Elementary Probability

35.2

Introduction

Probability is about the study of uncertainty. Engineers are expected to design and produce systems which are both useful and reliable. Essentially we are dealing with situations where 'chance' is at work and probability theory gives us the theoretical underpinning necessary for a full understanding of any experimental results we observe in practice. Probability theory also gives us the tools to set up mathematical models of systems and processes which are affected by random occurrences or 'chance'. In fact the study of probability enables engineers to discuss the reliability of the processes they use and the systems they produce in terms that other engineers, scientists and designers can understand. It is worth noting that 'chance' is taken to be responsible for variations in simple manufactured products such as screws, bolts, and light bulbs as well as complex products such as cars, ships and aircraft. In each of these products, small chance variations in raw materials and production processes may have a substantial effect on a product.

Prerequisites

Before starting this Section you should ...

① understand the ideas of sets and subsets. (Section 35.1)

Learning Outcomes

After completing this Section you should be able to ...

✓ explain the terms 'random experiment'and 'event'

✓ understand and calculate the probability of an event occurring

✓ calculate the probability that an event does not occur

1. Introductory Probability

Probability as an informal idea is something you will have been familiar with for a long time. In conversation with friends, you must have used sentences such as

- ‘It might start raining soon’
- ‘I might be lucky and pass all my examinations’
- ‘It is very unlikely that my team will not win the Premiership this year’
- ‘Getting a good degree will improve my chances of getting a good job’

Essentially, when you are talking about whether some event is likely to happen, you are using the concept of probability. In reality, we need to agree on some terminology so that misunderstanding may be avoided.

Terminology

To start with there are four terms – experiment, outcome, event and sample space – that need formal definition. There will, of course, be others as you progress through this Workbook.

1. **Experiment:** - an activity with an observable result, or set of results, for example

   (a) tossing a coin, the result being a head or a tail

   (b) testing a component, the result being a defective or non-defective component

   (c) maximum speed testing of standard production cars;

   (d) testing to destruction armour plating intended for use on tanks.

   Some of the experiments outlined above have a very limited set of results (tossing a coin) while others (destruction testing) may give a widely variable set of results. Also it is worth noting that destruction testing is not appropriate for all products. Companies manufacturing say trucks or explosives could not possibly test to destruction on a large scale - they would have little or nothing left to market!

2. **Outcome** - an outcome is simply an observable result of an experiment, for example

   (a) tossing a coin, the possible outcomes are a head or a tail

   (b) testing a component, the outcome being a defective or non-defective component

   (c) maximum speed testing of standard production cars, the outcomes being a set of numbers representing the maximum speed of a set of vehicles

   (d) testing to destruction armour plating intended for use on tanks, the outcomes might be (for example) the number of direct hits sustained before destruction.

3. **Event** - this is just an outcome or set of outcomes to an experiment of interest to the experimenter.

4. **Sample Space** - a sample space is the set of all possible outcomes of an experiment.

Everyday examples include games of chance.

Example Obtain the sample space of the experiment *throwing a single coin.*

Solution

Consider the experiment of throwing a coin which can land heads up (H) or tails up (T). We list the outcomes as a set $\{H,\ T\}$ − the order being unimportant. $\{H, T\}$ is the sample space. Clearly, on any particular throw of a coin, heads or tails are equally likely to occur. We say that, for a **fair** coin, H and T are **equally-likely outcomes**.

If the sample space can be written in the form of a list (possibly infinite) then it is called a **discrete sample space**. If this is not possible then it is called a **continuous sample space** (e.g. position where shells land in a tank battle).

List the equally likely outcomes to the experiments:
(a) throwing a fair die with six faces (b) throwing three fair coins.

Your solution

(a) $\{1, 2, 3, 4, 5, 6\}$ (b) $\{TTT,\ TTH,\ THT,\ THH,\ HTT,\ HTH,\ HHT,\ HHH\}$

For the following list of experiments, write out (if possible) a suitable sample space. If you cannot write out a suitable sample space, describe one in words.

