Trigonometric Functions

4.2

Introduction

Our discussion so far has been limited to right-angled triangles where, apart from the right-angle itself, all angles are necessarily less than 90°. We now extend the definitions of the trigonometric functions to any size angle, an extension which greatly broadens the range of applications of trigonometry.

Prerequisites

Before starting this Section you should . . .

① have a basic knowledge of the geometry of triangles

Learning Outcomes

After completing this Section you should be able to . . .

✓ define trigonometric functions generally

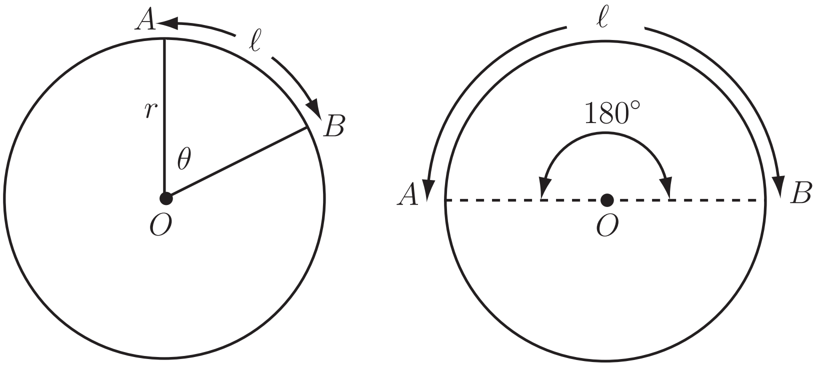
✓ express angles in degrees or radians

✓ be familiar with the graphs of the three main trigonometric functions

1. Trigonometric functions for any size angle

The radian

First we consider an alternative to measuring angles in degrees. Look at the circle shown in Figure 7(a). It has radius r and we have shown an arc AB of length ℓ (measured in the same units as r). As you can see the arc subtends an angle θ at the centre O of the circle.



(a) (b)

Figure 7

The angle θ in **radians** is defined as

$$\theta = \frac{\text{length of arc } AB}{\text{radius}} = \frac{\ell}{r}$$

So, for example, if $r = 10$ cm, $\ell = 20$ cm, the angle θ would be $\frac{20}{10} = 2$ radians.

The relation between the value of an angle in radians and its value in degrees is readily obtained as follows.

Referring to Figure 7(b) imagine that the arc AB extends to cover half the complete perimeter of the circle. The arc length is now πr (half the circumference of the circle) so the angle θ subtended by AB is now

$$\theta = \frac{\pi r}{r} = \pi \quad \text{radians}$$

But clearly this angle is 180°. Thus π radians is the same as 180°.

Key Point

$$\begin{aligned} 180^\circ &= \pi \ \text{ radians} \\ \text{By proportion} \qquad 360^\circ &= 2\pi \ \text{ radians} \\ 1^\circ &= \frac{\pi}{180} \ \text{ radians} \\ x^\circ &= \frac{\pi x}{180} \ \text{ radians} \end{aligned}$$

Write down the values in radians of $30°,\ 45°,\ 60°,\ 120°,\ 135°,\ 150°$ (Leave your answers as multiples of π).

Your solution

$$
\begin{aligned}
30° &= \pi \times \frac{30}{180} = \frac{\pi}{6} \text{ radians} \\
45° &= \pi \times \frac{45}{180} = \frac{\pi}{4} \text{ radians} \\
60° &= \pi \times \frac{60}{180} = \frac{\pi}{3} \text{ radians} \\
90° &= \frac{\pi}{2} \text{ radians} \\
120° &= \frac{2\pi}{3} \text{ radians} \\
135° &= \frac{3\pi}{4} \text{ radians} \\
150° &= \pi \times \frac{150}{180} = \frac{5\pi}{6} \text{ radians}
\end{aligned}
$$

Note conversely that since π radians $= 180°$ then 1 radian $= \dfrac{180}{\pi}$ degrees (about $57.3°$) and y radians $= \dfrac{180y}{\pi}$ degrees.

Write in degrees the following angles given in radians

$$\frac{\pi}{10},\quad \frac{\pi}{5},\quad \frac{7\pi}{10},\quad \frac{6\pi}{5},\quad \frac{23\pi}{12}$$

Your solution

$\frac{\pi}{10}$ radian equals $\frac{180}{\pi} \times \frac{\pi}{10} = 18°$

$\frac{\pi}{5}$ radian equals $\frac{180}{\pi} \times \frac{\pi}{5} = 36°$

$\frac{7\pi}{10}$ radian equals $126°$

$\frac{6\pi}{5}$ radian equals $216°$

$\frac{23\pi}{12}$ radian equals $\frac{180}{\pi} \times \frac{23\pi}{12} = 345°$

Put your calculator into **radian mode** (using the DRG button) to do this exercise: Verify the results

$$\sin 30° = \frac{1}{2} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 60° = \sqrt{3}$$

by using your calculator in radian mode. (Use the π button to obtain the value for π.)

