Contents 4

trigonometry **trigonometry**

Learning **outcomes**

needs doing.

Time **allocation**

You are expected to spend approximately thirteen hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Right-angled Triangles

4.1

Introduction

Right-angled triangles (that is triangles where one of the angles is 90°) are the easiest vehicle for introducing trigonometry. Since the sum of the three angles in a triangle is 180° it follows that in a right-angled triangle there are no obtuse angles (i.e. angles greater than 90°).

Prerequisites

Before starting this Section you should . . .

① have a basic knowledge of the geometry of triangles

Learning Outcomes

After completing this Section you should be able to . . .

✓ define trigonometric functions both in right-angled triangles and more generally

✓ express angles in degrees or radians

✓ obtain all the angles and sides in any right-angled triangle given certain information

1. Right angled triangles

Look at Figure 1 which could, for example, be a profile of a hill with a constant gradient.

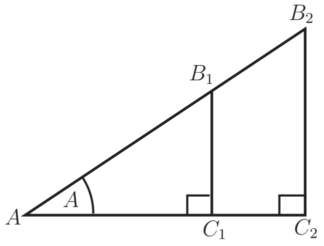


Figure 1

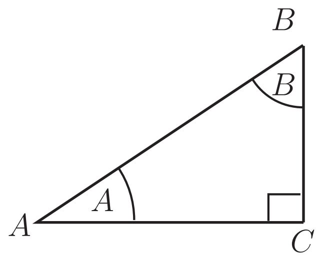
The two right-angled triangles AB_1C_1 and AB_2C_2 are **similar** (because the three angles of triangle AB_1C_1 are equal to the equivalent 3 angles of triangle AB_2C_2). From the basic properties of similar triangles corresponding sides have the same ratio. Thus, for example,

$$\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2} \qquad (1)$$

$$\frac{AC_1}{AB_1} = \frac{AC_2}{AB_2} \qquad (2)$$

The values of the ratios (1) and (2) will clearly depend on the angle A of inclination. These ratios are called, respectively, the **sine** and **cosine** of the angle A, these being abbreviated to $\sin A$ and $\cos A$ respectively.

Key Point



$$\sin A = \frac{BC}{AB}$$

$$\cos A = \frac{AC}{AB}$$

Figure 2

Now AC is the side **adjacent** to angle A and side BC is the side **opposite** to angle A. Also BC is the **hypotenuse** of the triangle (the longest side).

We can write, therefore, for any right-angled triangle containing an angle θ (not the right-angle)

$$\sin\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent to angle } \theta}{\text{length of hypotenuse}}$$

Referring again to Figure 2 write down the ratios which give $\sin B$ and $\cos B$.

Your solution

$$\sin B = \frac{AC}{AB} \qquad \cos B = \frac{BC}{AB}$$

Note that $\sin B = \cos A = \cos(90^\circ - B)$
$\cos B = \sin A = \sin(90^\circ - B)$

A third result of importance from Figure 1 is

$$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2}$$

These ratios is referred to as the **tangent** of the angle at A, written $\tan A$.

Key Point

In Figure 2

$$\tan A = \frac{BC}{AC} = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

In general, for an angle θ in any right-angled triangle (again not the right-angle)

$$\tan\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta}$$

Summary

Using Opp to denote the length of the side opposite to angle θ, Adj to denote length of side adjacent to angle θ, and Hyp to denote length of hypotenuse: we have, for any right-angled triangle,

$$\sin\theta = \frac{Opp}{Hyp} \qquad \cos\theta = \frac{Adj}{Hyp} \qquad \tan\theta = \frac{Opp}{Adj}$$

These are sometimes memorised as SOH, CAH and TOA respectively.
These three ratios are called **trigonometric ratios**

Write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Your solution

$$\tan\theta = \frac{Opp}{Adj} = \frac{Opp}{Adj}\cdot\frac{Hyp}{Hyp} = \frac{Opp}{Hyp}\cdot\frac{Hyp}{Adj} = \frac{Opp}{Hyp}\Big/\frac{Adj}{Hyp}$$

i.e. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Example Use the isosceles 45°, 45°, 90° triangle in Figure 3 to obtain the sine, cosine and tangent of 45°.