(a) Test a light switch

(b) Count the daily traffic accidents in Loughborough involving cyclists

(c) Measure the tensile strength of small gauge steel wire

(d) Test the maximum current carrying capacity of household mains cabling

(e) Test the number of on-off switchings that a new type of fluorescent tube will cope with before failure

(f) Pressure test an underwater TV camera.

Your solution

Sample spaces might be

(a) {works, fails}

(b) {0,1, 2, 3, . . . }, hopefully a small upper limit!

(c) {Suitable continuous range (0 →) depending on the wire}

(d) {Suitable continuous range (0 →) depending on the type of cable}

(e) {0,1, 2, 3, ... }, hopefully a high upper limit!

(f) {Suitable continuous range (0 →) }

Example A car manufacturer offers certain options on its family cars. Customers may order:

(a) either automatic gearboxes or manual gearboxes

(b) either sunroof or air-conditioning

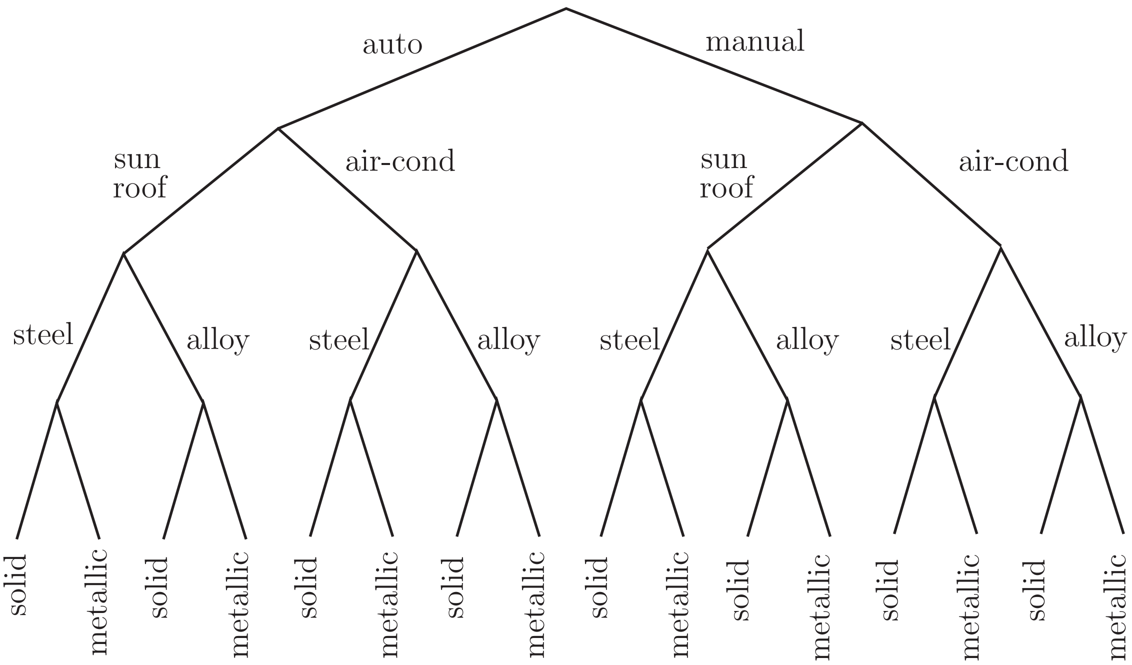
(c) either steel wheels or allow wheels

(d) either solid colour paint or metallic paint

Find the number of outcomes in the sample space of options that it is possible to order and represent them using a suitable diagram.

Solution

A suitable diagram is shown below. The diagram makes it easy to find the number of outcomes simply by counting. It also points the way to a formula for calculating the number of outcomes.



In this case there is a total of 16 outcomes in the sample space of options. Note that in each case the customer makes two choices. This implies that there are

$$2 \times 2 \times 2 \times 2 = 16$$

options in total. Diagrams such as the one above are called *tree diagrams*.

Events

As we have already noted a collection of some or all of the outcomes of an experiment is called an **event**. So an event is a **subset** of the sample space. For example, if we throw a die then the sample space is $\{1, 2, 3, 4, 5, 6\}$ and two possible events are

(a) a score of 3 or more, represented by the set: $\{3, 4, 5, 6\}$

(b) a score which is even, represented by the set: $\{2, 4, 6\}$.