Your solution

$$\sin 30° = \sin\left(\frac{\pi}{6}\right) = 0.5 = \frac{1}{2}$$

$$\cos 45° = \cos\left(\frac{\pi}{4}\right) = 0.7071 = \frac{1}{\sqrt{2}}$$

$$\tan 60° = \tan\left(\frac{\pi}{3}\right) = 1.7320 = \sqrt{3}$$

2. General definitions of trigonometric functions

We now define the trigonometric functions in a more general way than in terms of ratios of sides of a right-angled triangle. To do this we consider a circle of **unit radius** whose centre is at the origin of a Cartesian coordinate system and an arrow (or **radius vector**) OP from the centre to a point P on the circumference of this circle. We are interested in the angle θ that the arrow makes with the **positive** x-axis. See Figure 8.

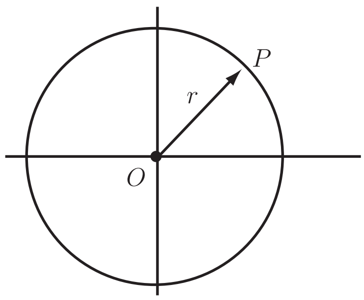
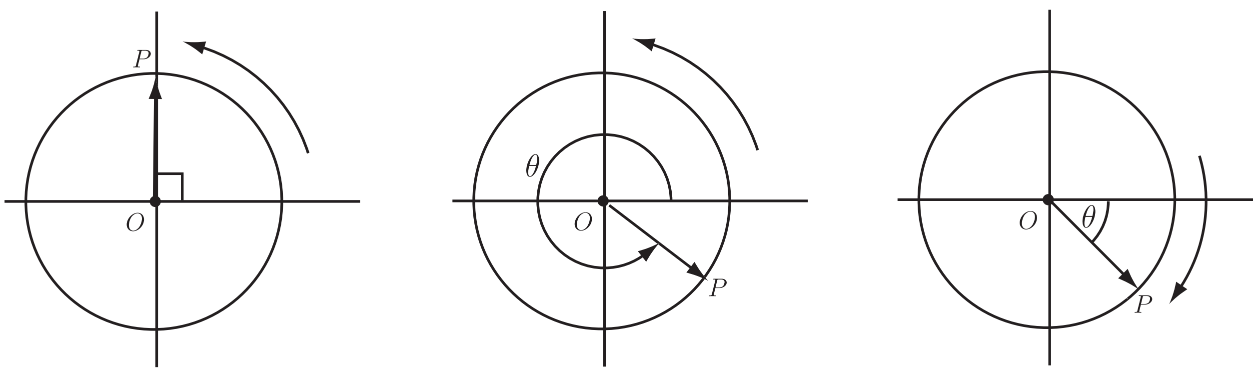


Figure 8

Imagine that the vector OP rotates in **anti-clockwise direction**. With this sense of rotation the angle θ is taken as positive whereas a **clockwise** rotation is taken as negative. See examples in Figure 9.



$\theta = 90° = \dfrac{\pi}{2}$ rads $\qquad \theta = 315° = \dfrac{7\pi}{4}$ rads $\qquad \theta = -45° = -\dfrac{\pi}{4}$ rads

Figure 9

The sine and cosine of an angle

For $0 \le \theta \le \dfrac{\pi}{2}$ (called the **first** quadrant) we have the following situation with our unit radius circle. See Figure 10.

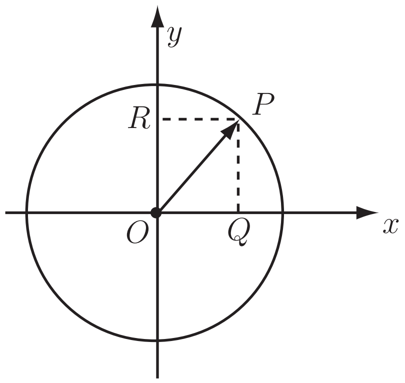


Figure 10

The **projection** of OP along the positive $x-$axis is OQ. But, in the right-angled triangle OPQ

$$\cos\theta = \frac{OQ}{OP}$$

or
$$OQ = OP\cos\theta = \cos\theta \qquad (3)$$

(since OP has unit length).