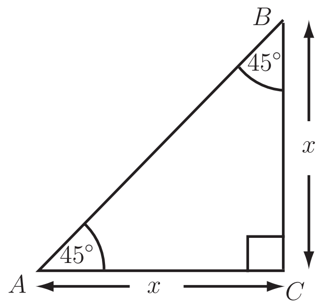


Figure 3

Solution

By Pythagoras' Theorem

$$(AB)^2 = x^2 + x^2 = 2x^2$$

so

$$AB = x\sqrt{2}$$

Hence

$$\sin 45° = \frac{BC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{BC}{AC} = \frac{x}{x} = 1$$

(or, of course, $\tan 45° = \dfrac{\sin 45°}{\cos 45°} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$)

Using the triangle ABC in Figure 4 which can be regarded as one half of the equilateral triangle ABD, calculate the three trigonometric ratios for the angles 30° and 60°.

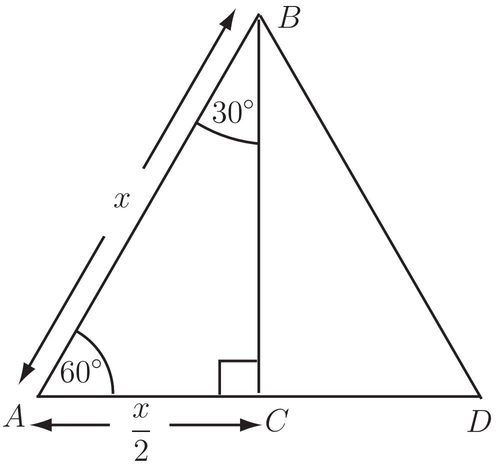


Figure 4

Your solution

By Pythagoras' Theorem:

$$(BC)^2 = (AB)^2 - (AC)^2 = x^2 - \frac{x^2}{4} = \frac{3x^2}{4} \quad \text{so} \quad BC = \frac{\sqrt{3}}{2}x$$

Hence

$$\sin 60^\circ = \frac{BC}{AB} = \frac{x\frac{\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2} \qquad \sin 30^\circ = \frac{AC}{AB} = \frac{\frac{x}{2}}{x} = \frac{1}{2}$$

$$\cos 60^\circ = \frac{AC}{AB} = \frac{1}{2} \qquad \cos 30^\circ = \frac{BC}{AB} = \frac{x\frac{\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can of course be obtained by calculator. When entering the angle in degrees (e.g. 30°) the calculator must be in degree mode. (Typically this is ensured by pressing the DRG button until 'DEG' is shown on the display). The keystrokes for $\sin 30^\circ$ are usually simply sin 30

or, on some calculators, 30 sin perhaps followed by =.

Use your calculator to check the values of $\sin 45°,\ \cos 30°$ and $\tan 60°$ obtained above.
Also obtain $\sin 3.2°,\ \cos 86.8°,\ \tan 28°15'$.
($'$ denotes a minute where 1 minute $= \dfrac{1}{60}^{\circ}$)

Your solution

$\sin 3.2° = \cos 86.8° = 0.0558$ (to 4 decimal places)

Since $15' = 0.25°$ $\quad \tan 28°15' = \tan 28.25° = 0.5373$ to 4 d.p.

Inverse Trigonometric functions (A first look)

Consider, by way of example, a right-angled triangle with sides 3, 4 and 5 cm, see Figure 5.

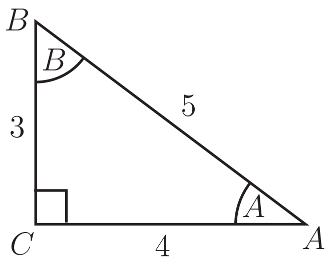


Figure 5

Suppose we wish to find the angles at A and B.
Clearly $\sin A = \dfrac{3}{5},\ \cos A = \dfrac{4}{5},\ \tan A = \dfrac{3}{4}$.

so we need to solve one of the above three equations to find A.
Using $\sin A = \dfrac{3}{5}$ we write

$$A = \sin^{-1}\left(\frac{3}{5}\right)$$

(read as 'A is the inverse sine of $\frac{3}{5}$')

The value of A can be obtained by calculator using the '$\sin^{-1}$' button (often a second function to the sin function and accessed using a SHIFT or INV or SECOND FUNCTION key.)

