Numerical differentiation

31.3

Introduction

In this Section we will look at ways in which derivatives of a function may be approximated numerically.

Prerequisites

Before starting this Section you should . . .

① review previous material concerning differentiation

Learning Outcomes

After completing this Section you should be able to . . .

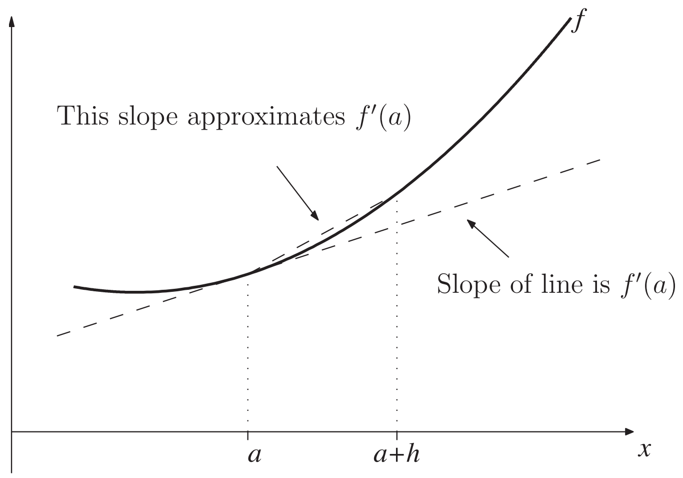
✓ obtain numerical approximations to the first and second derivatives of certain functions

1. Numerical differentiation

This Section deals with ways of numerically approximating derivatives of functions. One reason for dealing with this now is that we will use it briefly in the next Section. But as we shall see in these next few pages, the technique is useful in itself.

2. First derivatives

Our aim is to approximate the slope of a curve f at a particular point $x = a$ in terms of $f(a)$ and the value of f at a nearby point where $x = a + h$. The shorter broken line in the following diagram may be thought of as giving a decent approximation to the required slope (shown by the longer broken line), if h is small enough.



So we might approximate

$$f'(a) \approx \text{slope of short broken line} = \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} = \frac{f(a+h) - f(a)}{h}.$$

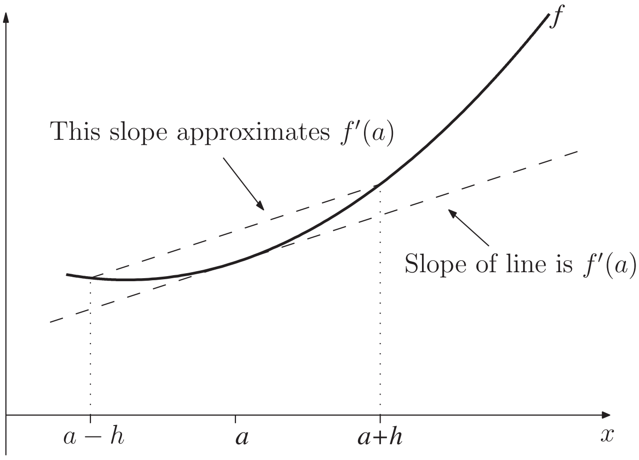
This is called a **one-sided** or **forward** difference approximation to the derivative of f.

A second version of this arises on considering a point to the left of a, rather than to the right as we did above. In this case we obtain the approximation

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}$$

This is another **one-sided** difference, called a **backward** difference, approximation to $f'(a)$.

A third method for approximating the first derivative of f can be seen in the next diagram.



Here we approximate as follows

$$f'(a) \approx \text{slope of short broken line} = \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} = \frac{f(x+h)-f(x-h)}{2h}$$

This is called a **central difference** approximation to $f'(a)$.

Key Point

Three approximations to the derivative $f'(a)$ are

1. the one sided (forward) difference $\quad \dfrac{f(a+h)-f(a)}{h}$

2. the one sided (backward) difference $\quad \dfrac{f(a)-f(a-h)}{h}$

3. the central difference $\quad \dfrac{f(a+h)-f(a-h)}{2h}$

In practice, the central difference formula is the most accurate.

These first, rather artificial, examples will help fix our ideas before we move on to more realistic applications.

Example Use a forward difference, and the values of h shown, to approximate the derivative of $\cos(x)$ at $x = \pi/3$.

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$

Work to 8 decimal places throughout.

Solution

(a) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.41104381 - 0.5}{0.1} = -0.88956192$

(b) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49131489 - 0.5}{0.01} = -0.86851095$

(c) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49913372 - 0.5}{0.001} = -0.86627526$

(d) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49991339 - 0.5}{0.0001} = -0.86605040$

One advantage of doing a simple example first is that we can compare these approximations with the exact value which is

$$f'(a) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2} = -0.86602540$$

to 8 decimal places. Notice that second to fourth approximations in the example above have one extra accurate decimal place when compared with the previous approximation.

Example Use a central difference, and the value of h shown, to approximate the derivative of $\cos(x)$ at $x = \pi/3$.

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$

Work to 8 decimal places throughout.

Solution

(a) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a-h)}{2h} = \dfrac{0.41104381 - 0.58396036}{0.2} = -0.86458275$

(b) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a-h)}{2h} = \dfrac{0.49131489 - 0.50863511}{0.02} = -0.86601097$

(c) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a-h)}{2h} = \dfrac{0.49913372 - 0.50086578}{0.002} = -0.86602526$

(d) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a-h)}{2h} = \dfrac{0.49991339 - 0.50008660}{0.0002} = -0.86602540$

This time each successive approximation has *two* extra accurate decimal places. This pattern continues in the following exercise.