Similarly in this right-angled triangle

$$\sin\theta = \frac{PQ}{OP}$$

or
$$PQ = OP\sin\theta = \sin\theta$$

But $PQ = OR$

so
$$\sin\theta = OR \qquad (4)$$

Equation (3) tells us that we can interpret $\mathbf{cos}\,\theta$ as the projection of OP along the **positive *x*-axis** and $\mathbf{sin}\,\theta$ as the projection of OP along the **positive *y*-axis**.

We shall use these interpretations as the **definition** of $\sin\theta$ and $\cos\theta$ for **any** values of the angle θ.

Key Point

For a radius vector OP of a circle of unit radius making an angle θ with the positive $x-$axis as illustrated, for the first quadrant, in Figure 10 earlier:

$\cos\theta =$ projection OQ of OP along the positive $x-$axis

$\sin\theta =$ projection OR of OP along the positive $y-$axis

Sine and cosine in the four quadrants

First quadrant ($0 \le \theta \le 90°$).

See Figure 11.

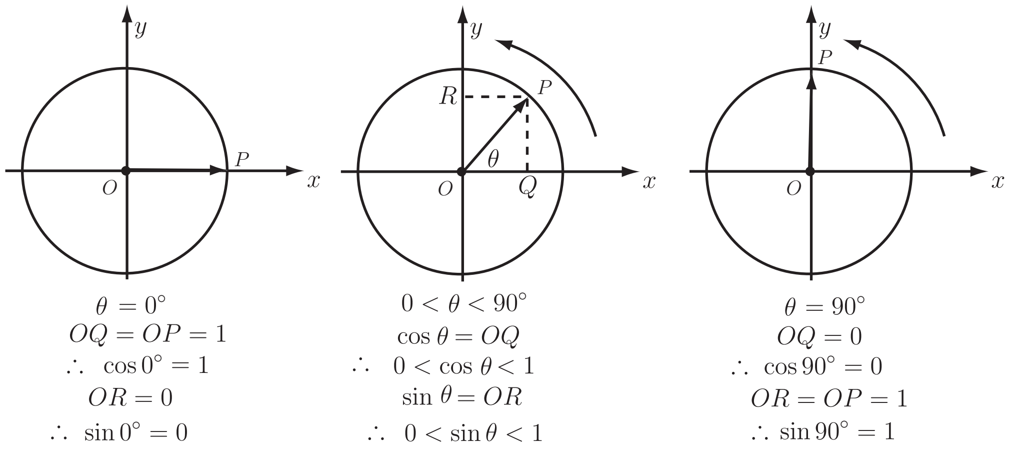


Figure 11

It follows from Figure 11 that $\cos\theta$ decreases from 1 to 0 as OP rotates from the horizontal position to the vertical, i.e. as θ increases from $0°$ to $90°$.

$\sin\theta = OR$ increases from 0 (when $\theta = 0$) to 1 (when $\theta = 90°$).

Second quadrant ($90° \le \theta \le 180°$).

Referring to Figure 12, remember that it is the projections along the **positive** x and y axes that are used to define $\cos\theta$ and $\sin\theta$ respectively.

It follows that as θ increases from $90°$ to $180°$, $\cos\theta$ decreases from 0 to -1 and $\sin\theta$ decreases from 1 to 0.

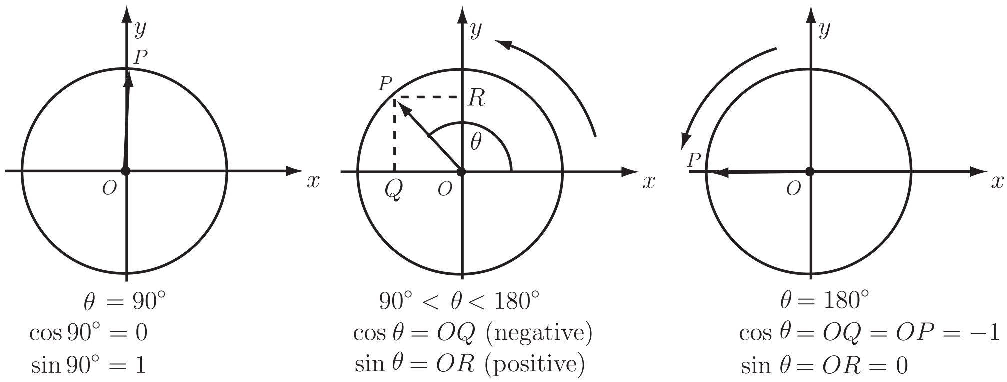


Figure 12

Considering for example an angle of $135°$, referring to Figure 13 (where we have not drawn the unit circle), by symmetry we have:

$$\sin 135° = OR = \sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 135° = OQ_2 = -OQ_1 = -\cos 45° = -\frac{1}{\sqrt{2}}$$

