Trigonometric Identities

4.3

Introduction

A trigonometric identity is a relation between trigonometric expressions which is true for all values of the variables (usually angles). There are a very large number of such identities. In this Section we discuss only the most important and widely used. Any engineer using trigonometry in an application is likely to encounter some of these identities.

Prerequisites

Before starting this Section you should ...

① have a basic knowledge of the geometry of triangles

Learning Outcomes

After completing this Section you should be able to ...

✓ be familiar with the main trigonometric identities

✓ use trigonometric identities to combine trigonometric functions

1. Trigonometric Identities

A trigonometric identity is simply a relation which is always true.

Using the familiar values $\sin 30^\circ = \dfrac{1}{2}$ etc., evaluate $\sin^2\theta + \cos^2\theta$ for

(i) $\theta = 30^\circ$ (ii) $\theta = 45^\circ$ (iii) $\theta = 60^\circ$.

[Note that $\sin^2\theta \equiv (\sin\theta)^2, \ \cos^2\theta \equiv (\cos\theta)^2$]

Your solution

$$\sin^2 30^\circ + \cos^2 30^\circ = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\sin^2 45^\circ + \cos^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\sin^2 60^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

Key Point

For any value of θ

$$\sin^2\theta + \cos^2\theta = 1 \qquad (5)$$

One way of demonstrating this result is to use the definitions of $\sin\theta$ and $\cos\theta$ obtained from the circle of unit radius. Refer back to Figure 10.
Recall that $\cos\theta = OQ, \ \sin\theta = OR = PQ$ but, by Pythagoras' Theorem

$$(OQ)^2 + (QP)^2 = (OP)^2 = 1$$

hence $\cos^2\theta + \sin^2\theta = 1$.
We have demonstrated the result (5) using an angle θ in the first quadrant but the result is true for any θ i.e. it is indeed an identity.

By dividing the identity

$$\sin^2\theta + \cos^2\theta = 1$$

by (i) $\sin^2\theta$ (ii) $\cos^2\theta$ obtain further identities.
Hint: Recall the definitions of $\operatorname{cosec}\theta$, $\sec\theta$, $\cot\theta$.

Your solution

(i)

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

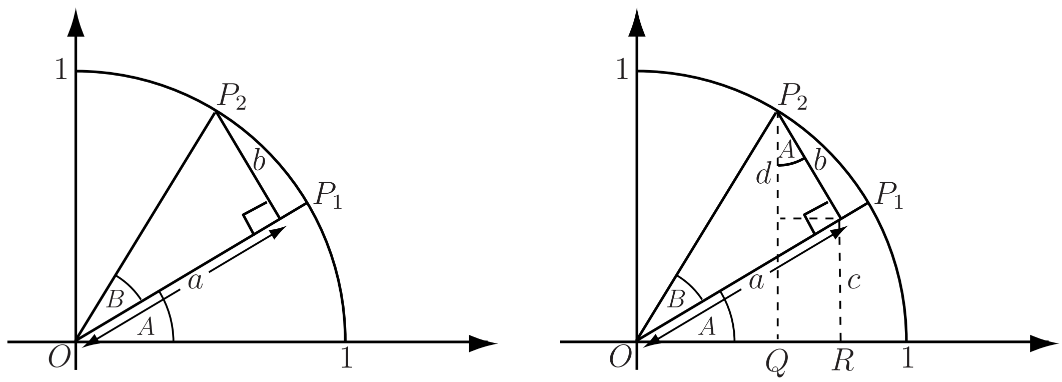
$$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

(ii)

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

We now prove a trigonometric identity from which most of the other important identities can be easily obtained. We shall adopt the notation for a triangle of denoting by a the length of the side opposite angle A, b the length of side opposite angle B and c that side opposite angle C. Refer to Figure 19 where we have shown a quarter of a circle of unit radius.



(a) (b)

Figure 19

Clearly in the right-angled triangle in Figure 19 (a) we have

$$\sin B = \frac{b}{OP_2} = b \qquad \cos B = \frac{a}{OP_2} = a$$

If the sum of angles A and B is denoted by C we have, from Figure 19(b)

$$\begin{aligned} \sin C = P_2Q &= c + d \\ &= a \sin A + b \cos A \end{aligned}$$

Substituting for a and b gives

$$\sin(A + B) = \sin A \cos B + \sin B \cos A.$$

Carry out a similar exercise using the same diagrams to obtain the corresponding formula for $\cos(A + B)$.

Your solution

We have, from Figure 19(b)

$$\begin{aligned} \cos C = OQ &= OR - QR \\ &= a \cos A - b \sin A \\ &= \cos B \cos A - \sin B \sin A \end{aligned}$$

The following are key identities

Key Point

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (6)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad (7)$$

Note carefully the addition sign in (6) but the subtraction sign in (7).

