Addition and Multiplication Laws of Probability

35.3

Introduction

When we require the probability of two events occurring simultaneously or the probability of one or the other or both of two events occurring then we need probability laws to carry out the calculations. For example, if a traffic management engineer looking at accident rates wishes to know the probability that cyclists and motorcyclists are injured during a particular period in a city, he or she must take into account the fact that a cyclist and a motorcyclist might collide. (Both events can happen simultaneously.)

Prerequisites

Before starting this Section you should . . .

① understand the ideas of sets and subsets. (Section 35.1)

② understand the concepts of probability and events (Section 35.2)

Learning Outcomes

After completing this Section you should be able to . . .

✓ state and use the addition law of probability

✓ understand the term 'independent events'

✓ state and use the multiplication law of probability

✓ understand the idea of conditional probability

1. The Addition Law

As we have already noted the sample space S is the set of all possible outcomes of a given experiment. Certain events A and B are subsets of S. In the previous block we defined what was meant by $P(A)$, $P(B)$ and their complements in the particular case in which the experiment had equally likely outcomes.
Events, like sets, can be combined to produce new events.

- $A \cup B$ denotes the event that A or B (or both) occur when the experiment is performed.
- $A \cap B$ denotes the event that both A and B occur together.

In this block we obtain expressions for determining the probabilities of these combined events, which are written $P(A \cup B)$ and $P(A \cap B)$ respectively.

Types of Events

There are two types of events you will need to able to identify and work with: mutually exclusive events and independent events.

Mutually Exclusive Events

Mutually exclusive events are events that by definition cannot happen together. For example, when tossing a coin, the events 'head' and 'tail' are mutually exclusive, when testing a switch 'operate' and 'fail' are mutually exclusive and when testing the tensile strength of a piece of wire, 'hold' and 'snap' are mutually exclusive. In such cases, the probability of both events occurring (call the events A and B) must be zero. Hence, using the usual set theory notation, we may write:

$$P(A \cap B) = 0, \qquad \text{provided that } A \text{ and } B \text{ are mutually exclusive events}$$

Decide which of the following pairs (A and B) of events arising from the experiments described are mutually exclusive.

(a) Two cards are drawn from a pack

$A = \{\text{a red card is drawn}\}$
$B = \{\text{a picture card is drawn}\}$

(b) The daily traffic accidents in Loughborough involving pedal cyclists and motor cyclists are counted

$A = \{\text{three motor cyclists are injured in collisions with cars}\}$
$B = \{\text{one pedal cyclist is injured when hit by a bus}\}$

(c) A box contains 20 nuts, some have a metric thread, some have a British Standard Fine (BSF) thread and some have a British Standard Whitworth (BSW) thread.

$A = \{\text{first nut picked out of the box is BSF}\}$
$B = \{\text{second nut picked out of the box is metric}\}$

Your solution

(a) A and B are not mutually exclusive.

(b) A and B are mutually exclusive.

(c) A and B are not mutually exclusive.

Key Point

The Addition Law of Probability - Simple Case

If two events A and B are mutually exclusive then

$$P(A \cup B) = P(A) + P(B)$$

Key Point

The Addition Law of Probability - General Case

If two events are A and B then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Of course if $A \cap B = \emptyset$ i.e. A and B are mutually exclusive, then, $P(A \cap B) = P(\emptyset) = 0$ and this general expression reduces to the simpler case.
This rule can be extended to three or more events, for example:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Example Consider a pack of 52 playing cards. A card is selected at random. What is the probability that the card is **either** a diamond **or** a ten?

Solution

If A is the event $\{$*a diamond is selected*$\}$ and B is the event $\{$*a ten is selected*$\}$ then obviously $P(A) = \frac{13}{52}$ and $P(B) = \frac{4}{52}$. The intersection event $A \cap B$ consists of only one member - the ten of diamonds - which gets counted twice hence $P(A \cap B) = \frac{1}{52}$.

Therefore $P(A \cup B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$.

A bag contains 20 balls, 3 are coloured red, 6 are coloured green, 4 are coloured blue, 2 are coloured white and 5 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.

(a) the ball is either red or green

(b) the ball is not blue

(c) the ball is either red or white or blue. (Hint: consider the complementary event.)

Your solution

Note that a ball can only have one colour, which are designated by the letters R, G, B, W, Y.

(a) $P(R \cup G) = P(R) + P(G) = \frac{3}{20} + \frac{6}{20} = \frac{9}{20}$.

(b) $P(B') = 1 - P(B) = 1 - \frac{4}{20} = \frac{16}{20} = \frac{4}{5}$.