Example Two coins are thrown. List the ordered outcomes for the event when just one tail is obtained.

Solution

$\{H, T\}$, $\{T, H\}$

Three coins are thrown. List the ordered outcomes which belong to each of the following events.

(a) two tails are obtained (b) at least two tails are obtained
(c) at most two tails are obtained

State the relationship between (a) and (b) and between (a) and (c).

Your solution

(a) $\{TTH, THT, HTT\}$ (b) $\{TTT, TTH, THT, HTT\}$

(a) is a subset of (b).

(c) $\{TTH, THT, HTT, THH, HTH, HHT, HHH\}$

(a) is a subset of (c).

A new type of paint to be used in the manufacture of garden equipment is tested for impact shock resistance to damage and scratch resistance to damage. The results (50 samples) are as follows

		Shock Resistance	
		Good	**Poor**
Scratch Resistance	**Good**	20	15
	Poor	12	3

If A is the event {High Shock Resistance} and B is the event {High Scratch Resistance}, describe the following events and determine the number of samples in each event.

(a) $A \cup B$ (b) $A \cap B$ (c) A' (d) B'

Your solution

The event $A \cup B$ consists of those samples which have either good shock and good scratch resistance (or both). $n(A \cup B) = 47$.

The event $A \cap B$ consists of those samples which have both good shock and good scratch resistance. $n(A \cap B) = 20$

The event A' consists of those samples which do not have good shock resistance. $n(A') = 18$.

The event B' consists of those samples which do not have good scratch resistance. $n(B') = 15$.

Complement

We have met the complement before (Section 4.1) in relation to sets. We consider it again here in relation to sample spaces and events. The **complement** of an event is the set of outcomes which are not members of the event.
For example, the experiment of throwing a 6-faced die has sample space $S = \{1, 2, 3, 4, 5, 6\}$.

The event "score of 3 or more is obtained" is the set $\{3, 4, 5, 6\}$.

The complement of this event is $\{1, 2\}$ which can be described in words as "score of 3 or more is **not** obtained" or "score of 1 or 2 is obtained".

The event: "even score is obtained" is the set $\{2, 4, 6\}$.

The complement of this event is $\{1, 3, 5\}$ or, in words "odd score is obtained".

In the last guided exercise the complement of the event (a) is $\{TTT,\ THH,\ HTH,\ HHT,\ HHH\}$, for event (b) the complement is $\{THH,\ HTH,\ HHT,\ HHH\}$ and for event (c) it is $\{TTT\}$.

State, in words, what are the complements of each of the following events in relation to the experiment of throwing three coins (avoid using the word *not*):
(a) two heads are obtained (b) at least two heads are obtained (c) at most two heads are obtained.

Your solution

(a) no heads, one head or three heads (b) no heads or one head (c) three heads

Notation: It is customary to use a capital letter to denote an event. For example, $A = \{$two heads are thrown$\}$. The complementary event is denoted A'.
Hence, in the case where $A = \{$at least two heads are thrown$\}$ A' is the event $\{$fewer than two heads are thrown$\}$.

2. Definitions of Probability

Relative Frequency applied to Probability

Consider the experiment of throwing a single coin many times.

Suppose we throw a coin 10 times and obtain six heads and four tails; does this suggest that the coin is biased? What about the case when we obtain 9 heads and 1 tail?
We conducted an experiment in which a coin was thrown 100 times and the result recorded each time as 1 if a head appeared face up and 0 if a tail appeared. In Figure 1 we have plotted the average score $\frac{r}{n}$, where r is the number of heads and n is the total number of throws, against n for $n = 10, 20, \ldots, 100$. The quantity $\frac{r}{n}$ is called the **relative frequency** of heads.