Further identities can readily be obtained from (6) and (7).
Replacing B by $-B$ and remembering that $\cos(-B) = \cos B,\ \sin(-B) = -\sin B$ we find

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad (8)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad (9)$$

Dividing (6) by (7) we obtain

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)}\\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}\end{aligned}$$

Dividing every term by $\cos A\ \cos B$ we obtain

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad (10)$$

By dividing (8) by (9) obtain a similar expansion of $\tan(A - B)$

Your solution

$$\tan(A-B) = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Dividing every term by $\cos A\ \cos B$ gives

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad (11)$$

Example Obtain expressions for $\cos\theta$ in terms of the sine function and for $\sin\theta$ in terms of the cosine function.

Solution

Using (9) with $A = \theta,\ B = \frac{\pi}{2}$ we obtain

$$\cos\left(\theta - \tfrac{\pi}{2}\right) = \cos\theta\ \cos\left(\tfrac{\pi}{2}\right) + \sin\theta\ \sin\left(\tfrac{\pi}{2}\right) = (\cos\theta)\ (0) + \sin\theta\ (1)$$

i.e. $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \theta\right)$

This result explains why the graph of $\sin\theta$ has exactly the same shape as the graph of $\cos\theta$ but it is shifted to the right by $\frac{\pi}{2}$. (See Figure 17). A similar calculation using (6) yields the result $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.

Double angle formulae

If we put $B = A$ in the identity given in (6) we obtain

$$\begin{aligned} \sin 2A &= \sin A \cos A + \cos A \sin A \\ &= 2 \sin A \cos A \qquad (12) \end{aligned}$$

Put $B = A$ in (7) to obtain an identity for $\cos 2A$. Using $\sin^2 A + \cos^2 A = 1$ obtain two alternative forms of the identity.

Your solution

Using (7) with $A = B$

$$\begin{aligned} \cos(2A) &= (\cos A)(\cos A) - (\sin A)(\sin A) \\ \therefore \quad \cos(2A) &= \cos^2 A - \sin^2 A \qquad (13) \end{aligned}$$

Substituting for $\sin^2 A$ in (13) we obtain

$$\begin{aligned} \cos 2A &= \cos^2 A - (1 - \cos^2 A) \\ &= 2\cos^2 A - 1 \qquad (14) \end{aligned}$$

Alternatively substituting for $\cos^2 A$ in (13)

$$\begin{aligned} \cos 2A &= (1 - \sin^2 A) - \sin^2 A \\ \cos 2A &= 1 - 2\sin^2 A \qquad (15) \end{aligned}$$

Use (14) and (15) to obtain, respectively, $\cos^2 A$ and $\sin^2 A$ in terms of $\cos 2A$

Your solution

From (14) $\cos^2 A = \dfrac{1}{2}(1 + \cos 2A)$

From (15) $\sin^2 A = \dfrac{1}{2}(1 - \cos 2A)$

Use (12) and (13) to obtain an identity for $\tan 2A$ in terms of $\tan A$.

Your solution

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Dividing numerator and denominator by $\cos^2 A$ we obtain

$$\tan 2A = \frac{2\frac{\sin A}{\cos A}}{1 - \frac{\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1 - \tan^2 A} \qquad (16)$$

Half-angle formula

If we simply replace A by $\dfrac{A}{2}$ and, consequently $2A$ by A, in (12) we obtain

$$\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \qquad (17)$$

Similarly from (13)

$$\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1. \qquad (18)$$

These are examples of half-angle formulae. We can obtain a half-angle formula for $\tan A$ using (16). Replacing A by $\frac{A}{2}$, $2A$ by A in (16) we obtain

$$\tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)} \qquad (19)$$

Other formulae, useful for integration when trigonometric functions are present, can be obtained using (19). For example if we let $t = \tan\left(\dfrac{A}{2}\right)$ then from (19)

$$\tan A = \frac{2t}{1-t^2}$$

Also, from (17)

$$\begin{aligned}
\sin A &= 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\\
&= 2\tan\left(\frac{A}{2}\right)\cos^2\left(\frac{A}{2}\right) \qquad \text{(multiplying top and bottom by } \cos\left(\frac{A}{2}\right)\text{)}\\
&= 2\frac{\tan\left(\frac{A}{2}\right)}{\sec^2\left(\frac{A}{2}\right)} = \frac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} = \frac{2t}{1+t^2}
\end{aligned}$$

