The Exponential Distribution

38.3

Introduction

If an engineer is responsible for the quality of, say, copper wire for use in domestic wiring systems, he or she might be interested in knowing both the number of faults in a given length of wire and also the distances between such faults. While the number of faults may be analysed by using the Poisson distribution, the distances between faults along the wire may be shown to give rise to the exponential distribution defined and used in this Section.

Prerequisites

Before starting this Section you should ...

① understand the concepts of probability

② be familiar with the concepts of expectation and variance

③ be familiar with the concepts of continuous distributions, in particular the Poisson distribution.

Learning Outcomes

After completing this Section you should be able to ...

✓ understand what is meant by the term 'exponential distribution'

✓ calculate the mean and variance of an exponential distribution

✓ use the exponential distribution to solve simple practical problems

1. The Exponential Distribution

The exponential distribution is defined by

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0 \qquad \lambda \text{ a constant}$$

or sometimes (see the Section on Reliability in Workbook 46) by

$$f(t) = \frac{1}{\mu} e^{-t/\mu} \qquad t \geq 0 \qquad \mu \text{ a constant}$$

The advantage of this latter representation is that it may be shown that the mean of the distribution is μ.

Example The lifetime T (years) of an electronic component is a continuous random variable with a probability density function given by

$$f(t) = e^{-t} \qquad t \geq 0 \qquad (\text{i.e. } \lambda = 1 \text{ or } \mu = 1)$$

Find the lifetime L which a typical component is 60% certain to exceed. If five components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years.

Solution

We require $P(T > L) = 0.6$. We know that this probability is given by the relationship

$$P(T > L) = \int_L^{\infty} e^{-t} dt = \left[-e^{-t}\right]_L^{\infty} = e^{-L}$$

Solving $e^{-L} = 0.6$ for the least value of L we obtain $L = 0.51$ years.
Assuming that the lifetime of each component is independent we have

$$\begin{aligned} &P(\text{at least one component has a lifetime less than 0.51 years}) \\ &\quad = 1 - P(\text{no component has a lifetime less than 0.51 years}) \\ &\quad = 1 - 0.6^5 \\ &\quad = 0.92 \end{aligned}$$

Commonly, car cooling systems are controlled by electrically driven fans. Assuming that the lifetime T in hours of a particular make of fan can be modelled by an exponential distribution with $\lambda = 0.0003$ find the proportion of fans which will give at least 10000 hours service. If the fan is redesigned so that its lifetime may be modelled by an exponential distribution with $\lambda = 0.00035$, would you expect more fans or less to give at least 10000 hours service?

Your solution

We know that $f(t) = 0.0003e^{-0.0003t}$ so that the probability that a fan will give at least 10000 hours service is given by the expression

$$P(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.0003e^{-0.0003t}\,dt = -\left[e^{-0.0003t}\right]_{10000}^{\infty} = e^{-3} \approx 0.0498$$

Hence about 5% of the fans may be expected to give at least 10000 hours service. After the redesign, the calculation becomes

$$P(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.00035e^{-0.00035t}\,dt = -\left[e^{-0.00035t}\right]_{10000}^{\infty} = e^{-3.5} \approx 0.0302$$

and so only about 3.% of the fans may be expected to give at least 10000 hours service. Hence, after the redesign we expect *less* fans to give 10000 hours service.

Exercises

1. In the manufacture of petroleum the distilling temperature (T^oC) is crucial in determining the quality of the final product. T can be considered as a random variable uniformly distributed over 150^{o}C to 300^{o}C. It costs £C_1 to produce 1 gallon of petroleum. If the oil distills at temperatures less than 200^{0}C the product sells for £C_2 per gallon. If it distills at a temperature greater than 200^{o}C it sells for £C_3 per gallon. Find the expected net profit per gallon.

2. A target is made of three concentric circles of radii $1/\sqrt{3}$, 1 and $\sqrt{3}$ metres. Shots within the inner circle count 4 points, in the next ring 3 points and within the third ring 2 points. (Shots outside the target count zero.) The distance of a shot from the centre of the target is a random variable R with density function. $f(r) = \frac{2}{\pi(1+r^2)}$, $r > 0$. Calculate the expected value of the score after five shots.