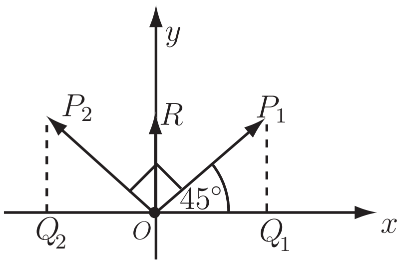


Figure 13

Key Point

In general

$$\sin(180 - x) = \sin x \qquad \text{and} \qquad \cos(180 - x) = -\cos x$$

Without using a calculator write down the values of

$$\sin 120°,\ \sin 150°,\ \cos 120°,\ \cos 150°,\ \tan 120°,\ \tan 150°.$$

(Assume that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for any value of θ)

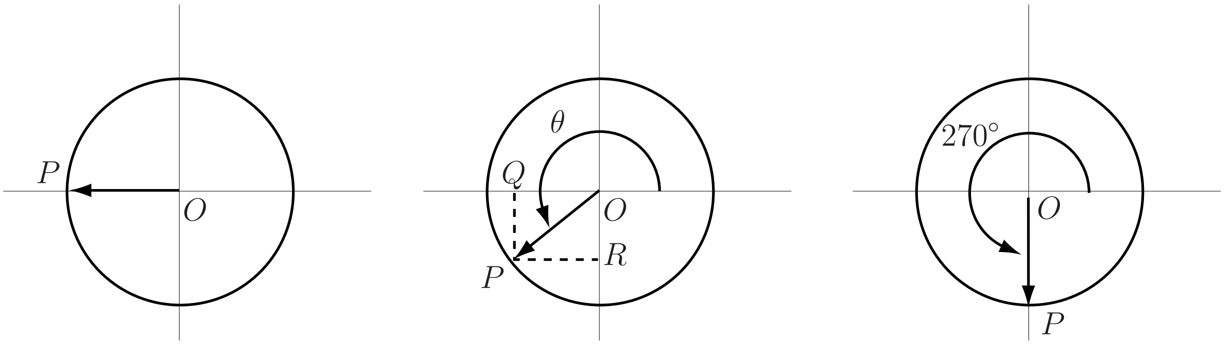
Your solution

$$\sin 120^\circ = \sin(180-60) = \sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \sin 150^\circ = \sin(180-30) = \sin 30^\circ = \frac{1}{2}$$

$$\cos 120^\circ = -\cos 60 = -\frac{1}{2} \qquad \cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 120^\circ = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3} \qquad \tan 150^\circ = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Third quadrant ($180^\circ \le \theta \le 270^\circ$).



$\cos 180^\circ = -1$
$\sin 180^\circ = 0$

$180^\circ < \theta < 270^\circ$
$\cos\theta = OQ$ (negative)
$\sin\theta = OR$ (negative)

$\theta = 270^\circ$
$\cos\theta = ?$
$\sin\theta = ?$

Figure 14

Using the projection definition write down the values of $\cos 270^\circ$ and $\sin 270^\circ$.

Your solution

$\cos 270^\circ = 0$ (OP has zero projection along postive $x-$axis)

$\sin 270^\circ = -1$ (OP is directed along the negative axis)

Thus in the third quadrant, as θ increases from 180° to 270° so $\cos\theta$ increases from -1 to 0 whereas $\sin\theta$ decreases from 0 to -1.