Thus to obtain $\sin^{-1}\left(\frac{3}{5}\right)$ we might use the following keystrokes:

INV SIN 0.6 (=)

or

3 ÷ 5 INV SIN (=)

We find $\sin^{-1}\frac{3}{5} = 36.87°$ (to 4 significant figures).

Key Point

In general

$$\text{if} \quad \sin\theta = x \quad \text{then} \quad \theta = \sin^{-1} x$$

Similarly

$$\cos\theta = y \quad \text{implies} \quad \theta = \cos^{-1} y \qquad \text{and} \qquad \tan\theta = z \quad \text{implies} \quad \theta = \tan^{-1} z$$

(The notations arc sin, arc cos, arc tan are sometimes used for these **inverse trigonometric functions**).

Check the value of the angle at A in Figure 5 above using the $\cos^{-1}$ and $\tan^{-1}$ functions on your calculator. Obtain the angle at B similarly (clearly $A + B = 90°$). Give answers to 4 sig. figs.

Your solution

$$A = \cos^{-1}\frac{4}{5} = 36.87° \qquad B = \cos^{-1}\frac{3}{5} = 53.13°$$

$$A = \tan^{-1}\frac{3}{4} = 36.87° \qquad B = \tan^{-1}\frac{4}{3} = 53.13°$$

You should note carefully that $\sin^{-1} x$ does **not** mean $\dfrac{1}{\sin x}$.

Indeed the function $\dfrac{1}{\sin x}$ has a special name – the **cosecant** of x, written $\text{cosec}\, x$.

Similarly

$$\sec x \equiv \frac{1}{\cos x} \qquad \text{(the } \textbf{secant} \text{ function)}$$

$$\cot x \equiv \frac{1}{\tan x} \qquad \text{(the } \textbf{cotangent} \text{ function)}$$

are further (but less used) trigonometric functions.

Use your calculator to obtain

$$\text{cosec}\, 38.5° \qquad \sec 22.6° \qquad \cot 88.32°$$

(Use the sin, cos or tan buttons unless your calculator has specific buttons for cosec , sec and cot.)

Your solution

$$\text{cosec}\, 38.5° = \frac{1}{\sin 38.5°} = \frac{1}{0.6225} = 1.6064$$

$$\sec 22.6° = \frac{1}{\cos 22.6°} = \frac{1}{0.9232} = 1.0832$$

$$\cot 88.32° = \frac{1}{\tan 88.32°} = \frac{1}{34.09} = 0.0293$$

2. Solving right-angled triangles

The title phrase means obtaining the values of all the angles and all the sides of a given right-angled triangle using the trigonometric functions (and, if necessary, the inverse trigonometric function) and, perhaps, Pythagoras' Theorem.

There are three cases to be considered:

Case 1 Given the hypotenuse and an angle: we use the sin or cosine as appropriate:

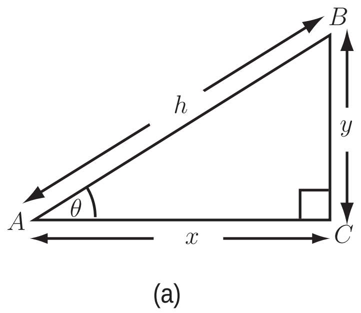


Figure 6(a)

Assuming h and θ in Figure 6(a) are given then

$$\cos\theta = \frac{x}{h} \text{ whichgives, on transposing, } x = h\cos\theta$$

from which x can be calculated.

Also

$$\sin\theta = \frac{y}{h}$$

allows y to be calculated as $y = h\sin\theta$.

Clearly the third angle of this triangle (at B) is $90^\circ - \theta$.

Case 2 Given a side other than the hypotenuse and also an angle we use the tan:
If x and θ are known then, in Figure 6(a),

$$\tan\theta = \frac{y}{x} \qquad \text{so} \quad y = x\tan\theta$$

which enables us to calculate y.
If y and θ are known then

$$\tan\theta = \frac{y}{x} \quad \text{gives} \quad x = \frac{y}{\tan\theta}$$

from which x can be calculated.