3. A continuous random variable T has the following probability density function.

$$f_T(u) = \begin{cases} 0 & (u < 0) \\ 3(1 - u/k) & (0 \leq u \leq k) \\ 0 & (u > k) \end{cases} .$$

   Find

   (a) k.

   (b) $\mathrm{E}(T)$.

   (c) $\mathrm{E}(T^2)$.

   (d) $\mathrm{var}(T)$.

4. A continuous random variable X has the following probability density function

$$f_X(u) = \begin{cases} 0 & (u < 0) \\ ku & (0 \leq u \leq 1) \\ 0 & (u > 1) \end{cases}$$

   (a) Find k.

   (b) Find the distribution function $F_X(u)$.

   (c) Find $\mathrm{E}(X)$.

   (d) Find $\mathrm{var}(X)$.

   (e) Find $\mathrm{E}(e^X)$.

   (f) Find $\mathrm{var}(e^X)$.

   (g) Find the distribution function of e^X. (Hint: For what values of X is $e^X < u$?)

   (h) Find the probability density function of e^X.

   (i) Sketch $f_X(u)$.

   (j) Sketch $F_X(u)$.

Exercises Continued

5. It is believed that the time X for a worker to complete a certain task has probability density function $f_X(x)$ where

$$f_X(x) = \begin{cases} 0 & (x \leq 0) \\ kx^2 e^{-\lambda x} & (x > 0) \end{cases}$$

where λ is a parameter, the value of which is unknown, and k is a constant which depends on λ.

(a) Show that if

$$I_n = \int_0^\infty x^n e^{-\lambda x}.dx$$

then

$$I_n = \frac{n}{\lambda} I_{n-1},$$

where $n > 0$ and $\lambda > 0$. Evaluate

$$I_0 = \int_0^\infty e^{-\lambda x}.dx$$

and hence find a general expression for I_n. This result can be used in the rest of this question.

(b) Find, in terms of λ, the value of k.

(c) Find, in terms of λ, the expected value of X.

(d) Find, in terms of λ, the variance of X.

(e) Write down the expected value and variance of the sample mean of a sample of n independent observations on X.

(f) Find, in terms of λ, the expected value of X^{-1}.

Answers 1.

$$P(X < 200) = 50.\frac{1}{150} = \frac{1}{3} \qquad P(X > 200) = \frac{2}{3}$$

Let F be a random variable defining profit.

F can take two values £$(C_2 - C_1)$ or £$(C_3 - C_1)$

x	$C_2 - C_1$	$C_3 - C_1$
$P(F = x)$	$^1/_3$	$^2/_3$

$$E(F) = \left[\frac{C_2 - C_1}{3}\right] + \frac{2}{3}[C_3 - C_1] = \frac{C_2 - 3C_1 + 2C_3}{3}$$

2.

$$\begin{aligned} P\ (\text{inner hit}) &= P\left(0 < r < \frac{1}{\sqrt{3}}\right) = \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{\pi(1+r^2)}\mathrm{d}r = \frac{2}{\pi}[\tan^{-1} r]_0^{\frac{1}{\sqrt{3}}} \\ &= \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{2}{\pi}\left(\frac{\pi}{6}\right) = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} P\ (\text{middle band}) &= P\left(\frac{1}{\sqrt{3}} < r < 1\right) \\ &= \int_{\frac{1}{\sqrt{3}}} \frac{2}{\pi(1+r^2)}\mathrm{d}r = \frac{2}{\pi}[\tan^{-1} r]_{\frac{1}{\sqrt{3}}}^{1} = \frac{2}{\pi}\tan^{-1} 1 - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

$$\begin{aligned} P(\text{outer band}) &= P(1 < r < \sqrt{3}) = \frac{2}{\pi}[\tan^{-1} r]_1^{\sqrt{3}} = \frac{2}{\pi}\tan^{-1}\sqrt{3} - \frac{1}{2} = \frac{1}{6} \\ P(\text{miss target}) &= 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{3} = \frac{1}{3} \end{aligned}$$