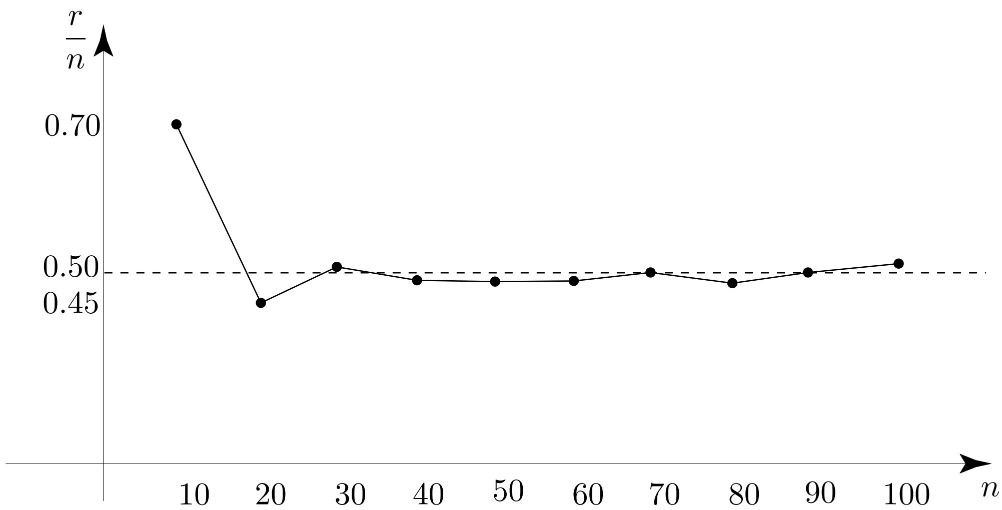


Figure 1

As n increases the relative frequency settles down near the value $\frac{1}{2}$. This is an experimental estimate of the likelihood of throwing a head with this particular coin. Note that when $n = 50$ this estimate was 0.49 and when $n = 100$ this estimate was 0.51. When we repeated the whole experiment again, the value of $\frac{r}{n}$ when $n = 100$ was 0.46. Hence the use of the word **estimate**. Normally, as the number of trials is increased the estimate tends to settle down but this is not certain to occur.
Theoretically, the likelihood of obtaining a head when a fair coin is thrown is $\frac{1}{2}$. Experimentally, we **expect** the relative frequency to approach $\frac{1}{2}$ as n increases.

The Principle of Equally Likely Outcomes

One definition of probability is based on the so-called Principle of Equally Likely Outcomes. In order to understand this principle you first of all need to consider a sample space which consists of simple outcomes. (By *simple* outcome is meant an outcome which is not made up of more than one outcome).

(a) Tossing a coin: the sample space S is, using an obvious notation:

$S = \{H, T\}$

H and T are the two simple outcomes and are equally likely to occur.

(b) Rolling a die: a sample space is:

$S = \{1, 2, 3, 4, 5, 6\}$

Each number from 1 to 6 is a simple outcome and is equally likely to occur.

(c) Tossing three coins: a sample space comprised of simple outcomes is:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Each of the outcomes stated is equally likely to occur.

(d) Tossing three coins, another sample space, not comprised of simple outcomes, is S={ 3 Heads, 2 Heads, 1 Head, 0 Heads}. These outcomes are **not** equally likely.

The sample spaces above all consist of a set of elementary or simple outcomes, that is, no outcome is a combination of other outcomes.

Key Point

The Principle of Equally Likely Outcomes states that each elementary or simple outcome is equally likely to occur. This principle enables us to deduce the probabilities that simple events (and hence more complicated events which are combinations of simple events) occur.

Referring to the examples above we may immediately deduce that

(a) $\mathrm{P}\{H\} = \mathrm{P}\{T\} = \frac{1}{2}$

(b) $\mathrm{P}\{1\} = \mathrm{P}\{2\} = \mathrm{P}\{3\} = \mathrm{P}\{4\} = \mathrm{P}\{5\} = \mathrm{P}\{6\} = \frac{1}{6}$

(c) $\mathrm{P}\{HHH\} = \mathrm{P}\{HHT\} = \mathrm{P}\{HTH\} = \mathrm{P}\{HTT\} =$

$\mathrm{P}\{THH\} = \mathrm{P}\{THT\} = \mathrm{P}\{TTH\} = \mathrm{P}\{TTT\} = \frac{1}{8}$

Definition

We can now define probability using the Principle of Equally Likely Outcomes as follows:
If a sample space S consists of n simple outcomes which are equally likely and an event A consists of m of those simple outcomes, then

$$\mathrm{P}(A) = \frac{m}{n} = \frac{\text{the number of simple outcomes in } A}{\text{the number of simple outcomes in S}}$$

It follows from this definition that $0 \leq P(A) \leq 1$.

