Contents

37 discrete probability distributions

Learning outcomes

In this workbook you will learn what a discrete random variable is. You will find how to calculate the expectation and variance of a discrete random variable. You will then examine two of the most important examples of discrete random variables: the binomial and Poisson distributions. The Poisson distribution can be deduced from the binomial distribution and is often used as a way of finding good approximations to the binomial probabilities. The binomial is a finite discrete random variable whereas the Poisson distribution has an infinite number of possibilities. Finally you will learn about another important distribution - the hypergeometric.

Time allocation

You are expected to spend approximately five hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Discrete Probability Distributions

37.1

Introduction

It is often possible to model real systems by using the same or similar random experiments and their associated random variables. Numerical random variables may be classified in two broad but distinct categories called discrete random variables and continuous random variables. Often, discrete random variables are associated with counting while continuous random variables are associated with measuring. In Workbook 42. you will meet contingency tables and deal with non-numerical random variables. Generally speaking, discrete random variables can take values which are separate and can be listed. Strictly speaking, the real situation is a little more complex but it is sufficient for our purposes to equate the word discrete with a finite list. In contrast, continuous random variables can take values anywhere within a specified range. This Section will familiarize you with the idea of a discrete random variable and the associated probability distributions. The Workbook makes no attempt to cover the whole of this large and important branch of statistics but concentrates on the discrete distributions most commonly met in engineering. These are the binomial, Poisson and hypergeometric distributions.

Prerequisites

Before starting this Section you should . . .

① understand the concepts of probability

Learning Outcomes

After completing this Section you should be able to . . .

- ✓ understand what is meant by the term discrete random variable
- ✓ understand what is meant by the term discrete probability distribution
- ✓ be able to use some of the discrete distributions which are important to engineers

1. Discrete Probability Distributions

We shall look at discrete distributions in this Workbook and continuous distributions in Workbook 23. In order to get a good understanding of discrete distributions it is advisable to familiarise yourself with two related topics: permutations and combinations. Essentially we shall be using this area of mathematics as a calculating device which will enable us to deal sensibly with situations where *choice* leads to the use of very large numbers of possibilities. We shall use combinations to express and manipulate these numbers in a compact and efficient way.

Permutations and Combinations

You may recall from Section 4.2 concerned with probability that if we define the probability that an event A occurs by using the definition:

$$P(A) = \frac{\text{The number of experimental outcomes favourable to } A}{\text{The total number of outcomes forming the sample space}} = \frac{a}{n}$$

then we can only find $\mathrm{P}(A)$ provided that we can find both a and n. In practice, these numbers can be very large and difficult if not impossible to find by a simple counting process. Permutations and combinations help us to calculate probabilities in cases where counting is simply not a realistic possibility.
Before discussing permutations, we will look briefly at the idea and notation of a factorial.

Factorials

The factorial of an integer n commonly called 'factorial n' and written $n!$ is defined as follows:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \qquad n \geq 1$$

Simple examples are:

$$3! = 3\times2\times1 = 24 \qquad 5! = 5\times4\times3\times2\times1 = 120 \qquad 8! = 8\times7\times6\times5\times4\times3\times2\times1 = 40320$$

As you can see, factorial notation enables us to express large numbers in a very compact format. You will see that this characteristic is very useful when we discuss the topic of permutations. A further point is that the definition falls down when $n = 0$ and we define

$$0! = 1$$

Permutations

A *permutation* of a set of distinct objects places the objects in *order*. For example the set of three numbers $\{1, 2, 3\}$ can be placed in the following orders:

1,2,3 1,3,2 2,1,3 2,3,1 3,2,1 3,1,2

Note that we can choose the first item in 3 ways, the second in 2 ways and the third in 1 way This gives us $3 \times 2 \times 1 = 3! = 6$ distinct orders. We say that the set $\{1, 2, 3\}$ has the distinct permutations

1,2,3 1,3,2 2,1,3 2,3,1 3,2,1 3,1,2

Example Write out the possible permutations of the letters A, B, C and D

Solution

The possible permutations are

$$\begin{array}{cccccc} ABCD & ABDC & ADBC & ADCB & ACBD & ACDB \\ BADC & BACD & BCDA & BCAD & BDAC & BDCA \\ CABD & CADB & CDBA & CDAB & CBAD & CBDA \\ DABC & DACB & DCAB & DCBA & DBAC & DBCA \end{array}$$

There are $4! = 24$ permutations of the four letters A, B, C and D.