It follows, see Figure 15, with $\theta = 180° + x$

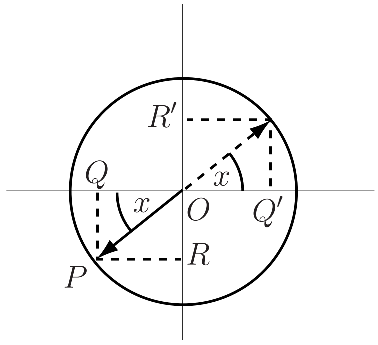


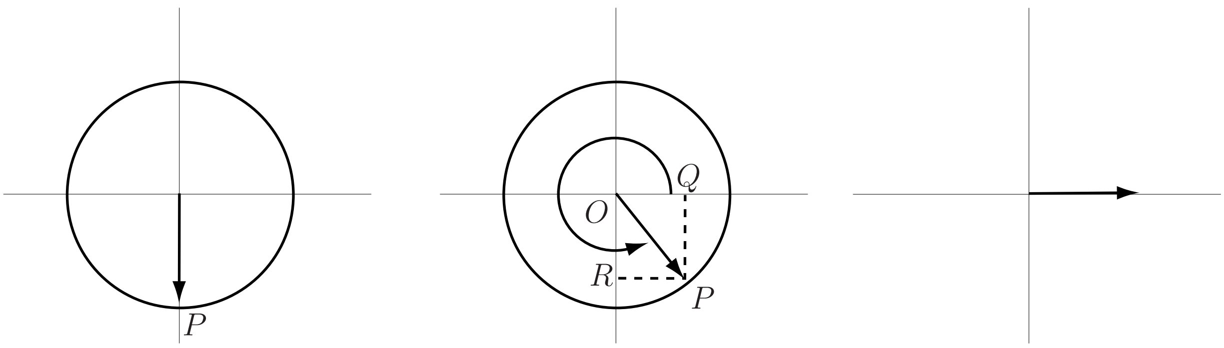
Figure 15

$$\begin{aligned} \sin\theta &= \sin(180+x) &= OR = -OR' &= -\sin x \\ \cos\theta &= \cos(180+x) &= OQ = -OQ' &= -\cos x. \end{aligned}$$

Key Point

$$\sin(180+x) = -\sin x$$
$$\cos(180+x) = -\cos x$$
$$\text{Hence} \quad \tan(180+x) = \frac{\sin(180°+x)}{\cos(180°+x)} = \frac{\sin x}{\cos x} = +\tan x$$

Fourth quadrant ($270° \leq \theta \leq 360°$)..



$\theta = 270°$
$\cos\theta = 0$
$\sin\theta = -1$

$270° < \theta < 360°$
(alternatively $-90° < \theta < 0$)
$\cos\theta = OQ > 0$
$\sin\theta = OR < 0$

$360°$
(results as for $0°$)

Figure 16

From Figure 16 the following results should be clear:

$$
\begin{aligned}
\cos(-\theta) &= \cos\theta \qquad \text{(both equal to } OQ\text{)}\\
\sin(-\theta) &= -\sin\theta\\
\text{Hence} \qquad \tan(-\theta) &= -\frac{\sin\theta}{\cos\theta} = -\tan\theta.
\end{aligned}
$$

Write down (no calculator) the values of

$$\sin 300^\circ \qquad \sin(-60^\circ) \qquad \cos 330^\circ \qquad \cos(-30^\circ)$$

Describe the behaviour of $\cos\theta$ and $\sin\theta$ as θ increases from 270° to 360°.

Your solution

$$\sin 300^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2} \qquad \cos 330^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin(-60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2} \qquad \cos(-30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$\cos\theta$ increases from 0 to 1, $\sin\theta$ increases from -1 to 0 as θ increases from 270° to 360°.

If the vector OP continues to rotate around the circle of unit radius then in the next complete rotation θ increases from 360° to 720°. However, a θ value of, say, 405° is indistinguishable from one of 45° (just one extra complete revolution is involved).
So

$$\sin(405^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos(405^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

In general

$$
\begin{aligned}
\sin(360^\circ + x^\circ) &= \sin x^\circ\\
\cos(360^\circ + x^\circ) &= \cos x^\circ
\end{aligned}
$$

Key Point

If n is any positive integer

$$\sin(360n^\circ + x^\circ) = \sin x^\circ \qquad \cos(360n^\circ + x^\circ) = \cos x^\circ$$

or, since $360^\circ = 2\pi$ radians,

$$\sin(x + 2n\pi) = \sin x \qquad \cos(x + 2n\pi) = \cos x$$

We say that the functions $\sin x$ and $\cos x$ are **periodic** with period (in radian measure) of 2π.

Graphs of $\sin\theta$, $\cos\theta, \tan\theta$

Since we have defined both $\sin\theta$ and $\cos\theta$ in terms of the projections of the radius vector OP of a circle of **unit radius** it follows immediately that

$$-1 \leq \sin\theta \leq +1 \qquad \text{and} \qquad -1 \leq \cos\theta \leq +1$$

for any value of θ

We have discussed above the behaviour of $\sin\theta$ and $\cos\theta$ in each of the four quadrants.

The behaviour is summarised in the following diagram

$0 \leq \sin\theta \leq 1$ $-1 \leq \cos\theta \leq 0$	$0 \leq \sin\theta \leq 1$ $0 \leq \cos \leq 1$
$-1 \leq \sin\theta \leq 0$ $-1 \leq \cos\theta \leq 0$	? ?

Complete the above diagram by writing suitable inequalities for the fourth quadrant.