Let S be the random variable equal to 'score'.

s	0	2	3	4
$P(S = s)$	$^1/_3$	$^1/_6$	$^1/_6$	$^1/_3$

$$E(S) = \tfrac{2}{6} + \tfrac{3}{6} + \tfrac{4}{3} = \tfrac{13}{6}$$

The expected score after 5 shots is this value times 5 which is: $= 5\left(\frac{13}{6}\right) = 10.83$.

Continued 3.

(a)

$$1 = \int_0^k 3(1 - u/k).du = \left[3\left(u - \frac{u^2}{2k}\right)\right]_0^k = 3(k - k/2)$$

So $k = 2/3$.

(b)

$$\begin{aligned} \mathrm{E}(T) &= \int_0^{2/3} 3u(1 - 3u/2).du = 3\int_0^{2/3} u - 3u^2/2.du \\ &= 3\left[\frac{u^2}{2} - \frac{u^3}{2}\right]_0^{2/3} = 3\left(\frac{2}{9} - \frac{4}{27}\right) \\ &= 3\left(\frac{6-4}{27}\right) = \frac{2}{9}. \end{aligned}$$

(c)

$$\begin{aligned} \mathrm{E}(T^2) &= \int_0^{2/3} 3u^2(1 - 3u/2).du = 3\int_0^{2/3} u^2 - 3u^3/2.du \\ &= 3\left[\frac{u^3}{3} - \frac{3u^4}{8}\right]_0^{2/3} = 3\left(\frac{8}{81} - \frac{6}{81}\right) \\ &= 3\left(\frac{8-6}{81}\right) = \frac{2}{27} \end{aligned}$$

(d)

$$\mathrm{var}(T) = \mathrm{E}(T^2) - [\mathrm{E}(T)]^2 = \frac{2}{27} - \frac{4}{81} = \frac{2}{81}.$$

Continued

4.

(a)

$$1 = \int_0^1 ku.du = \left[\frac{ku^2}{2}\right]_0^1 = \frac{k}{2}.$$

So $k = 2$.

(b)

$$F_X(u) = \begin{cases} 0 & (u < 0) \\ u^2 & (0 \le u \le 1) \\ 1 & (1 < u) \end{cases} .$$

(c)

$$\text{E}(X) = \int_0^1 2u^2.du = \left[\frac{2u^3}{3}\right]_0^1 = \frac{2}{3}.$$

(d)

$$\text{E}(X^2) = \int_0^1 2u^3.du = \left[\frac{2u^4}{4}\right]_0^1 = \frac{1}{2}.$$

So

$$\text{var}(X) = \text{E}(X^2) - [\text{E}X]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

(e)

$$\begin{aligned} \text{E}(e^X) &= \int_0^1 2ue^u.du = [2ue^u]_0^1 - 2\int_0^1 e^u.du \\ &= [2ue^u - 2e^u]_0^1 = 2e - 2e + 2 = 2 \end{aligned}$$

(f)

$$\begin{aligned} \text{E}(e^{2X}) &= \int_0^1 2ue^{2u}.du = [ue^{2u}]_0^1 - \int_0^1 e^{2u}.du \\ &= [ue^{2u} - e^{2u}/2]_0^1 = e^2 = e^2/2 + 1/2 \\ &= (e^2 + 1)/2 \end{aligned}$$