(c) The complementary event is $G \cup Y$ and $P(G \cup Y) = \frac{6}{20} + \frac{5}{20} = \frac{11}{20}$.

Hence $P(R \cup W \cup B) = 1 - \frac{11}{20} = \frac{9}{20}$

In the last example (part (c)) we could alternatively have used an obvious extension of the law of addition for mutually exclusive events:

$$P(R \cup W \cup B) = P(R) + P(W) + P(B) = \frac{3}{20} + \frac{2}{20} + \frac{4}{20} = \frac{9}{20}.$$

Figure 1 shows a simplified circuit in which two independent components a and b are connected in parallel.

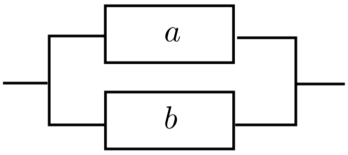


Figure 1

The circuit functions if either or both of the components are operational. It is known that if A is the event 'component a is operating' and B is the event 'component b is operating' then $P(A) = 0.99$, $P(B) = 0.98$ and $P(A \cap B) = 0.9702$. Find the probability that the circuit is functioning.

Your solution

The probability that the circuit is functioning is $P(A \cup B)$. In words: either a or b or both must be functioning if the circuit is to function. Using the keypoint:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.99 + 0.98 - 0.9702 = 0.9998 \end{aligned}$$

Not surprisingly the probability that the circuit functions is greater than the probability that either of the individual components functions.

Exercises

1. The following people are in a room: 5 men aged 21 and over, 4 men under 21, 6 women aged 21 and over, and 3 women under 21. One person is chosen at random. The following events are defined: $A = \{$the person is aged 21 and over$\}$; $B = \{$the person is under 21$\}$; $C = \{$the person is male$\}$; $D = \{$ the person is female$\}$. Evalute the following:

   (a) $P(B \cup D)$

   (b) $P(A' \cap C')$
   Express the meaning of these quantities in words.

2. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a face card (i.e, K,Q,J)?

3. In a single throw of two dice, what is the probability that neither a double nor a sum of 9 will appear?

Answers

1. (a) $P(B \cup D) = P(B) + P(D) - P(B \cap D)$

   $p(B) = \frac{7}{18}, \quad p(D) = \frac{9}{18}, \quad P(B \cap D) = \frac{3}{18}$

   $\therefore \quad P(B \cup D) = \frac{7}{18} + \frac{9}{18} - \frac{3}{18} = \frac{13}{18}$

   (b) $P(A' \cap C')$ $\quad A' = \{\text{people under } 21\} \quad C' = \{\text{people who are female}\}$

   $\therefore \quad P(A' \cap C') = \frac{3}{18} = \frac{1}{6}$

2. $F = \{\text{face card}\} \quad A = \{\text{card is ace}\} \quad P(F) = \frac{12}{52}, \quad P(A) = \frac{4}{52}$

   $\therefore \quad P(F \cup A) = P(F) + P(A) - P(F \cap A) = \frac{12}{52} + \frac{4}{52} - 0 = \frac{16}{52}$

3. $D = \{\text{double is thrown}\} \qquad N = \{\text{sum is } 9\}$

   $P(D) = \frac{6}{36}$ (36 possible outcomes in an experiment in which all the outcomes are equally probable).

   $P(N) = P\{(6 \cap 3) \cup (5 \cap 4) \cup (4 \cap 5) \cup (3 \cap 6)\} = \frac{4}{36}$

   $P(D \cup N) = p(D) + P(N) - P(D \cap N) = \frac{6}{36} + \frac{4}{36} - 0 = \frac{10}{36}$

   $P((D \cup N)') = 1 - P(D \cup N) = 1 - \frac{10}{36} = \frac{26}{36}$

2. Conditional Probability

Suppose a bag contains 6 balls, 3 Red and 3 White. Two balls are chosen (without replacement) at random, one after the other. Consider the two events R, W:

R is event "first ball chosen is Red"
W is event "second ball chosen is White"

We easily find $P(R) = \frac{3}{6} = \frac{1}{2}$. However, determining the probability of W is not quite so straightforward. If the first ball chosen is red then the bag subsequently contains 2 red balls and 3 white. In this case $P(W) = \frac{3}{5}$. However, if the first ball chosen is white then the bag subsequently contains 3 red balls and 2 white. In this case $P(W) = \frac{2}{5}$. What this example shows is that the probability that W occurs is clearly **dependent** upon whether or not the event R has occurred. The probability of W occurring is **conditional** on the occurrence or otherwise of R.