In general we can order n distinct objects in $n!$ ways.
Suppose we have r different types of object. It follows that if we have n_1 objects of one kind, n_2 of another kind and so on then the n_1 objects can be ordered in $n_1!$ ways, the n_2 objects in $n_2!$ ways and so on. If $n_1 + n_2 + \cdots + n_r = n$ and if p is the number of permutations possible from n objects we may write

$$p \times (n_1! \times n_2! \times \cdots \times n_r!) = n!$$

and so p is given by the formula

$$p = \frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$$

Very often we will find it useful to be able to calculate the number of permutations of n objects taken r at a time. Assuming that we do not allow repetitions, we may choose the first object in n ways, the second in $n-1$ ways, the third in $n-2$ ways and so on so that the r^{th} object may be chosen in $n-r+1$ ways.

Example Find the number of permutations of the four letters A, B, C and D taken three at a time.

Solution

We may choose the first letter in 4 ways, either A, B, C or D. Suppose, for the purposes of illustration we choose A. We may choose the second letter in 3 ways, either B, C or D. Suppose, for the purposes of illustration we choose B. We may choose the third letter in 2 ways, either C or D. Suppose, for the purposes of illustration we choose C. The total number of choices made is $4 \times 3 \times 2 = 24$.

In general the numbers of permutations of n objects taken r at a time is

$$n(n-1)(n-2)\ldots(n-r+1) \qquad \text{which is the same as} \qquad \frac{n!}{(n-r)!}$$

This is usually denoted by ${}^nP_r$ so that

$${}^nP_r = \frac{n!}{(n-r)!}$$

If we allow repetitions the number of permutations becomes n^r (can you see why?).

Example Find the number of permutations of the four letters A, B, C and D taken two at a time.

Solution

We may choose the first letter in 4 ways and the second letter in 3 ways giving us

$$4 \times 3 = \frac{4 \times 3 \times 2 \times 1}{1 \times 2} = \frac{4!}{2!} = 12 \quad \text{permutations}$$

Combinations

A *combination* of objects takes *no account of order*; a permutation as we know does. The formula ${}^nP_r = \dfrac{n!}{(n-r)!}$ gives us the number of ordered sets of r objects chosen from n. Suppose the number of sets of r objects (taken from n objects) in which order is not taken into account is C. It follows that

$$C \times r! = \frac{n!}{(n-r)!} \text{ and so } C \text{ is given by the formula} \qquad C = \frac{n!}{r!(n-r)!}$$

We normally denote the right-hand side of this expression by ${}^nC_r$ so that

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

A common alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Example How many car registrations are there beginning with $NP02$ followed by three letters? Note that, conventionally, I, O and Q may not be chosen.

Solution

We have to choose 3 letters from 23 allowing repetition. Hence the number of registrations beginning with $NP02$ must be $23^3 = 12167$. Note that if a further requirement is made that all three letters must be distinct, the number of registrations reduces to $23 \times 22 \times 21 = 10626$

How many different signals consisting of five symbols can be sent using the dot and dash of Morse code? How many can be sent if five symbols *or less* can be sent?