Your solution

For $270^\circ \leq \theta \leq 360^\circ$ $\qquad -1 \leq \sin\theta \leq 0$ $\qquad 0 \leq \cos\theta \leq 1.$

Using all the above results we can draw the graphs of these two trigonometric functions. See Figure 17. We have labelled the horizontal axis using radians and have shown 2 **periods** in each case

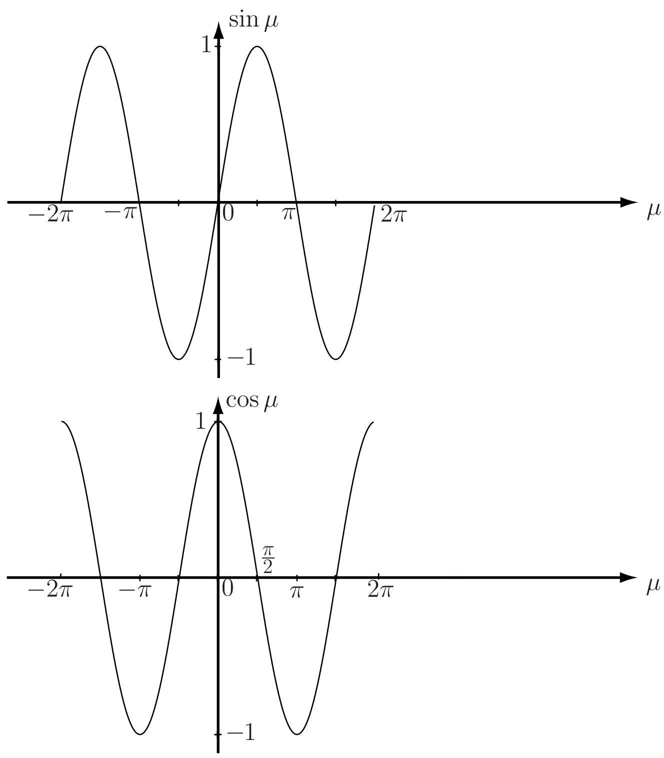


Figure 17

You should be able to see on these graphs the behaviour that we have already discussed for both $\sin\theta$ and $\cos\theta$ in each of the four quadrants in $0 \leq \theta \leq 2\pi$.
Note that we have extended the graphs to negative values of θ using the relations $\sin(-\theta) = \sin\theta, \quad \cos(-\theta) = \cos\theta$.

Both graphs could be extended indefinitely to the left ($\theta \to -\infty$) and to the right ($\theta \to +\infty$).

The Graph of $\tan\theta$

Recall that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Using the above graphs of $\cos\theta$ and $\sin\theta$ write down the values of $\tan 0$, $\tan\pi$, $\tan 2\pi$. For what values of θ is $\tan\theta$ undefined? State whether $\tan\theta$ is positive or negative in each of the 4 quadrants.

Your solution

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0 \qquad \tan\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$$

$$\tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = \frac{0}{1} = 0$$

$\tan\theta$ will not be defined when $\cos\theta = 0$ i.e. when $\theta = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2}$

1st quadrant:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{positive}}{\text{positive}} = \text{positive}$$

2nd quadrant

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{positive}}{\text{negative}} = \text{negative}$$

3rd quadrant

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{negative}}{\text{negative}} = \text{positive}$$

4th quadrant

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{negative}}{\text{positive}} = \text{negative}$$

The graph of $\tan\theta$ against θ, for $-2\pi \le \theta \le 2\pi$ is then as follows:

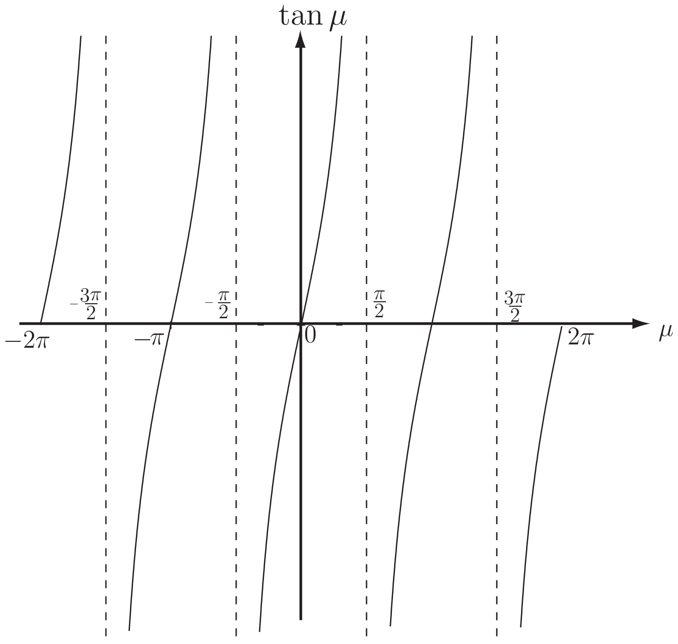
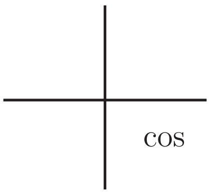


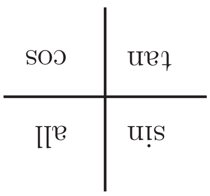
Figure 18

Note that whereas $\sin\theta$ and $\cos\theta$ have period 2π, $\tan\theta$ has period π.