Let $f(x) = \ln(x)$ and $a = 3$. Using both a forward and a central difference, and working to 8 decimal places, approximate $f'(a)$ using $h = 0.1$ and $h = 0.01$. (Note that this is another example where we can work out the exact answer, which in this case is $\frac{1}{3}$.)

Your solution

Using the forward difference we find, for $h = 0.1$

$$f'(a) \approx \frac{\ln(a+h) - \ln(a)}{h} = \frac{1.13140211 - 1.09861229}{0.1} = 0.32789823$$

and for $h = 0.01$ we obtain

$$f'(a) \approx \frac{\ln(a+h) - \ln(a)}{h} = \frac{1.10194008 - 1.09861229}{0.01} = 0.33277901$$

Using central differences the two approximations to $f'(a)$ are

$$f'(a) \approx \frac{\ln(a+h) - \ln(a-h)}{2h} = \frac{1.13140211 - 1.06471074}{0.2} = 0.33345687$$

and

$$f'(a) \approx \frac{\ln(a+h) - \ln(a-h)}{2h} = \frac{1.10194008 - 1.09527339}{0.02} = 0.33333457$$

There is clearly little point in studying this technique if all we ever do is approximate quantities we could find exactly in another way. The following example is one in which this so-called **differencing** is the best approach.

Example The distance x of a runner from a fixed point is measured (in metres) at intervals of half a second. The data obtained is

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use central differences to approximate the runner's velocity at times $t = 0.5$s and $t = 1.25$s.

Solution

Our aim here is to approximate $x'(t)$. The choice of h is dictated by the available data. Using data with $t = 0.5$s at its centre we obtain

$$x'(0.5) \approx \frac{x(1.0) - x(0.0)}{2 \times 0.5} = 6.80\text{m/s}.$$

Data centred at $t = 1.25$s gives us the approximation

$$x'(1.25) \approx \frac{x(1.5) - x(1.0)}{2 \times 0.25} = 6.20\text{m/s}.$$

Note the value of h used.

The velocity v (in m/s) of a rocket measured at half second intervals is

t	0.0	0.5	1.0	1.5	2.0
v	0.000	11.860	26.335	41.075	59.051

Use central differences to approximate the acceleration of the rocket at times $t = 1.0$s and $t = 1.75$s.

Your solution

Using data with $t = 1.0$s at its centre we obtain

$$v'(1.0) \approx \frac{v(1.5) - v(0.5)}{1.0} = 29.215\text{m/s}^2.$$

Data centred at $t = 1.75$s gives us the approximation

$$v'(1.75) \approx \frac{v(2.0) - v(1.5)}{0.5} = 35.952\text{m/s}^2.$$

3. Second derivatives

An approach which has been found to work well for second derivatives involves applying the notion of a central difference three times. We begin with

$$f''(a) \approx \frac{f'(a+\frac{1}{2}h) - f'(a-\frac{1}{2}h)}{h}.$$

Next we approximate the two derivatives in the numerator of this expression using central differences as follows:

$$f'(a+\tfrac{1}{2}h) \approx \frac{f'(a+h) - f'(a)}{h} \qquad \text{and} \qquad f'(a-\tfrac{1}{2}h) \approx \frac{f'(a) - f'(a-h)}{h}.$$

Combining these three results gives

$$\begin{aligned} f''(a) &\approx \frac{f'(a+\frac{1}{2}h) - f'(a-\frac{1}{2}h)}{h} \\ &\approx \frac{1}{h}\left(\left(\frac{f'(a+h) - f'(a)}{h}\right) - \left(\frac{f'(a) - f'(a-h)}{h}\right)\right) \\ &= \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \end{aligned}$$

Key Point

A central difference approximation to the second derivative $f''(a)$ is

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Example The distance x of a runner from a fixed point is measured (in metres) at intervals of half a second. The data obtained is

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use a central difference to approximate the runner's acceleration at time $t =$ 1.5s.

Solution

Our aim here is to approximate $x''(t)$.
Using data with $t = 1.5$s at its centre we obtain

$$x''(1.5) \approx \frac{x(2.0) - 2x(1.5) + x(1.0)}{0.5^2} = -3.40\text{m/s}^2,$$

from which we see that the runner is slowing down.

Exercises

1. Let $f(x) = \cosh(x)$ and $a = 2$. Let $h = 0.01$ and approximate $f'(a)$ using forward, backward and central differences. Work to 8 decimal places and compare your answers with the exact result, which is $\sinh(2)$.

2. The distance x, measured in metres, of a downhill skier from a fixed point is measured at intervals of 0.25 s. The data gathered is

t	0	0.25	0.5	0.75	1	1.25	1.5
x	0	4.3	10.2	17.2	26.2	33.1	39.1

Use a central difference to approximate the skier's velocity and acceleration at the times t =0.25s, 0.75s and 1.25s. Give your answers to 1 decimal place.

Answers

1. Forward: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a)}{h} = \dfrac{3.79865301 - 3.76219569}{0.01} = 3.64573199$

   Backward: $f'(a) \approx \dfrac{\cosh(a) - \cosh(a-h)}{h} = \dfrac{3.76219569 - 3.72611459}{0.01} = 3.60810972$

   Central: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a-h)}{2h} = \dfrac{3.79865301 - 3.72611459}{0.02} = 3.62692086$

   The exact result is $\sinh(2) = 3.62686041$.

2. Velocities at the given times approximated by a central difference are 20.4m/s, 32.0m/s and 25.8m/s.
   Accelerations at these times are approximated to be 0.4m/s^2, 0.3m/s^2 and -0.1m/s^2.