Your solution

Clearly, in this example, the order of the symbols is important. We can choose each symbol in two ways, either a dot or a dash. The number of distinct signals is

$$2 \times 2 \times 2 \times 2 \times = 2^5 = 32$$

If five or less symbols may be used, the total number of signals may be calculated as follows:

Using one symbol:	2 ways
Using two symbols:	$2 \times 2 = 4$ ways
Using three symbols:	$2 \times 2 \times 2 = 8$ ways
Using four symbols:	$2 \times 2 \times 2 \times 2 = 16$ ways
Using five symbols:	$2 \times 2 \times 2 \times 2 \times 2 = 32$ ways

The total number of signals which may be sent is 62.

A box contains 50 resistors, 20 are deemed to be 'good' , 20 'average' and 10 'poor.' In how many ways can a batch of 5 resistors be chosen if it is to contain 2 'good' , 2 'average' and 1 'poor' resistor?

Your solution

Answers The order in which the resistors are chosen does not matter so that the number of ways in which the batch of 5 can be chosen is:

$$^{20}C_2 \times ^{20}C_2 \times ^{10}C_1 = \frac{20!}{18! \times 2!} \times \frac{20!}{18! \times 2!} \times \frac{10!}{9! \times 1!} = \frac{20 \times 19}{1 \times 2} \times \frac{20 \times 19}{1 \times 2} \times \frac{10}{1} = 361000$$

2. Random Variables

A random variable X is a quantity whose value cannot be predicted with certainty. We assume that for every real number a the probability $P(X = a)$ in a trial is well-defined. In practice engineers are often concerned with two broad types of variables and their probability distributions, discrete random variables and their distributions, and continuous random variables and their distributions. Discrete distributions arise from experiments involving counting, for example, road deaths, car production and car sales, while continuous distributions arise from experiments involving measurement, for example, voltage, current and oil pressure.

Discrete Random Variables and Probability Distributions

A random variable X and its distribution are said to be discrete if the values of X can be presented as an ordered list say $x_1, x_2, x_3, \ldots$ with probability values $p_1, p_2, p_3, \ldots$. That is $P(X = x_i) = p_i$. For example, the number of times a particular machine fails during the course of one calendar year is a discrete random variable.
More generally a discrete distribution $f(x)$ may be defined by:

$$f(x) = \begin{cases} p_i & \text{if } x = x_i \quad i = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The distribution function $F(x)$ (sometimes called the cumulative distribution function) is obtained by taking sums as defined by

$$F(x) = \sum_{x_i \leq x} f(x_i) = \sum_{x_i \leq x} p_i$$

We sum the probabilities p_i for which x_i is less than or equal to x. This gives a step function with jumps of size p_i at each value x_i of X. The step function is defined for all values, not just the values x_i of X.

Key Point

Let X be a random variable associated with an experiment. Let the values of X be denoted by $x_1, x_2, \ldots, x_n$ and let $P(X = x_i)$ be the probability that x_i occurs. We have two necessary conditions for a valid probability distribution

- $P(X = x_i) \geq 0 \quad \text{for all} \quad x_i$
- $\displaystyle\sum_{i=1}^{n} P(X = x_i) = 1$

Note that n may be uncountably large (infinite).

(These two statements are sufficient to guarantee that $P(X = x_i) \leq 1$ for all x_i)

Example Turbo Generators plc manufacture seven large turbines for a customer. Three of these turbines do not meet the customer's specification. Quality control inspectors choose two turbines at random. Let the discrete random variable X be defined to be the number of turbines inspected which meet the customer's specification. Find the probabilities that X takes the values $0, 1$ or 2. Find and graph the cumulative distribution function.

