $\frac{1}{(b-a)h^2} \frac{\sqrt{2\pi}}{12} > 0.005$, for 2 decimal place accuracy. This means that $h^2 > 0.150398$ and therefore, since $N = 1/h$, it is necessary for $N = 3$ for the error bound to guarantee 2 decimal place accuracy.

(b) To carry out the composite trapezium rule, with $h = \frac{3}{2}$ we need to evaluate $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ at $x = 0, h, 2h, 1$. This evaluation gives

$$f(0) = f_0 = 0.39894, \quad f(h) = f_1 = 0.37738, \quad f(2h) = f_2 = 0.31945$$

and $f(1) = f_3 = 0.24197$,
 all to 5 decimal places. It follows that

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx \frac{1}{2}h(f_0 + f_3 + 2[f_1 + f_2]) = 0.33910.$$

We know from part (a) that this approximation is in error by less than the error in rounding to two decimal places.

4. Other methods for approximating integrals

Here we briefly describe other methods that you may have heard, or get to hear, about. In the end they all boil down to the same sort of thing, that is we sample the integrand f at a few points in the integration interval and then take a weighted average of all these f values. All that is needed to implement any of these methods is the list of sampling points and the weight that should be attached to each evaluation. Lists of these points and weights can be found in many books on the subject.

Simpson's rule

This is based on passing a quadratic through three equally spaced points, rather than passing a straight line through two points as we did for the simple trapezium rule. The composite Simpson's rule is given in the following Key Point.



Key Point

The composite Simpson's rule for approximating $\int_a^b f(x)dx$ is carried out as follows:

1. Choose N , the even number of subintervals,
2. (Just to be clear, let us restate that N must be an even number)
3.
$$\int_a^b f(x)dx \approx \frac{1}{3}h \left(f_0 + 4[f_1 + f_3 + f_5 + \cdots + f_{N-1}] + 2[f_2 + f_4 + f_6 + \cdots + f_{N-2}] + f_N \right)$$

where

$$h = \frac{b-a}{N}, \quad f_0 = f(a), \quad f_1 = f(a+h), \dots, f_n = f(a+nh), \dots, \\ \text{and } f_N = f(a+Nh) = f(b).$$

The formula in this Key Point is slightly more complicated than the corresponding one for composite Trapezium rule. One way of remembering the rule is to learn the pattern

$$1 \ 4 \ 2 \ 4 \ 2 \ 4 \ 2 \ \dots \ 4 \ 2 \ 4 \ 2 \ 4 \ 1$$

which show that the end point values are multiplied by 1, the values with odd-numbered subscripts are multiplied by 4 and the *interior* values with even subscripts are multiplied by 2.

Example Using 4 subintervals in the composite Simpson rule approximate

$$\int_0^2 \cosh(x) dx.$$

Solution

In this case $h = (2 - 0)/4 = 0.5$.

We require $\cosh(x)$ evaluated at 5 x -values and the results are tabulated below

x_n	$f_n = \cosh(x_n)$
0	1
0.5	1.127626
1	1.543081
1.5	2.35241
2	3.762196

It follows that

$$\begin{aligned} \int_0^2 \cosh(x) dx &\approx \frac{1}{3}h (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \\ &= \frac{1}{3}(0.5) (1 + 4 \times 1.127626 + 2 \times 1.543081 + 4 \times 2.35241 + 3.762196) \\ &= 3.628083, \end{aligned}$$

where this approximation is given to 6 decimal places.

This approximation to $\int_0^2 \cosh(x) dx$ is closer to the true value of $\sinh(2) = 3.626860$ (to 6 decimal places) than we obtained when using the composite Trapezium rule with the same number of subintervals.



Using 4 subintervals in the composite Simpson rule approximate

$$\int_1^2 \ln(x) dx.$$

Your solution

$$\begin{aligned} & \approx \frac{3}{1}h (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \\ & = \frac{3}{1}(0.25) (0 + 4 \times 0.223144 + 2 \times 0.405465 + 4 \times 0.559616 + 0.693147) \\ & = 0.386260. \end{aligned}$$

It follows that

x_n	$f^n = \ln(x_n)$
1	0.000000
1.25	0.223144
1.5	0.405465
1.75	0.559616
2	0.693147

In this case $h = (2 - 1)/4 = 0.25$. We require $\ln(x)$ evaluated at 5 x -values and the results are tabulated below

How good is it?

On page ?? of this Workbook we saw a formula for an upper bound on the error in the composite trapezium method. A corresponding result for the composite Simpson's rule also exists and is given in the following Key Point.



Key Point

The error in the N -subinterval composite Simpson approximation to $\int_a^b f(x)dx$ is bounded above by

$$\max_{a \leq x \leq b} \left| f^{(iv)}(x) \right| \frac{(b-a)h^4}{180}$$

(Here $f^{(iv)}$ is the fourth derivative of f .)

This formula can be used to decide how many sub-intervals to use to guarantee a specific accuracy.

Example The function f is known to have a fourth derivative with the property that

$$|f^{(iv)}(x)| < 5$$

for x between 1 and 5 . Determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_1^5 f(x)dx$$

incurs an error that is less than 0.005 .

Solution

We require that

$$5 \times \frac{4^4 h^4}{180} < 0.005$$

This implies that $h^4 < 0.000703$ and therefore $h < 0.162839$.