The **conditional probability** of an event B occurring given that event A has occurred is written $P(B|A)$. In this particular example

$$P(W|R) = \frac{3}{5} \qquad \text{and} \qquad P(W|R') = \frac{2}{5}.$$

Consider, more generally, the performance of an experiment in which the outcome is a member of an event A. We can therefore say that the event A has occurred.

What is the probability that B then occurs? That is what is $P(B|A)$? In a sense we have a new sample space which is the event A. For B to occur some of its members must also be members of event A. So $P(B|A)$ must be the number of outcomes in $A \cap B$ divided by the number of outcomes in A. That is

$$P(B|A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}.$$

Now if we divide both the top and bottom of this fraction by the total number of outcomes of the experiment we obtain an expression for the conditional probability of B occurring given that A has occurred:

Key Point

Conditional Probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad \text{or, equivalently} \qquad P(A \cap B) = P(B|A)P(A)$$

To illustrate the use of conditional probability concepts we return to the example of the bag containing 3 red and 3 white balls in which we consider two events:

- R is event "first ball is Red"
- W is event "second ball is White"

Let the red balls be numbered 1 to 3 and the white balls 4 to 6. If, for example, (3,5) represents the fact that the first ball is 3 (red) and the second ball is 5 (white) then we see that there are $6 \times 5 = 30$ possible outcomes to the experiment (no ball can be selected twice.)
If the first ball is red then only the fifteen outcomes $(1, x)$, $(2, y)$, $(3, z)$ are then possible (here $x \neq 1$, $y \neq 2$ and $z \neq 3$). Of these fifteen the six outcomes $\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$ will produce the required result, ie the event in which both balls chosen are red, giving a probability: $P(B|A) = \frac{6}{15} = \frac{2}{5}$.

Example A box contains six 10 Ω resistors and ten 30 Ω resistors. The resistors are all unmarked and are of the same physical size.

1. One resistor is picked at random from the box; find the probability that:

   (a) It is a 10 Ω resistor.

   (b) It is a 30 Ω resistor.

2. At the start, two resistors are selected from the box. Find the probability that:

   (a) Both are 10 Ω resistors.

   (b) The first is a 10 Ω resistor and the second is a 30 Ω resistor.

   (c) Both are 30 Ω resistors.

Solution

1. (a) As there are six 10 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$P(10\Omega) = \frac{6}{16} = \frac{3}{8}$$

(b) As there are ten 30 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$P(30\ \Omega) = \frac{10}{16} = \frac{5}{8}$$

2. (a) As there are six 10 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$P(\text{1st selected is a 10 }\Omega\text{ resistor}) = \frac{6}{16} = \frac{3}{8}$$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are only five 10 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors.

Hence, $\quad P(\text{2nd selected is also a 10 }\Omega\text{ resistor}) = \dfrac{5}{15} = \dfrac{1}{3}$

And, $\quad P(\text{both are 10 }\Omega\text{ resistors}) = \dfrac{3}{8} \times \dfrac{1}{3} = \dfrac{1}{8}$

(b) As before, $P(\text{1st selected is a 10 }\Omega\text{ resistor}) = \dfrac{6}{18} = \dfrac{3}{8}$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are still ten 30 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors. Hence, $P(\text{2nd selected is a 30 }\Omega\text{ resistor}) = \dfrac{10}{15} = \dfrac{2}{3}$

And, $P(\text{1st was a 10 }\Omega\text{ resistor and 2nd was a 30 }\Omega\text{ resistor}) = \dfrac{3}{8} \times \dfrac{2}{3} = \dfrac{1}{4}$

(c) As there are ten 30 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$P(\text{1st selected is a 30 }\Omega\text{ resistor}) = \dfrac{10}{16} \times \dfrac{5}{8}$

If the first resistor selected was a 30 Ω one, then when the second resistor is selected, there are only nine 30 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors.

Hence, $P(\text{2nd selected is also a 30 }\Omega\text{ resistor}) = \dfrac{9}{15} = \dfrac{3}{5}$

And, $P(\text{both are 30 }\Omega\text{ resistors}) = \dfrac{5}{8} \times \dfrac{3}{5} = \dfrac{3}{8}$

3. Independent events

If the occurrence of one event A does not affect, nor is affected by, the occurrence of another event B then we say that A and B are **independent** events. Clearly, if A and B are independent then

$$P(B|A) = P(B) \qquad \text{and} \qquad P(A|B) = P(A)$$

Then, using the last key point formula P$(A \cup B)$ =P$(B|A)P(A)$ we have, for independent events:

Key Point

The Multiplication Law

If A and B are independent events then

$$P(A \cap B) = P(A) \times P(B)$$

In words

'The probability of independent events A and B occurring is the product of the probabilities of the events occurring separately.'