- If $P(A) = 1$ we say that the event A is **certain** because A is identical to S.
- If $P(A) = 0$ we say that the event A is **impossible** because A is empty.

The set with no outcomes in it is called the **empty set** and written $\emptyset$; therefore $P(\emptyset) = 0$.

For each of the following events $A,\ B,\ C$, list and count the number of outcomes it contains and hence calculate the probability of $A,\ B$ or C occurring.

(a) $A =$ "throwing 3 or higher with one die",

(b) $B =$ "throwing exactly two heads with three coins",

(c) $C =$ "throwing a total score of 14 with two dice".

Your solution

(a) There are six possible equally likely outcomes of the experiment and four of them, $\{3, 4, 5, 6\}$, are in the event A; hence $P(A) = \frac{4}{6} = \frac{2}{3}$.

(b) There are eight equally likely outcomes of which three, $\{HHT, HTH, THH\}$ are in B; hence $P(B) = \frac{3}{8}$.

(c) It is impossible to throw a total higher than 12 so that $C = \emptyset$ and $P(C) = 0$.

Not surprisingly, the probabilities of an event A and its complement are related. The probability of the event A' is easily found from the identity

$$\frac{\text{number of outcomes in } A}{\text{total number of outcomes}} + \frac{\text{number of outcomes not in } A}{\text{total number of outcomes}} \equiv 1$$

so that $P(A) + P(A') = 1$

Key Point

The **complement rule**

$$P(A') = 1 - P(A)$$

in words:

the probability of the complement of A occurring
is equal to 1 minus the probability of A occurring.

For the events in (a) and (b) of the previous exercise find $P(A')$ and $P(B')$. Describe in words what A' and of B' are in this case.

Your solution

(a) $P(A) = \frac{2}{3}$ so that $P(A') = \frac{1}{3}$.

A' is the event of throwing a score of less than 3 on one die.

(b) $P(B) = \frac{3}{8}$ so that $P(B') = \frac{5}{8}$

B' is the event of throwing no heads, exactly one head or exactly three heads with three coins.

The use of the event A' can sometimes simplify the calculation of the probability $P(A)$. For example, suppose that two dice are thrown and we require the probability of the event

A: that we obtain a score of at least four.

There are many combinations that produce a score of at least four; however there are only 3 combinations that produce a score of two or three which is the complementary event to the one of interest. The event $A' = \{(1,1),\ (1,2),\ (2,1)\}$ (where we use an obvious notation of stating the score on the first die followed by the score on the second die) is the complement of A. Now $P(A') = \frac{3}{36}$ since there are 6×6 possible combinations in throwing two dice. Thus

$$P(A) = 1 - P(A') = 1 - \frac{3}{36} = \frac{33}{36} = \frac{11}{12}.$$

Find the probability of obtaining a total score of at least five when three dice are thrown. Hint: identify A and A', then calculate $P(A')$, then $P(A)$.

Your solution

There are $6 \times 6 \times 6 = 216$ possible outcomes. If, for example, $(1, 3, 6)$ denotes the scores of 1 on die one, 3 on die two and 6 on die three and if A is the event 'a total score of five or more' then A' is the event 'a total score of less than 5' i.e.

$A' = \{(1, 1, 1),\ (2, 1, 1),\ (1, 2, 1),\ (1, 1, 2)\}$

There are four outcomes in A' and hence $P(A') = \frac{4}{216}$ so that $P(A) = \frac{212}{216} = \frac{53}{54}$.

Exercises

1. For each of the following experiments, state whether the variable is discrete or continuous. In each case state the sample space.

   (a) The number of defective items in a batch of twenty is noted.

   (b) The weight, in kg., of lubricating oil drained from a machine is determined using a spring balance.