Solution

The possible values of X are clearly $0, 1$ or 2 and may occur as follows:

Sample Space	Value of X
Turbine faulty, Turbine faulty	0
Turbine faulty, Turbine good	1
Turbine good, Turbine faulty	1
Turbine good, Turbine good	2

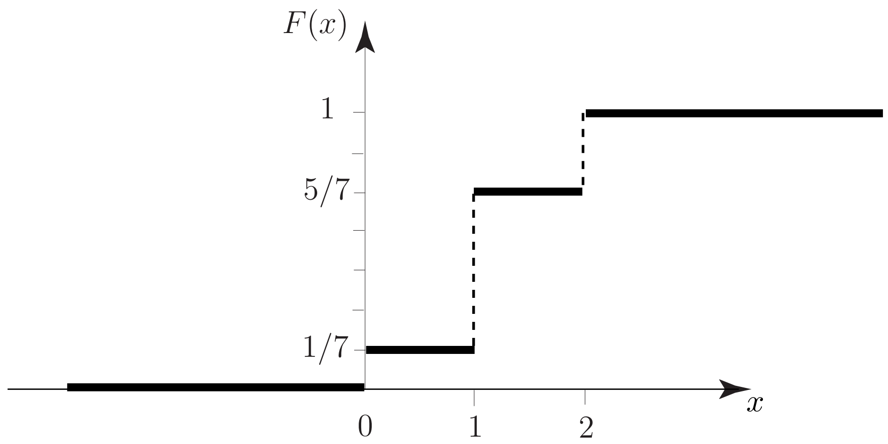
We can easily calculate the probability that X takes the values $0, 1$ or 2 as follows:

$$P(X=0) = \frac{3}{7}\times\frac{2}{6} = \frac{1}{7} \qquad P(X=1) = \frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6} = \frac{4}{7} \qquad P(X=2) = \frac{4}{7}\times\frac{3}{6} = \frac{2}{7}$$

The values of $F(x) = \sum_{x_i \le x} P(X = x_i)$ are clearly

$$F(0) = \frac{1}{7} \qquad F(1) = \frac{5}{7} \qquad \text{and} \qquad F(2) = \frac{7}{7} = 1$$

The graph of the step function $F(x)$ is shown below.



3. Mean and Variance of a Discrete Probability Distribution

If an experiment is performed N times in which the n possible outcomes $X = x_1, x_2, x_3, \ldots, x_n$ are observed with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively, we know that the mean of the distribution of outcomes is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i = \sum_{i=1}^{n}\left(\frac{f_i}{N}\right)x_i$$

(Note that $\sum_{i=1}^{n} f_i = f_1 + f_2 + \cdots + f_n = N$.)

The quantity $\dfrac{f_i}{N}$ is called the *relative frequency* of the observation x_i. Relative frequencies may be thought of as akin to probabilities, informally we would say that the chance of observing the outcome x_i is $\dfrac{f_i}{N}$. Formally, we consider what happens as the number of experiments becomes very large. In order to give meaning to the quantity $\dfrac{f_i}{N}$ we consider the limit (if it exists) of the quantity $\dfrac{f_i}{N}$ as $N \to \infty$. Essentially, we define the probability p_i as

$$\lim_{N\to\infty} \frac{f_i}{N} = p_i$$

Replacing $\dfrac{f_i}{N}$ with the probability p_i leads to the following definition of the mean or **expectation** of the discrete random variable X.

Key Point

The Expectation of a Random Variable

Let X be a random variable with values $x_1, x_2, \ldots, x_n$. Let the probability that X takes the value x_i (i.e. $P(X = x_i)$) be denoted by p_i. The mean or **expected value** of X, which is written $E(X)$ is defined as:

$$E(X) = \sum_{i=1}^{n} P(X = x_i) = p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

The symbol μ is sometimes used to denote $E(X)$.

The expectation $E(X)$ of X is the value of X which we expect on average. In a similar way we can write down the expected value of the function $g(X)$ as $E[g(X)]$, the value of $g(X)$ we expect on average. We have

$$E[g(X)] = \sum_{i}^{n} g(x_i)f(x_i)$$

In particular if $g(X) = X^2$, we obtain $\qquad E[X^2] = \sum_{i}^{n} x_i^2 f(x_i)$

The variance is usually written as σ^2. For a frequency distribution it is:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \mu)^2 \qquad \text{where } \mu \text{ is the mean value}$$

and can be expanded and simplified to appear as:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \mu)^2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \mu^2$$

This is often quoted in words:

The variance is equal to the mean of the squares minus the square of the mean.