In either of these cases the hypotenuse could be calculated using Pythagoras' theorem viz.

$$h = \sqrt{x^2 + y^2}$$

Case 3 Obtaining an angle if two of the sides are given:

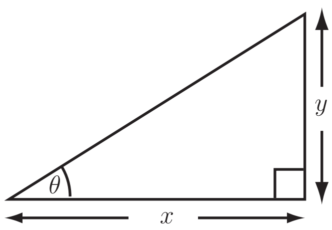


Figure 6(b)

$$\tan\theta = \frac{y}{x} \qquad \text{so} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

enables us to obtain θ.

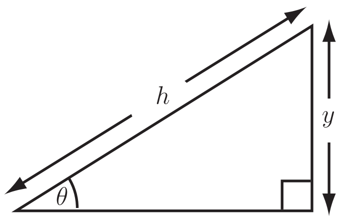


Figure 6(c)

$$\sin\theta = \frac{y}{h} \qquad \text{so} \qquad \theta = \sin^{-1}\left(\frac{y}{h}\right)$$

can be used to find θ.

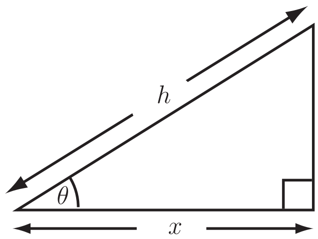


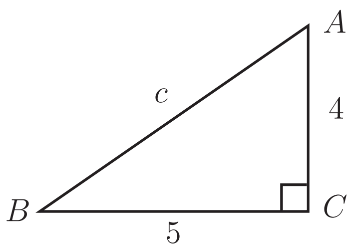
Figure 6(d)

$$\cos\theta = \frac{x}{h} \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{x}{h}\right)$$

is the relevant formula.

Note: in this third case (i.e. when two sides are given) we can use Pythagoras' Theorem to obtain the length of the third side at the outset.

Obtain all the angles and the remaining side for the triangle shown:



Your solution

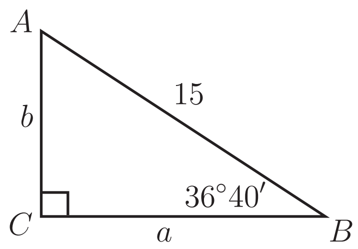
This is case 3 above.

To obtain the angle at B we use $\tan B = \frac{4}{5}$ so $B = \tan^{-1}(0.8) = 38.66^\circ$.

Then the angle at A is $180^\circ - (90^\circ - 38.66^\circ) = 51.34^\circ$.

By Pythagoras' Theorem $c = \sqrt{4^2 + 5^2} = \sqrt{41} \approx 6.40$.

Obtain the remaining sides and angles in the triangle shown.



Your solution

This is case 1 above.

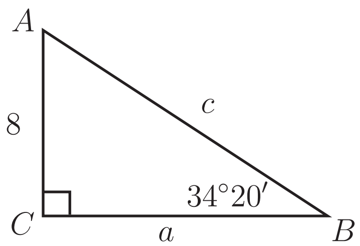
In this case since $31^\circ 40' = 31.6731^\circ$ then $\cos 31.6731^\circ = \frac{a}{15}$ so $a = 15 \cos 31.6731^\circ = 12.77$

The angle at A is $180^\circ - (90 + 31.6731^\circ) = 58.3331^\circ$.

Finally $\sin 31.6731^\circ = \frac{b}{15}$ $\therefore$ $b = 15 \sin 31.6731^\circ = 7.85$.

(Alternatively of course Pythagoras' Theorem could be used to calculate the length b).

Obtain the remaining sides and angles of the following triangle.



Your solution

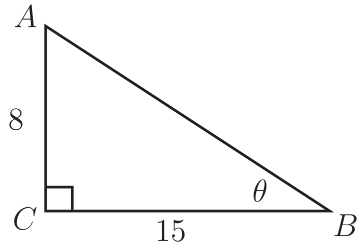
This is case 2 above.