   (c) The natural logarithm of the weight, in kg., according to a spring balance, of lubricating oil drained from a machine, is noted.

2. An experiment consists of throwing two four-faced die with faces labelled 1,2,3,4.

   (a) Write down the sample space of this experiment.

   (b) If A is the event 'total score is at least 4' list the outcomes belonging to A'.

   (c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?

   (d) What is the probability that a double: i.e. $(1,1),(2,2),(3,3),(4,4)$ will *not* be thrown?

   (e) What is the probability that a double is not thrown and the score is less than 6?

3. A **lot** consists of 10 good articles, 4 articles with minor defects and 2 with major defects. One article is chosen at random. Find the probability that:

   (a) it has no defects,

   (b) it has no major defects,

   (c) it is either good or has major defects.

4. Propeller shafts for marine applications are inspected to ensure that they satisfy both diameter requirements and surface finish requirements. The results of 400 inspections are as follows:

		Diameter Requirements	
		Good	**Poor**
Surface Finish	**Good**	200	50
	Poor	80	70

   (a) What is the probability that a shaft selected at random satisfies the surface finish requirements?

   (b) What is the probability that a shaft selected at random satisfies both diameter and surface finish requirements?

   (c) What is the probability that a shaft selected at random satisfies either the diameter or the surface finish requirements?

   (d) What is the probability that a shaft selected at random satisfies neither the diameter nor the surface finish requirements?

Answers

1. (a) The variable is discrete. The sample space is $\{1, 2, \dots, 20\}$.
   (b) The variable is continuous. The sample space is the set of real numbers x such that $0 \leq x < \infty$.
   (c) The variable is continuous. The sample space is the set of real numbers x such that $-\infty < x < \infty$.

2. (a) $S = \{(1,1), (1,2), (1,3), (1,4)$

   $(2,1), (2,2), (2,3), (2,4)$

   $(3,1), (3,2), (3,3), (3,4)$

   $(4,1), (4,2), (4,3), (4,4)\}$

   (b) $A' = \{(1,1), (1,2), (2,1)\}$
   (c) The outcomes in the event are $\{(2,4), (3,3), (3,4), (4,2), (4,3), (4,4)\}$ so the probability of this event occurring is $\frac{6}{16} = \frac{3}{8}$. The probability of the complement event is $1 - \dfrac{3}{8} = \dfrac{5}{8}$.
   (d) The probability of a double occurring is $\dfrac{4}{16}$ so the probability of the complement (i.e, double not thrown) is $1 - \dfrac{4}{16} = \dfrac{3}{4}$.
   (e) Here, consider the sample space in (a). If the doubles and those outcomes with a score greater than 6 are removed we have left the event :

   $$\{(1,2), (1,3), (1,4), (2,1), (2,3), (3,1), (3,2), (4,1)\}.$$

   Hence the probability of this event occurring is $\dfrac{8}{16} = \dfrac{1}{2}$.

3. Let G be the event 'article is good' , M_n be the event 'article has minor defect' and M_j be the event 'article has major defect'

   (a) Here we require $P(G)$. Obviously $P(G) = \dfrac{10}{16} = \dfrac{5}{8}$
   (b) We require $\mathrm{P}(M_j') = 1 - \mathrm{P}(M_j) = 1 - \dfrac{2}{16} = \dfrac{7}{8}$
   (c) The event we require is the complement of the event M_n.

   Since $\mathrm{P}(M_n) = \dfrac{4}{16} = \dfrac{1}{4}$ we have $\mathrm{P}(M_n') = \mathrm{P}(G \text{ or } M_j) = 1 - \frac{1}{4} = \frac{3}{4}$.

   Equivalently $\mathrm{P}(G) + \mathrm{P}(M_n) = \dfrac{10}{16} + \dfrac{2}{16} = \dfrac{12}{16} = \dfrac{3}{4}$

4. (a) $\dfrac{250}{400} = 0.625$ (b) $\dfrac{200}{400} = 0.5$ (c) $\dfrac{330}{400} = 0.825$ (d) $\dfrac{70}{400} = 0.175$