

The Uniform Distribution

38.2



Introduction

This Section introduces the simplest type of continuous uniform distribution which features a continuous random variable X with probability density function $f(x)$ which assumes a constant value over a finite interval.



Prerequisites

Before starting this Section you should ...

- ① understand the concepts of probability
- ② be familiar with the concepts of expectation and variance
- ③ be familiar with the concept of continuous probability distribution



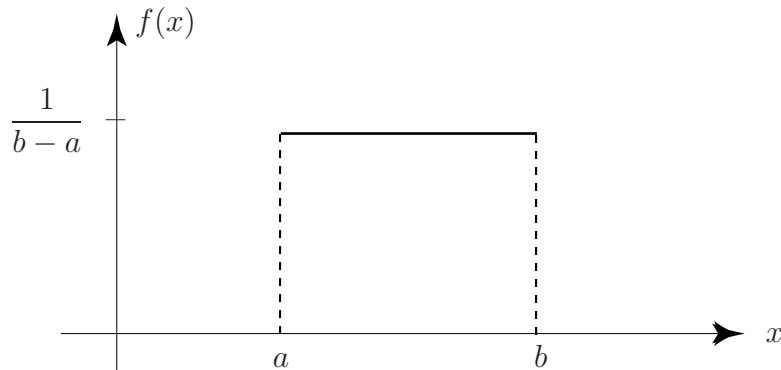
Learning Outcomes

After completing this Section you should be able to ...

- ✓ understand what is meant by the term ‘uniform distribution’
- ✓ be able to calculate the mean and variance of a uniform distribution

1. The Uniform Distribution

The Uniform or Rectangular distribution has random variable X restricted to a finite interval $[a, b]$ and has $f(x)$ has constant density over the interval. An illustration is



The function $f(x)$ is defined by:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Mean and Variance of a Uniform Distribution

Using the definitions of expectation and variance leads to the following calculations. As you might expect, for a uniform distribution, the calculations are not difficult.

Using the basic definition of expectation we may write:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{2(b-a)} [x^2]_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

Using the formula for the variance, we may write:

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= \int_a^b x^2 \cdot \frac{1}{b-a} dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{3(b-a)} [x^3]_a^b - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$



Key Point

The Uniform random variable X whose density function $f(x)$ is defined by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

has expectation and variance given by the formulae

$$E(X) = \frac{b+a}{2} \quad \text{and} \quad V(X) = \frac{(b-a)^2}{12}$$

Example The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval $[0, 25]$. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$.

Solution

Over the interval $[0, 25]$ the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} \frac{1}{25-0} = 0.04, & 0 \leq x \leq 25 \\ 0 & \text{otherwise} \end{cases}$$

Using the formulae developed for the mean and variance gives

$$E(X) = \frac{25+0}{2} = 12.5 \text{ mA} \quad \text{and} \quad V(X) = \frac{(25-0)^2}{12} = 52.08 \text{ mA}^2$$

The cumulative distribution function is obtained by integrating the probability density function as shown below.

$$F(x) = \int_{-\infty}^x f(t) dt$$

Hence, choosing the three distinct regions $x < 0$, $0 \leq x \leq 25$ and $x > 25$ in turn gives:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{25} & 0 \leq x \leq 25 \\ 1 & x > 25 \end{cases}$$



The thickness x of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval $[20, 40]$ microns. Find the mean, standard deviation and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick.

Your solution

Over the interval $[20, 40]$ the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} 0.05, & 20 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

Using the formulae developed for the mean and variance gives

$$E(X) = 10 \mu\text{m} \quad \text{and} \quad \sigma = \sqrt{V(X)} = \frac{\sqrt{12}}{20} = 5.77 \mu\text{m}$$

The cumulative distribution function is given by

$$F(x) = \int_x^{-\infty} f(x) dx$$

Hence, choosing appropriate ranges for x , the cumulative distribution function is obtained as:

$$F(x) = \begin{cases} 0, & x < 20 \\ \frac{x-20}{20}, & 20 \leq x \leq 40 \\ 1, & x \geq 40 \end{cases}$$

Hence the probability that the coating is less than 35 microns thick is

$$F(x < 35) = \frac{35 - 20}{20} = 0.75$$