On the following diagram showing the four quadrants mark which trigonometric quantities $\cos(\theta)$, $\sin(\theta)$, $\tan(\theta)$, or all are positive in the four quadrants. One quadrant has been done already.



Your solution



Exercises

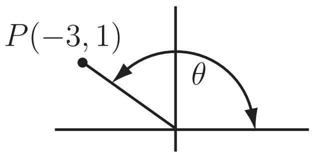
1. Express the following angles in radians (as multiples of π)

   (a) $120°$ (b) $20°$ (c) $135°$ (d) $300°$ (e) $-90°$ (f) $720°$

2. Express in degrees the following quantities which are in radians

   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{6}$ (d) $\frac{11\pi}{9}$ (e) $-\frac{\pi}{8}$ (f) $\frac{1}{\pi}$

3. Obtain the **precise** values of all 6 trigonometric functions of the angle θ for the situation shown in the figure:



4. Obtain all the values of x between 0 and 2π such that

   (a) $\sin x = \frac{1}{\sqrt{2}}$ (b) $\cos x = \frac{1}{2}$ (c) $\sin x = -\frac{\sqrt{3}}{2}$ (d) $\cos x = -\frac{1}{\sqrt{2}}$ (e) $\tan x = 2$

   (f) $\tan x = -\frac{1}{2}$ (g) $\cos(2x + 60°) = 2$ (h) $\cos(2x + 60°) = \frac{1}{2}$

5. Obtain all the values of θ in the given domain satisfying the following quadratic equations

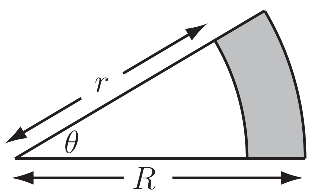
   (a) $2\sin^2\theta - \sin\theta = 0$ $\quad 0 \le \theta \le 360°$
   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0$ $\quad 0 \le \theta \le 360°$
   (c) $4\sin^2\theta - 1 = 0$

6. (a) Show that the area A of a sector formed by a central angle θ radians in a circle of radius r is given by

$$A = \frac{1}{2}r^2\theta.$$

   (Hint: By proportionality the ratio of the area of the sector to the total area of the circle equals the ratio of θ to the total angle at the centre of the circle.)

   (b) What is the value of the shaded area shown in the figure if θ is measured (i) in radians, (ii) in degrees?



7. Sketch, over $0 < \theta < 2\pi$, the graph of (a) $\sin 2\theta$ (b) $\sin\frac{1}{2}\theta$ (c) $\cos 2\theta$ (d) $\cos\frac{1}{2}\theta$.

   Mark the horizontal axis in radians in each case. Write down the period of $\sin 2\theta$ and the period of $\cos\frac{1}{2}\theta$.

1. (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{9}$ (c) $\dfrac{3\pi}{4}$ (d) $\dfrac{5\pi}{3}$ (e) $-\dfrac{\pi}{2}$ (f) 4π

2. (a) 15° (b) 270° (c) 150° (d) 220° (e) -22.5° (f) $\dfrac{180^\circ}{\pi^2}$

3. The distance of the point P from the origin is $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$. Then, since P lies on a circle radius $\sqrt{10}$ rather than a circle of unit radius:

$$\sin\theta = \frac{1}{\sqrt{10}} \qquad \operatorname{cosec}\theta = \sqrt{10}$$

$$\cos\theta = -\frac{3}{\sqrt{10}} \qquad \sec\theta = -\frac{\sqrt{10}}{3}$$

$$\tan\theta = \frac{1}{-3} = -\frac{1}{3} \qquad \cot\theta = -3$$

4. (a) $x = 45^\circ \left(\dfrac{\pi}{4} \text{ radians}\right)$ $x = 135^\circ \left(\dfrac{3\pi}{4}\right)$ (recall $\sin(180 - x) = \sin x$)
   (b) $x = 60^\circ \left(\dfrac{\pi}{3}\right)$ $x = 300^\circ \left(\dfrac{5\pi}{3}\right)$
   (c) $x = 240^\circ \left(\dfrac{4\pi}{3}\right)$ $x = 300^\circ \left(\dfrac{5\pi}{3}\right)$
   (d) $x = 135^\circ \left(\dfrac{3\pi}{4}\right)$ $x = 225^\circ \left(\dfrac{5\pi}{4}\right)$
   (e) $x = 63.43^\circ$ $x = 243.43^\circ$ (remember $\tan x$ has period 180° or π radians)
   (f) $x = 153.43^\circ$ $x = 333.43^\circ$
   (g) No solution !
   (h) $x = 0^\circ,\ 120^\circ,\ 180^\circ,\ 300^\circ,\ 360^\circ$