Use (18) to obtain $\cos A$ in terms of $t = \tan\left(\frac{A}{2}\right)$. Hint: use the identity $\tan^2\theta + 1 = \sec^2\theta$ with $\theta = \dfrac{A}{2}$

Your solution

$$\begin{aligned}
\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1 &= \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)\\
&= \cos^2\frac{A}{2}\left(1 - \tan^2\left(\frac{A}{2}\right)\right)\\
&= \frac{1-\tan^2\left(\frac{A}{2}\right)}{\sec^2\left(\frac{A}{2}\right)} = \frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}
\end{aligned}$$

$$\text{So} \qquad \cos A = \frac{1-t^2}{1+t^2}$$

Key Point

If $t = \tan\left(\frac{A}{2}\right)$ then

$$\sin A = \frac{2t}{1+t^2} \qquad \cos A = \frac{1-t^2}{1+t^2} \qquad \tan A = \frac{2t}{1-t^2}$$

Sum of two sines and of two cosines

Finally, in this Section , we obtain results that are widely used in areas of science and engineering such as vibration theory, wave theory and electric circuit theory.
We return to the identities (6) and (8)

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}$$

Adding these identities gives

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \qquad (20)$$

Subtracting the identities produces

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B \qquad (21)$$

It is now convenient to let $C = A + B$ $D = A - B$ so that

$$A = \frac{C+D}{2} \qquad B = \frac{C-D}{2}$$

Hence (20) becomes

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right) \qquad (22)$$

(or, in words, the sum of 2 sines = twice the sine of half the sum (of the angles) multiplied the cosine of half the difference (of the angles).
Similarly (21) becomes

$$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right) \qquad (23)$$

[Note that (23) can alternatively be obtained by replacing D by $-D$ in (22).]

Use (7) and (9) in a similar way to the above to obtain results for the sum and for the difference of two cosines.

Your solution

$$\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
\cos(A-B) &= \cos A\cos B + \sin A\sin B\\
\therefore\quad \cos(A+B)+\cos(A-B) &= 2\cos A\cos B\\
\cos(A+B)-\cos(A-B) &= -2\sin A\sin B
\end{aligned}$$

Hence with $C = A+B \quad D = A-B$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (24)$$

$$\begin{aligned}
\cos C - \cos D &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
&= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \qquad (25)
\end{aligned}$$

Summary

We have covered in this Section a large number of trigonometric identities. The most important of them and probably the ones most worth memorising are given in the following key-point.

Key Point

$$
\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\sin 2\theta &= 2\sin\theta\cos\theta\\
\cos 2\theta &= \cos^2\theta - \sin^2\theta\\
&= 2\cos^2\theta - 1\\
&= 1 - 2\sin^2\theta\\
\sin C + \sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\
\sin C - \sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\cos C - \cos D &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\cos C + \cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)
\end{aligned}
$$

Exercises

1. Show that $\sin t \sec t = \tan t$.

2. Show that $(1 + \sin t)(1 + \sin(-t)) = \cos^2 t$.

3. Show that $\dfrac{1}{\tan\theta + \cot\theta} = \frac{1}{2}\sin 2\theta$.

4. Show that $\sin^2(A + B) - \sin^2(A - B) = \sin 2A \sin 2B$.

   (Hint: the left hand side is the difference of the squared quantities.)

5. Show that $\dfrac{\sin 4\theta + \sin 2\theta}{\cos 4\theta + \cos 2\theta} = \tan 3\theta$.

6. Show that $\cos^4 A - \sin^4 A = \cos 2A$

7. Express each of the following as the sum (or difference) of 2 sines (or cosines)

   (a) $\sin 5x \cos 2x$ (b) $8 \cos 6x \cos 4x$ (c) $\dfrac{1}{3}\sin\dfrac{1}{2}x\cos\dfrac{3}{2}x$

8. Express (a) $\sin 3\theta$ in terms of $\cos\theta$. (b) $\cos 3\theta$ in terms of $\cos\theta$.

9. By writing $\cos 4x$ as $\cos 2(2x)$, or otherwise, express $\cos 4x$ in terms of $\cos x$.

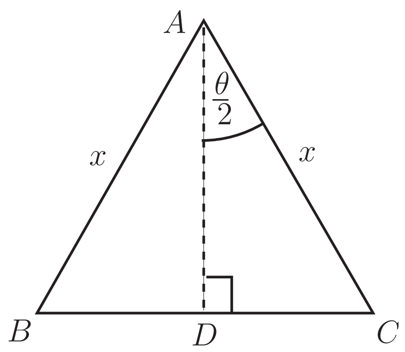
10. Show that $\tan 2t = \dfrac{2\tan t}{2 - \sec^2 t}$.