Now $N = 4/h$ and it follows that

$$N > 24.56416$$

For the composite Simpson's rule N must be an even whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.005 is $N = 26$.



The function f is known to have a fourth derivative with the property that

$$|f^{(iv)}(x)| < 12$$

for x between 2 and 6 . Determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_2^6 f(x)dx$$

incurs an error that is less than 0.0005 .

Your solution

We require that

$$12 \times \frac{180}{4^4 h^4} < 0.0005$$

This implies that $h^4 > 2.93E-05$ and therefore $h > 0.073571$.
 Now $N = 4/h$ and it follows that

$$N > 54.36942$$

N must be an *even* whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.0005 is $N = 56$.

The following exercise is similar to one that we saw earlier in this Section. Using the composite Simpson's rule we can achieve greater accuracy, for a similar amount of effort, than we managed using the composite trapezium rule.



It is given that the function $e^{-x^2/2}$ has a fourth derivative that is never greater than 3 in absolute value.

- (a) Use this fact to determine how many subintervals are required for the composite Simpson's rule to deliver an approximation to

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

that is guaranteed accurate to 4 decimal places.

- (b) Find an approximation to the integral that is accurate to 4 decimal places.

Your solution

We know from part (a) that this approximation is in error by less than the error in rounding here this approximation is given to 6 decimal places.

$$\int_1^0 \frac{1}{e^{-x^2/2}} dx \approx \frac{3}{1} h (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) = \frac{3}{1} (0.25) (0.398942 + 4 \times 0.386668 + 2 \times 0.352065 + 4 \times 0.301137 + 0.241971) = 0.341355$$

It follows that

x_n	$\frac{1}{e^{-x_n^2/2}}$
0	0.398942
0.25	0.386668
0.5	0.352065
0.75	0.301137
1	0.241971

the results are tabulated below

(b) In this case $h = (1 - 0)/4 = 0.25$. We require $\frac{\sqrt{2\pi}}{1} e^{-x^2/2}$ evaluated at five x -values and guarantee 4 decimal place accuracy.

(a) We require that $\frac{3}{(b-a)^4 h^4} \frac{\sqrt{2\pi}}{180} > 0.00005$, for 4 decimal place accuracy. This means that $h^4 > 0.00751988$ and therefore, since $N = 1/h$, it is necessary for $N = 4$ for the error bound to

Gaussian quadrature

The idea with this class of methods is to be a bit more flexible with *where* the function is evaluated. By “tuning” where the function is sampled it is possible to derive methods which have various desired properties. There are several different versions of Gaussian quadrature, each having different designed properties. For example there is (in alphabetical order) Gauss Chebychev, Gauss Hermite, Gauss Laguerre, Gauss Legendre, Gauss Lobatto, the list goes on and is increased somewhat by the various ways in which Chebychev’s name has been translated into English.

Exercises

1. Using 4 subintervals in the composite trapezium rule approximate

$$\int_1^5 \sqrt{x} dx.$$

2. The function f is known to have a second derivative with the property that

$$|f''(x)| < 12$$

for x between 2 and 3. Using the error bound given earlier in this Section determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_2^3 f(x) dx$$

can be guaranteed to have an error in it that is less than 0.001.

3. Using 4 subintervals in the composite Simpson rule approximate

$$\int_1^5 \sqrt{x} dx.$$

4. The function f is known to have a fourth derivative with the property that

$$|f^{(iv)}(x)| < 6$$

for x between -1 and 5 . Determine how many subintervals are required so that the composite trapezium rule used to approximate

$$\int_{-1}^5 f(x) dx$$

incurs an error that is less than 0.001 .

Answers

1. In this case $h = (5 - 1)/4 = 1$. We require \sqrt{x} evaluated at 5 x -values and the results are tabulated below

x_n	1
$f_n = \sqrt{x_n}$	1
	2
	1.414214
	3
	1.732051
	4
	2.000000
	5
	2.236068

It follows that

$$\int_5^1 \sqrt{x} dx \approx \frac{1}{2} h (f_0 + f_4 + 2[f_1 + f_2 + f_3])$$

$$= \frac{1}{2} (1) (1 + 2.236068 + 2[1.414214 + 1.732051 + 2]) = 6.764298.$$

2. We require that $12 \times \frac{(b-a)h^2}{12} > 0.001$. This implies that $h < 0.0316228$. Now $N = (b-a)/h = 1/h$ and it follows that

$$N > 31.6228$$

Clearly, N must be a whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.001 is $N = 32$.

3. In this case $h = (5 - 1)/4 = 1$.

We require \sqrt{x} evaluated at 5 x -values and the results are as tabulated in the solution to Exercise 1. It follows that

$$\int_5^1 \sqrt{x} dx \approx \frac{3}{1} h (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) = \frac{3}{1} (1) (1 + 4 \times 1.414214 + 2 \times 1.732051 + 4 \times 2.000000 + 2.236068) = 6.785675.$$

4. We require that $6 \times \frac{6^4 h^4}{180} > 0.001$. This implies that $h^4 > 2.31481 \times 10^{-5}$ and therefore $h > 0.069363$. Now $N = 6/h$ and it follows that $N > 86.50121$. We know that N must be an *even* whole number and we conclude that the smallest number of subintervals which *guarantees* an error smaller than 0.001 is $N = 88$.