In Figure 4 two components a and b are connected in series.

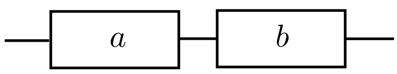


Figure 4

Define two events

- A is the event 'component a is operating'
- B is the event 'component b is operating'

Previous testing has indicated that $P(A) = 0.99$, and $P(B) = 0.98$. The circuit functions only if a and b are both operating simultaneously. The components are assumed to be independent. Then the probability that the circuit is operating is given by

$$P(A \cap B) = P(A)P(B) = 0.99 \times 0.98 = 0.9702$$

Note that this probability is smaller then either $P(A)$ or $P(B)$.

Decide which of the following pairs (A and B) of events arising from the experiments described are independent.

(a) One card is drawn from each of two packs

$A = \{$a red card is drawn from pack 1$\}$
$B = \{$a picture card is drawn from pack 2$\}$

(b) The daily traffic accidents in Loughborough involving pedal cyclists and motor cyclists are counted

$A = \{$three motor cyclists are injured in collisions with cars$\}$
$B = \{$one pedal cyclist is injured when hit by a bus$\}$

(c) Two boxes contains 20 nuts each, some have a metric thread, some have a British Standard Fine (BSF) threads and some have a British Standard Whitworth (BSW) thread. A nut is picked out of each box.

$A = \{$nut picked out of the first box is BSF$\}$
$B = \{$nut picked out of the second box is metric$\}$

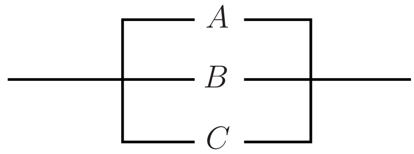
(d) A box contains 20 nuts, some have a metric thread, some have a British Standard Fine (BSF) threads and some have British Standard Whitworth (BSW) thread. Two nuts are picked out of the box.

$A = \{$first nut picked out of the box is BSF$\}$
$B = \{$second nut picked out of the box is metric$\}$

Your solution

(a) A and B are independent.
(b) A and B are independent.
(c) A and B are independent.
(d) A and B are not independent.

Example A circuit has three switches A, B and C wired in parallel as shown in the diagram below.



Current can only flow through the bank of switches if at least one of them is closed. The probability that any given switch is closed is 0.9. Calculate the probability that current can flow through the bank of switches.

Solution

Assume that the switches operate independently and that A is the event {switch A is closed}. Similarly for switches B and C. We require $P(A \cup B \cup C)$, the probability that at least one switch is closed. Using set theory,

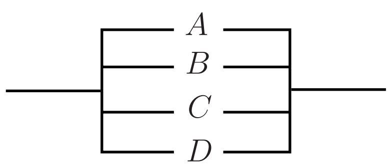
$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - [P(A \cap B) + P(B \cap C) + P(C \cap A)] \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

Using the fact that the switches operate independently,

$$\begin{aligned} P(A \cup B \cup C) &= 0.9 + 0.9 + 0.9 - [0.9 \times 0.9 + 0.9 \times 0.9 + 0.9 \times 0.9] \\ &\quad + 0.9 \times 0.9 \times 0.9 \\ &= 2.7 - 2.43 + 0.729 = 0.999 \end{aligned}$$

Note that the result implies that the system is more likely to allow current to flow than any single switch in the system. This is why duplication is built into systems requiring a high degree of reliability such as aircraft control systems.

A circuit has four switches A, B, C and D wired in parallel as shown in the diagram below.



Current can only flow through the bank of switches if at least one of them is closed. The probabilities that switches A, B, C and D are closed are 0.9, 0.8, 0.7 and 0.6 respectively. Calculate the probability that current can flow through the bank of switches.

Denoting the switches by A, B, C and D we have:

$$\begin{aligned} P(A \cup B \cup C \cup D) = {} & P(A) + P(B) + P(C) + P(D) \\ & -P(A \cap B) - P(B \cap C) - P(C \cap D) - P(D \cap A) - P(A \cap C) - P(B \cap D) \\ & +P(A \cap B \cap C) + P(B \cap C \cap D) + P(C \cap D \cap A) + P(D \cap A \cap B) \\ & -P(A \cap B \cap C \cap D) \end{aligned}$$

Assuming that the switches operate independently and substituting gives:

$$P(A \cup B \cup C \cup D) = 3 - 3.35 + 1.65 - 0.3024 = 0.9976$$

Hence, the probability that current can flow through the bank of switches is 0.9976.

Exercises

1. A box contains 4 bad and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

2. A man owns a house in town and a cottage in the country. In any one year the probability of the house being burgled is 0.01 and the probability of the cottage being burgled is 0.05. In any one year what is the probability that:

   (a) both will be burgled? (b) one or the other (but not both) will be burgled ?