Here $\tan 34.34^\circ = \frac{8}{a}$ so $a = \frac{8}{\tan 34.34^\circ} = 11.7$

Also $c = \sqrt{8^2 + 11.7^2} = 14.18$ and the angle at A is $180^\circ - (90^\circ + 34.34^\circ) = 55.66^\circ$.

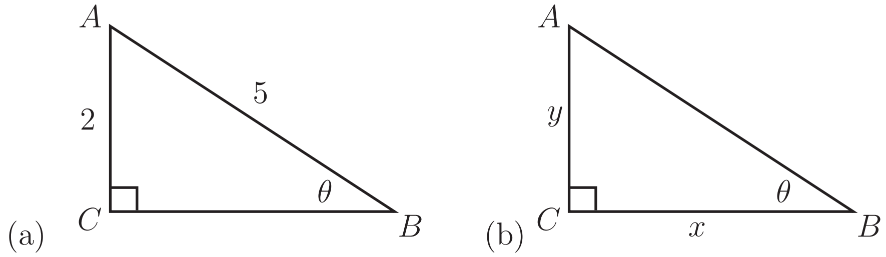
Exercises

1. Obtain $\sin\theta$, $\cos\theta$, $\tan\theta$, $\text{cosec}\,\theta$, $\sec\theta$, $\cot\theta$ in the following right-angled triangle.



Finally use your calculator to obtain the value of θ.

2. Write down the 6 trigonometric functions of the angle θ for each of the following triangles:

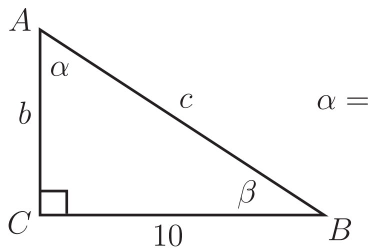


3. If θ is an acute angle such that $\sin\theta = 2/7$ obtain, without use of a calculator, $\cos\theta$ and $\tan\theta$.

4. Use your calculator to obtain the acute angles θ satisfying

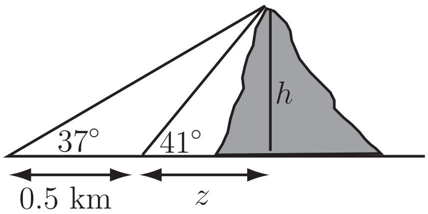
(a) $\sin\theta = 0.5260$, (b) $\tan\theta = 2.4$, (c) $\cos\theta = 0.2$

5. Solve the right-angled triangle shown:



$\alpha = 57.5°$

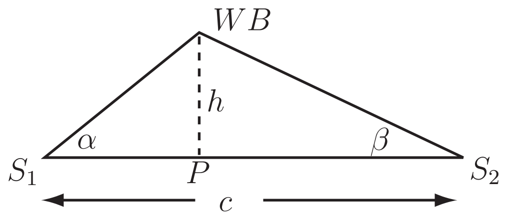
6. A surveyor measures the **angle of elevation** between the top of a mountain and ground level at two different points. The results are shown in the following figure. Use trigonometry to obtain the distance z (which cannot be measured) and then obtain the height h of the mountain.



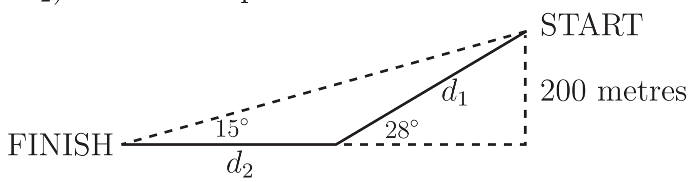
Exercises

7. As shown below two tracking stations S_1 and S_2 sight a weather balloon (WB) between them at elevation angles α and β respectively. Show that the height h of the balloon is given by

$$h = \frac{c}{\cot\alpha + \cot\beta}$$



8. An entry in a soap box derby rolls down a hill as shown in the figure. Find the total distance $(d_1 + d_2)$ that the soap box travels.