We now extend the concept of variance to a random variable.

Key Point

The Variance of a Random Variable

Let X be a random variable with values $x_1, x_2, \ldots, x_n$. The variance of X, which is written $V(X)$ is defined by

$$V(X) = \sum_{i=1}^{n} p_i(x_i - \mu)^2$$

where $\mu \equiv E(X)$. We note that $V(X)$ can be written in the alternative form

$$V(X) = E(X^2) - [E(X)]^2$$

The standard deviation σ of a random variable is then $\sqrt{V(X)}$.

Example A traffic engineer is interested in the number of vehicles reaching a particular crossroad during periods of relatively low traffic flow. The engineer finds that the number of vehicles X reaching the cross roads per minute is goverened by the probability distribution:

x	0	1	2	3	4
$P(X = x)$	0.37	0.39	0.19	0.04	0.01

Calculate the expected value, the variance and the standard deviation of the random variable X. Graph the probability distribution $P(X = x)$ and the corresponding cumulative probability distribution $F(x) = \sum_{x_i \leq x} P(X = x_i)$.

Solution

The expectation, variance and standard deviation and cumulative probability values are calculated as follows:

x	x^2	$P(X=x)$	$F(x)$
0	0	0.37	0.37
1	1	0.39	0.76
2	4	0.19	0.95
3	9	0.04	0.99
4	16	0.01	1.00

The expectation is given by

$$\begin{aligned} E(X) &= \sum_{x=0}^{4} xP(X=x) \\ &= 0 \times 0.37 + 1 \times 0.39 + 2 \times 0.19 + 3 \times 0.04 + 4 \times 0.01 \\ &= 0.93 \end{aligned}$$

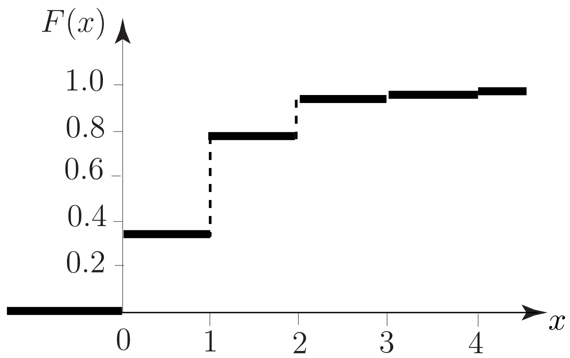
The variance is given by

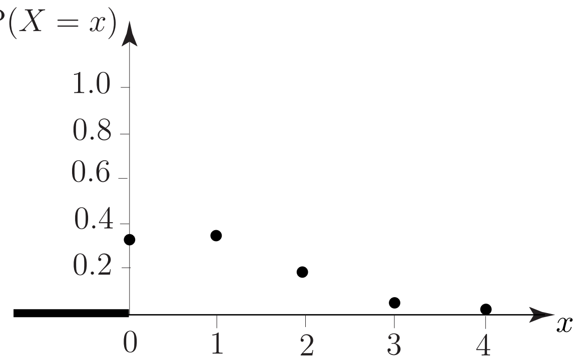
$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= \sum_{x=0}^{4} x^2 P(X=x) - \left[\sum_{x=0}^{4} xP(X=x)\right]^2 \\ &= 0 \times 0.37 + 1 \times 0.39 + 4 \times 0.19 + 9 \times 0.04 + 16 \times 0.01 - (0.93)^2 \\ &= 0.8051 \end{aligned}$$

The standard deviation is given by

$\sigma = \sqrt{V(X)} = 0.8973$

The two graphs required are:





Find the expectation, variance and standard deviation of the number of heads in the three-coin experiment. Refer to the previous guided exercise for the probability distribution.