So

$$\text{var}(e^X) = \text{E}(e^{2X}) - [\text{E}(e^X)]^2 = (e^2 + 1)/2 - 4.$$

Continued

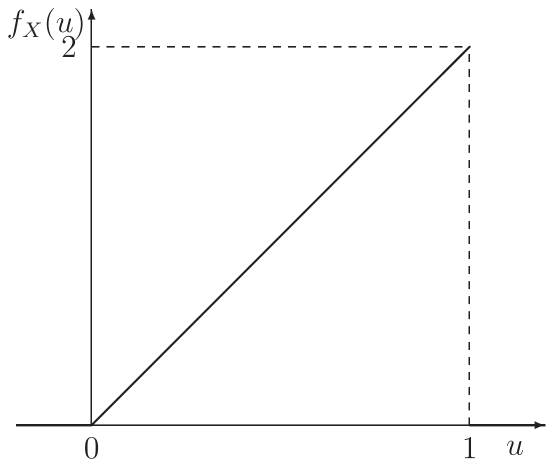
(g) $\Pr(e^X < u) = \Pr(X < \ln u) = (\ln u)^2$ for $0 < \ln u < 1$, i.e. $1 < u < e$.
Hence the distribution function of e^X is

$$F_{e^X}(u) = \begin{cases} 0 & (u < 1) \\ (\ln u)^2 & (\leq u \leq e) \\ 1 & (e < u) \end{cases}$$

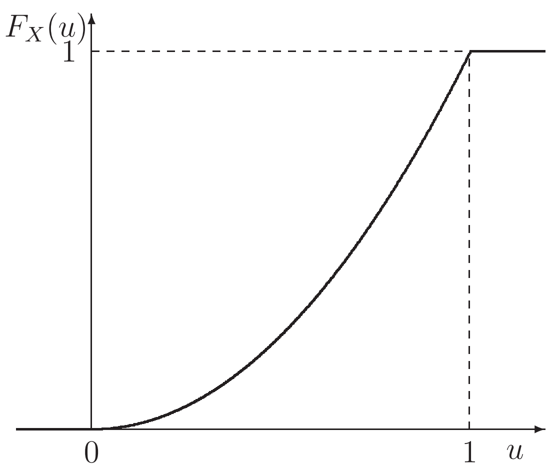
(h) The pdf of e^X is

$$f_{e^X}(u) = \begin{cases} 0 & (u < 1) \\ \frac{2 \ln u}{u} & (\leq u \leq e) \\ 0 & (e < u) \end{cases}$$

(i) Sketch of pdf:



(j) Sketch of distribution function:



Continued

5.

(a)

$$\begin{aligned} I_n = \int_0^\infty x^n e^{-\lambda x}.dx &= \left[-\frac{1}{\lambda}x^n e^{-\lambda x}\right]_0^\infty + \frac{n}{\lambda}\int_0^\infty x^{n-1}e^{-\lambda}.dx \\ &= \frac{n}{\lambda}I_{n-1} \\ I_0 = \int_0^\infty e^{-\lambda x}.dx &= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda} \end{aligned}$$

Hence

$$I_n = \frac{n!}{\lambda^{n+1}}.$$

(b)

$$\int_0^\infty kx^2 e^{-\lambda x}.dx = 1 \quad \Rightarrow \quad kI_2 = 1$$

$$\Rightarrow \quad k = \frac{1}{I_2} = \frac{\lambda^3}{2}$$

(c)

$$\mathrm{E}(X) = \int_0^\infty x f_X(x).dx = kI_3 = \frac{\lambda^3}{2}\frac{6}{\lambda^4} = \frac{3}{\lambda}$$

(d)

$$\mathrm{E}(X^2) = \int_0^\infty x^2 f_X(x).dx = kI_4 = \frac{\lambda^3}{2}\frac{24}{\lambda^5} = \frac{12}{\lambda^2}$$

So

$$\mathrm{var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{12}{\lambda^2} - \frac{9}{\lambda^2} = \frac{3}{\lambda^2}$$

(e)

$$\begin{aligned} \mathrm{E}(\bar{X}) &= \frac{3}{\lambda} \\ \mathrm{var}(\bar{X}) &= \frac{3}{n\lambda^2} \end{aligned}$$

(f)

$$\mathrm{E}\left(\frac{1}{X}\right) = \int_0^\infty \frac{1}{x} f_X(x).dx - kI_1 = \frac{\lambda^3}{2}\frac{1}{\lambda^2} = \frac{\lambda}{2}$$