1. By Pythagoras' Theorem the hypotenuse of the triangle has length

$$h = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$$

Then using the fundamental definitions of the trigonometric ratios

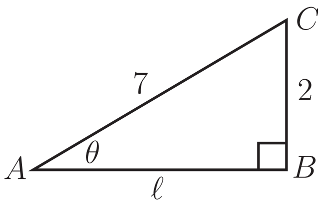
$$\sin\theta = \frac{8}{17} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{17}{8}$$

$$\cos\theta = \frac{15}{17} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{17}{15}$$

$$\tan\theta = \frac{8}{15} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{15}{8}$$

$$\theta = \sin^{-1}\frac{8}{17} \quad \text{(for example)} \qquad \therefore \quad \theta = 28.07^\circ$$

2. Referring to the following figure



Then the length of side AB is, using Pythagoras' Theorem $\ell = \sqrt{7^2 - 2^2} = \sqrt{45} = 3\sqrt{5}$.

Hence $\cos\theta = \dfrac{3\sqrt{5}}{7}$ $\quad \tan\theta = \dfrac{2}{3\sqrt{5}} = \dfrac{2\sqrt{5}}{15}$

3. (a)

$$\sin\theta = \frac{2}{5} \qquad \operatorname{cosec}\theta = \frac{5}{2}$$

$$\cos\theta = \frac{\sqrt{21}}{5} \qquad \sec\theta = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

$$\tan\theta = \frac{2\sqrt{21}}{21} \qquad \cot\theta = \frac{\sqrt{21}}{2}$$

(b)

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad \operatorname{cosec}\theta = \frac{\sqrt{x^2 + y^2}}{y}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sec\theta = \frac{\sqrt{x^2 + y^2}}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

4. (a) $\theta = \sin^{-1} 0.5260 = 31.73^\circ$ (b) $\theta = \tan^{-1} 2.4 = 67.38^\circ$ (c) $\theta = \cos^{-1} 0.2 = 78.46^\circ$

5. $\beta = 90 - \alpha = 32.5^\circ$. Now $\tan 57.5^\circ = \dfrac{10}{b}$ $\therefore$ $b = \dfrac{10}{\tan 57.5^\circ} = \dfrac{10}{1.56969} \simeq 6.37$

Also $\sin 57.5^\circ = \dfrac{10}{c}$ $\therefore$ $c = \dfrac{10}{\sin 57.5^\circ} \simeq 11.86$

6. We have 2 right-angled triangles both of which have one side h

$$\therefore \quad \tan 37^\circ = \frac{h}{z + 0.5} \qquad \tan 41^\circ = \frac{h}{z}$$

from which $h = (z + 0.5)\tan 37^\circ = \tan 41^\circ$. Solving the right-hand pair of equations for z

$$z \tan 37^\circ - z \tan 41^\circ = -0.5 \tan 37^\circ$$

$$\therefore \quad z = \frac{-0.5 \tan 37^\circ}{\tan 37^\circ - \tan 41^\circ} \simeq 3.2556\text{km. Then} \quad h = z \tan 41^\circ = 3.2556 \tan 41^\circ \simeq 2.83 \text{ km}$$

7. Since the required answer is in terms of $\cot\alpha$ and $\cot\beta$ we proceed as follows:

(Using x to denote the distance S_1P) $\cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{x}{h}$ $\qquad \cot\beta = \dfrac{1}{\tan\beta} = \dfrac{c - x}{h}$

Adding: $\cot\alpha + \cot\beta = \dfrac{x}{h} + \dfrac{c}{h} - \dfrac{x}{h} = \dfrac{c}{h}$ $\qquad \therefore \quad h = \dfrac{c}{\cot\alpha + \cot\beta}$ as required.

8. From the smaller right-angled triangle $d_1 = \dfrac{200}{\sin 28^\circ} = 426$ metres. The base of this triangle then has length

$$\ell = 426 \cos 28^\circ = 376.1 \text{ metres}$$

From the larger right-angled triangle the straight line distance from START to FINISH is $\dfrac{200}{\sin 15^\circ} = 772.7$ metres. Then, using Pythagoras' Theorem

$$(d_2 + \ell) = \sqrt{772.7^2 - 200^2} = 746.41 \text{ metres}$$

from which $d_2 = 370.3$ feet $\therefore$ $\quad d_1 + d_2 = 796.3$ feet