Your solution

$$
\begin{aligned}
E(X) &= \frac{1}{8}\times 0+\frac{3}{8}\times 1+\frac{3}{8}\times 2+\frac{1}{8}\times 3=\frac{12}{8}\\
\sum p_i x_i^2 &= \frac{1}{8}\times 0^2+\frac{3}{8}\times 1^2+\frac{3}{8}\times 2^2+\frac{1}{8}\times 3^2\\
&= \frac{1}{8}\times 0+\frac{3}{8}\times 1+\frac{3}{8}\times 4+\frac{1}{8}\times 9=3\\
V(X) &= 3-2.25=0.75=\frac{3}{4}\\
\text{s.d.} &= \frac{\sqrt{3}}{2}
\end{aligned}
$$

Important Discrete Probability Distributions

The ideas of a discrete random variable and a discrete probability distribution have been introduced from a general perspective. We will now look at three important discrete distributions which occur in the engineering applications of statistics. These distributions are

- The binomial distribution defined by $P(X=r)={}^nC_r(1-p)^{n-r}p^r$
- The Poisson distribution defined by $P(X=r)=\mathrm{e}^{-\lambda}\dfrac{\lambda^r}{r!}$
- The hypergeometric distribution defined by $P(X=r)=\dfrac{({}^mC_r)({}^{n-m}C_{n-r})}{({}^nC_m)}$

The hypergeometric enables us to deal with processes which involve sampling *without replacement* in contrast to the binomial distribution which describes processes involving sampling *with replacement*.
Note that the definitions given will be developed in the following three Sections. They are presented here only as a convenient summary that you may wish to refer back in the future.

Exercises

1. A machine is operated by two workers. There are sixteen workers available. How many possible teams of two workers are there?

2. A factory has 52 machines. Two of these have been given an experimental modification. In the first week after this modification, problems are reported with thirteen of the machines. What is the probability that, given that there are problems with thirteen of the 52 machines and assuming that all machines are equally likely to give problems, both of the modified machines are among the thirteen with problems?

3. A factory has 52 machines. Four of these have been given an experimental modification. In the first week after this modification, problems are reported with thirteen of the machines. What is the probability that, given that there are problems with thirteen of the 52 machines and assuming that all machines are equally likely to give problems, exactly two of the modified machines are among the thirteen with problems?

4. A random number generator produces sequences of independent digits, each of which is as likely to be any digit from 0 to 9 as any other. If X denotes any single digit find $E(X)$.

5. A hand-held calculator has a clock cycle time of 100 nanoseconds; these are positions numbered $0, 1, \ldots, 99$. Assume a flag is set during a particular cycle at a random position. Thus, if X is the position number at which the flag is set.

$$P(X = k) = \frac{1}{100} \qquad k = 0, 1, 2, \ldots, 99.$$

   Evaluate the average position number $E(X)$, and σ, the standard deviation.

   (Hint: The sum of the first k integers is $k(k+1)/2$ and the sum of their squares is: $k(k+1)(2k+1)/6$.)

6. Concentric circles of radii 1 cm and 3 cm are drawn on a circular target radius 5 cm. A darts player receives 10, 5 or 3 points for hitting the target inside the smaller circle, middle annular region and outer annular region respectively. The player has only a 50-50 chance of hitting the target at all but if he does hit it he is just as likely to hit any one point on it as any other. If X = 'number of points scored on a single throw of a dart' calculate the expected value of X.

Answers

1. The required number is

$$\binom{16}{2} = \frac{16 \times 15}{2 \times 1} = 120.$$

2. There are

$$\binom{52}{13}$$

possible different selections of 13 machines and all are equally likely. There is only

$$\binom{2}{2} = 1$$

way to pick two machines from those which were modified but there are

$$\binom{50}{11}$$

different choices for the 11 other machines with problems so this is the number of possible selections containing the 2 modified machines. Hence the required probability is

$$\frac{\binom{2}{2}\binom{50}{11}}{\binom{52}{13}} = \frac{\binom{50}{11}}{\binom{52}{13}} = \frac{50!/11!39!}{52!/13!39!} = \frac{50!13!}{52!11!} = \frac{13 \times 12}{52 \times 51} \approx 0.0588$$