5. (a) $2\sin^2\theta - \sin\theta = 0$ so $\sin\theta(2\sin\theta - 1) = 0$ so $\sin\theta = 0$
   giving $\theta = 0^\circ,\ 180^\circ,\ 360^\circ$ or $\sin\theta = \dfrac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ$
   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0$. With $x = \cos\theta$ we have $2x^2 + 7x + 3 = 0$ $(2x+1)(x+3) = 0$ (factorising) so $2x = -1$ or $x = -\dfrac{1}{2}$. The solution $x = -3$ is impossible since $x = \cos\theta$.
   The equation $x = \cos\theta = -\dfrac{1}{2}$ has solutions $\theta = 120^\circ,\ 240^\circ$
   (c) $4\sin^2\theta = 1$ so $\sin^2\theta = \dfrac{1}{4}$ i.e. $\sin\theta = \pm\dfrac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$

6.

(a) Using the hint,

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

from where we obtain $A = \dfrac{\pi r^2 \theta}{2\pi} = \dfrac{r^2\theta}{2}$

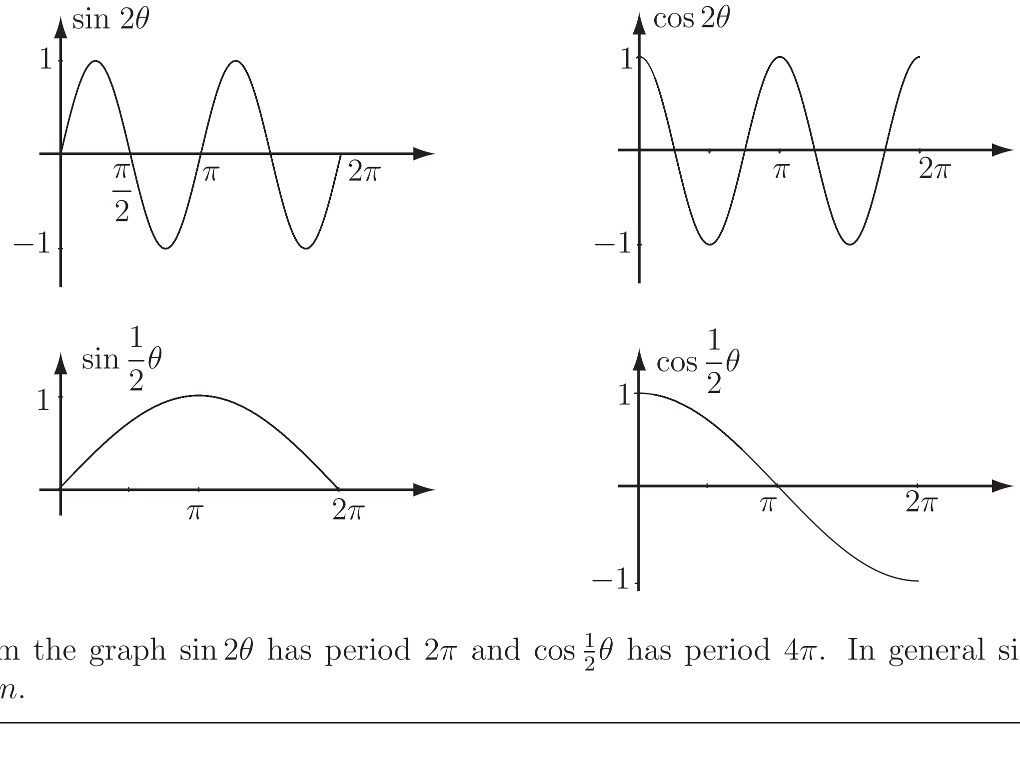
(b) With θ in radians the shaded area is

$$S = \frac{R^2\theta}{2} - \frac{r^2\theta}{2} = \frac{\theta}{2}(R^2 - r^2)$$

If θ is in degrees, then since x radians $= \dfrac{180x^\circ}{\pi}$ or $x^\circ = \dfrac{\pi x}{180}$ radians, we have

$$S = \frac{\pi\theta^\circ}{360^\circ}(R^2 - r^2)$$

7. (a) (c) The graphs of $\sin 2\theta$ and $\cos 2\theta$ are identical in form with those of $\sin\theta$ and $\cos\theta$ respectively but oscillates twice as rapidly. The graphs of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ oscillate half as rapidly as those of $\sin\theta$ and $\cos\theta$.



From the graph $\sin 2\theta$ has period 2π and $\cos\frac{1}{2}\theta$ has period 4π. In general $\sin n\theta$ has period $2\pi/n$.