3. In a Baseball Series, teams A and B play until one team has won 4 games. If team A has probability 2/3 of winning against B in a single game, what is the probability that the Series will end only after 7 games are played?

4. The probability that a single aircraft engine will fail during flight is q. A plane makes a successful flight if at least half its engines run. Assuming that the engines operate independently, find the values of q for which a two-engine plane is to be preferred to a four-engined one.

5. Current flows through a relay only if it is closed. The probability of any relay being closed is 0.95. Calculate the probability that a current will flow through a circuit composed of 3 relays in parallel. What assumption must be made?

6. A central heating installation and maintenance engineer keeps a record of the causes of failure of systems he is called out to repair. The causes of failure are classified as 'electrical', 'gas', or in rare cases 'other'. A summary of the records kept of failures involving either gas or electrical faults is as follows:

		Electrical	
		Yes	**No**
Gas	**Yes**	53	11
	No	23	13

   (a) Find the probability that failure involves gas given that it involves electricity.

   (b) Find the probability that failure involves electricity given that it involves gas.

Answers 1. Let $G_i = \{i^{th}$ tube is good$\}$ $B_i = \{i^{th}$ tube is bad$\}$

$P(G_2|G_1) = \frac{5}{9}$ (only 5 good tubes left out of 9).

2. (a) $H = \{$house is burgled$\}$ $C = \{$cottage is burgled$\}$

(b) $P(H \cap C) = P(H)P(C) = (0.01)(0.05) = 0.0005$ since events independent

$$\begin{aligned} P(\text{one or the other (but not both)}) &= P((H \cap C') \cup (H' \cap C)) = P(H \cap C') + P(H' \cap C) \\ &= P(H)P(C') + P(H')P(C) \\ &= (0.01)(0.95) + (0.99)(0.05) = 0.059. \end{aligned}$$

3. Let A_i be event $\{A$ wins the i^{th} game$\}$

req'd event is $\underbrace{\{A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') \cap (\ldots)}_{\text{no. of ways of arranging 3 in 6 i.e. } {}^6C_3}$

$P(\text{req'd event}) = {}^6C_3 P(A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') = {}^6C_3[P(A_1)]^3 P[A_1']^3 = \frac{160}{729}$

4. Let E_i be event $\{i^{th}$ engine success$\}$

2 engine plane: flight success if $\{(E_1 \cap E_2) \cup (E'_1 \cap E_2)(E_1 \cap E'_2)\}$ occurs

$$\begin{aligned} P(\text{req'd event}) &= P(E_1)P(E_2) + P(E_1')P(E_2) + P(E_1)P(E_2') \\ &= (1-q)^2 + 2q(1-q) = 1 - q^2 \end{aligned}$$

4 engine plane: success if following event occurs

$$\underbrace{\{E_1 \cap E_2 \cap E_3' \cap E_4'\}}_{{}^4C_2 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4'\}}_{{}^4C_1 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4\}}_{{}^4C_0 \text{ ways}}$$

req'd probability$= 6(1-q)^2q^2 + 4(1-q)^3q + (1-q)^4 = 3q^4 - 4q^3 + 1$

Two engine plane is preferred if

$$1 - q^2 > 3q^4 - 4q^3 + 1 \qquad \text{i.e. if} \quad 0 > q^2(3q-1)(q-1)$$

Let $y = (3q-1)(q-1)$. By drawing a graph of this quadratic you will quickly see that a two engine plane is preferred if $\frac{1}{3} < q < 1$

Answers

5. Let A be event {relay A is closed}: Similarly for $B,\ C$

req'd event is $\{A \cap B \cap C\} \cup \underbrace{\{A' \cap B \cap C\}}_{^3C_1} \ \cap\ \underbrace{\{A' \cap B' \cap C\}}_{^3C_2}$

$P(\text{req'd event}) = (0.95)^3 + 3(0.95)^2(0.05) + 3(0.95)(0.05)^2 = 0.999875$

(or $1 - p(\text{all relays open}) = 1 - (0.05)^3 = 0.999875$.)

The assumption is that relays operate independently.

6(a) A total of 76 failures involved electrical faults. Of the 76 some 53 involved gas. Hence

$$P\{\text{Gas Failure} \mid \text{Electrical Failure}\} = \frac{53}{76} = 0.697$$

6(b) A total of 64 failures involved electrical faults. Of the 64 some 53 involved gas. Hence

$$P\{\text{Electrical Failure} \mid \text{Gas Failure}\} = \frac{53}{64} = 0.828$$