Alternatively, let S be the event "first modified machine is in the group of 13" and C be the event "second modified machine is in the group of 13". Then the required probability is

$$\Pr(S)\Pr(C \mid S) = \frac{13}{52} \times \frac{12}{51}.$$

Continued

3. There are

$$\binom{52}{13}$$

different selections of 13,

$$\binom{4}{2}$$

different choices of two modified machines and

$$\binom{48}{11}$$

different choices of 11 non-modified machines.
Thus the required probability is

$$\frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = \frac{(4!/2!2!)(48!/11!37!)}{(52!/13!39!)} = \frac{4!48!13!39!}{52!2!2!11!37!} = \frac{4\times 3\times 13\times 12\times 39\times 38}{52\times 51\times 50\times 49\times 2} \approx 0.2135$$

Alternatively, let $I(i)$ be the event "modified machine i is in the group of 13" and $O(i)$ be the negation of this, for $i = 1, 2, 3, 4$. The number of choices of two modified machines is

$$\binom{4}{2}$$

so the required probability is

$$\binom{4}{2}\Pr\{I(1)\}\Pr\{I(2) \mid I(1)\}\Pr\{O(3) \mid I(1), I(2)\}\Pr\{O(4) \mid I(1)I(2)O(3)\}$$

$$= \binom{4}{2}\frac{13}{52}\times\frac{12}{51}\times\frac{39}{50}\times\frac{38}{49} = \frac{4\times 3\times 13\times 12\times 39\times 38}{52\times 51\times 50\times 49\times 2}$$

4.

x	0	1	2	3	4	5	6	7	8	9
$P(X = x)$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$	$^1/_{10}$

$$E(X) = \frac{1}{10}\{0 + 1 + 2 + 3 + \ldots + 9\} = 4.5$$

Continued

5. Same as Q.4 but with 100 positions

$$E(X) = \frac{1}{100}\{0+1+2+3+\ldots+99\} = \frac{1}{100}\left[\frac{99(99+1))}{2}\right] = 49.5$$

σ^2 = mean of squares - square of means

$$\begin{aligned}\therefore \qquad \sigma^2 &= \frac{1}{100}[1^2+2^2+\ldots+99^2] - (49.5)^2 \\ &= \frac{1}{100}\frac{[99(100)(199)]}{6} - 49.5^2 = 833.25\end{aligned}$$

so the standard deviation is $\sigma = \sqrt{833.25} = 28.87$

6. X can take 4 values 0, 3, 5 or 10

$$P(X=0) = 0.5 \qquad \text{[only 50/50 chance of hitting target.]}$$

The probability that a particular points score is obtained is related to the areas of the annular regions which are, from the centre: π, $(9\pi - \pi) = 8\pi$, $(25\pi - 9\pi) = 16\pi$

$$\begin{aligned}P(X=3) &= P[(3 \text{ is scored}) \cap (\text{target is hit})] \\ &= P(3 \text{ is scored} \mid \text{target is hit}).P(\text{target is hit}) \\ &= \frac{16\pi}{25\pi}.\frac{1}{2} = \frac{16}{50} \\ P(X=5) &= P(5 \text{ is scored} \mid \text{target is hit}).P(\text{target is hit}) \\ &= \frac{8\pi}{25\pi}.\frac{1}{2} = \frac{8}{50} \\ P(X)=10) &= P(10 \text{ is scored} \mid \text{target is hit}).P(\text{target is hit}) \\ &= \frac{\pi}{25\pi}.\frac{1}{2} = \frac{1}{50}\end{aligned}$$

x	0	3	5	10
$P(X=x)$	$^{25}/_{50}$	$^{16}/_{50}$	$^{8}/_{50}$	$^{1}/_{50}$

$\therefore \quad E(X) = \frac{48+40+10}{50} = 1.96.$