11. Show that $\dfrac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} = \tan t$.

12. Show that the area of an isosceles triangle with equal sides of length x is

$$A = \frac{x^2}{2}\sin\theta$$

   where θ is the angle between the two equal sides. Hint: Use the following figure:



1. $\sin t.\sec t = \sin t.\dfrac{1}{\cos t} = \dfrac{\sin t}{\cos t} = \tan t.$

2. Recall that $\sin(-t) = -\sin t$. (See the graph of $\sin t$ if you are unfamiliar with this result.) Hence

$$\begin{aligned}(1+\sin t)(1+\sin(-t)) &= (1+\sin t)(1-\sin t)\\ &= 1-\sin^2 t\\ &= \cos^2 t\end{aligned}$$

3.

$$\begin{aligned}\frac{1}{\tan\theta+\cos\theta} &= \frac{1}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}} = \frac{1}{\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}\\ &= \frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}\\ &= \sin\theta\cos\theta\\ &= \frac{1}{2}\sin 2\theta\end{aligned}$$

4. Using the hint and the identity

$$x^2-y^2 = (x-y)(x+y)$$

we have

$$\sin^2(A+B)-\sin^2(A-B) = (\sin(A+B)-\sin(A-B))(\sin(A+B)+\sin(A-B))$$

The first bracket gives

$$\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B) = 2\cos A\sin B$$

Similarly the second bracket gives $\quad 2\sin A\cos B$.

Multiplying we obtain $\;(2\cos A\sin A)(2\cos B\sin B) = \sin 2A.\sin 2B$

5. $\dfrac{\sin 4\theta+\sin 2\theta}{\cos 4\theta+\cos 2\theta} = \dfrac{2\sin 3\theta\cos\theta}{2\cos 3\theta\cos\theta} = \dfrac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta$

6.

$$\begin{aligned}\cos^4 A - \sin^4 A &= (\cos A)^4 - (\sin A)^4\\ &= (\cos^2 A)^2 - (\sin^2 A)^2\\ &= (\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A)\\ &= \cos^2 A - \sin^2 A = \cos 2A\end{aligned}$$

7.

(a) Using $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

Clearly here $\dfrac{A+B}{2} = 5x \qquad \dfrac{A-B}{2} = 2x$ giving $A = 7x \quad B = 3x$

$\therefore \qquad \sin 5x \cos 2x = \dfrac{1}{2}(\sin 7x + \sin 3x)$

(b) Using $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$. With $\dfrac{A+B}{2} = 6x$ $\dfrac{A-B}{2} = 4x$

giving $A = 10x \quad B = 2x \qquad \therefore \; 8\cos 6x \cos 4x = 4(\cos 6x + \cos 2x)$

(c) $\dfrac{1}{3}\sin\left(\dfrac{1}{2}x\right)\cos\left(\dfrac{3x}{2}\right) = \dfrac{1}{6}(\sin 2x - \sin x)$

8.

(a)

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta \sin\theta\\ &= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta = 3\sin\theta - 4\sin^3\theta\end{aligned}$$

(b)

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta\\ &= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta\end{aligned}$$

9.

$$\begin{aligned}\cos 4x = \cos 2(2x) &= 2\cos^2(2x) - 1\\ &= 2(\cos 2x)^2 - 1\\ &= 2(2\cos^2 x - 1)^2 - 1\\ &= 2(4\cos^4 x - 4\cos^2 x + 1) - 1 = 8\cos^4 x - 8\cos^2 x + 1.\end{aligned}$$

10.

$$\tan 2t = \frac{2\tan t}{1-\tan^2 t}$$

$$= \frac{2\tan t}{1-(\sec^2 t - 1)} = \frac{2\tan t}{2-\sec^2 t}$$

11. $\cos 10t - \cos 12t = 2\sin 11t \sin t \qquad \sin 10t + \sin 12t = 2\sin 11t\cos(-t)$

$$\therefore \quad \frac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} = \frac{\sin t}{\cos(-t)} = \frac{\sin t}{\cos t} = \tan t$$

12. The right-angled triangle ACD has area $\frac{1}{2}(CD)(AD)$

$$\text{But} \quad \sin\left(\frac{\theta}{2}\right) = \frac{CD}{x} \qquad \therefore \quad CD = x\sin\left(\frac{\theta}{2}\right)$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{AD}{x} \qquad \therefore \quad AD = x\cos\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \text{area of} \quad \Delta ACD = \frac{1}{2}x^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{4}x^2\sin\theta$$

$$\therefore \quad \text{area of} \quad \Delta ABC = 2\times\text{area of } \Delta ACD = \frac{1}{2}x^2\sin\theta$$