

# Applications of Trigonometry to Triangles

4.4



## Introduction

We originally introduced trigonometry using right-angled triangles. However, the subject has applications in dealing with **any** triangles such as those that might arise in surveying, navigation or mechanisms.

In this Section we show how, given certain information about a triangle, we can use appropriate rules, called the **sine rule** and the **cosine rule**, to fully ‘solve the triangle’ i.e. obtain the lengths of all the sides and the size of all the angles of that triangle.



## Prerequisites

Before starting this Section you should ...

- ① have a knowledge of the basics of trigonometry
- ② be aware of the standard trigonometric identities



## Learning Outcomes

After completing this Section you should be able to ...

- ✓ use trigonometry in everyday situations
- ✓ fully determine all the sides and angles and the area of any triangle from partial information

# 1. Applications of trigonometry to triangles

## Area of a triangle



Calculate an expression for the area of the right-angled triangle  $ABC$  shown in Figure 1.

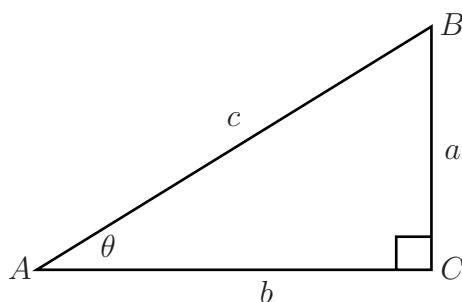


Figure 1

Your solution

The area  $S$  of any triangle is given by  $S = \frac{1}{2} \times (\text{base}) \times (\text{perpendicular height})$  where 'perpendicular height' means the perpendicular distance from the side called the 'base' to the opposite vertex. Thus for the right-angled triangle shown  $S = \frac{1}{2} b a$ .

Now consider the triangle  $ABC$  shown.

(This is an obtuse angled triangle but the formula obtained will hold for any triangle.)

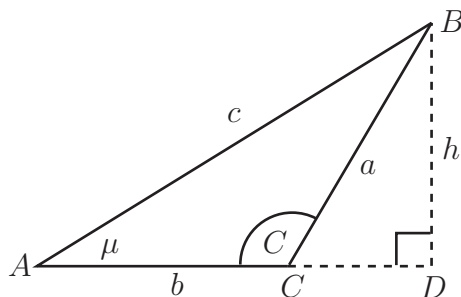


Figure 2

Using the construction shown in Figure 2, the area of triangle  $ABC$  is

$$S = \frac{1}{2} b h$$

If we use  $C$  to denote the angle  $ACB$  then

$$\sin(180 - C) = \frac{h}{a} \quad (\text{triangle } BCD \text{ is right-angled})$$

$$\begin{aligned} \therefore h &= a \sin(180 - C) \\ &= a \sin C \quad (\text{see the graph of the sine wave or expand } \sin(180 - c)) \end{aligned}$$

$$\therefore \boxed{S = \frac{1}{2} b a \sin C} \quad (1a)$$

By other similar constructions we could demonstrate that

$$\boxed{S = \frac{1}{2} a c \sin B \quad \text{and} \quad S = \frac{1}{2} b c \sin A} \quad (1b, c)$$

Note the pattern here: in each formula for the area the angle involved is the one **between** the sides whose lengths occur in that expression.

Clearly if  $C$  is a right-angle (so  $\sin C = 1$ ) then

$$S = \frac{1}{2} b a$$

as you deduced in the previous guided exercise.

## The Sine Rule

The sine rule is a formula which, if we are given certain information about a triangle, enables us to fully 'solve the triangle' i.e. obtain the lengths of all three sides and the value of all three angles.

To show the rule we note that from the formulae (1a) and (1b) for the area  $S$  of the triangle  $ABC$  in Figure 2 we have

$$ba \sin C = ac \sin B \text{ or}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly using (1b) and (1c)

$$ac \sin B = bc \sin A \text{ or}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



### Key Point

For **any** triangle  $ABC$  where  $a$  is the length of the side opposite angle  $A$ ,  $b$  the side length opposite angle  $B$  and  $c$  the side length opposite angle  $C$  the **sine rule** states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (2)$$

## Use of the sine rule:

To be able to fully determine all the angles and sides of a triangle it follows from the sine rule that we must know

**either** two angles and one side : (knowing two angles of a triangle really means that all three are known since the sum of the angles is  $180^\circ$ )

**or**

two sides and an angle **opposite** one of those two sides.

**Example** Solve the triangle  $ABC$  given that  $a = 32$  cm  $b = 46$  cm and angle  $B = 63^\circ 15' = 63.25^\circ$ .

### Solution

Using the first pair of equations in the sine rule (2) we have

$$\frac{32}{\sin A} = \frac{46}{\sin 63.25^\circ} \quad \therefore \quad \sin A = \frac{32}{46} \sin 63.25^\circ = 0.6212$$

$$\text{so} \quad A = \sin^{-1}(0.6212) = 38.4^\circ \quad (\text{by calculator})$$

You should, however, note carefully that because of the form of the graph of the sine function there are **two** angles between  $0^\circ$  and  $180^\circ$  which have the same value for their sine viz.  $x$  and  $(180 - x)$ . See Figure 3.

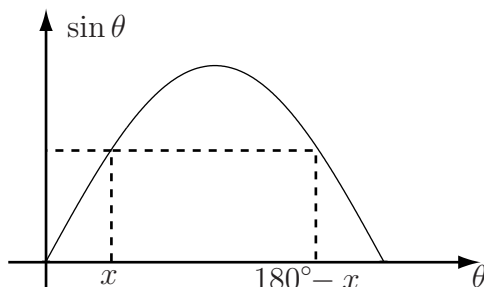


Figure 3

In our example

$$A = \sin^{-1}(0.6212) = 38.4^\circ$$

or

$$A = 180^\circ - 38.4^\circ = 141.6^\circ.$$

However since we are given that angle  $B$  is  $63.25^\circ$ , the value of  $141.6^\circ$  for angle  $A$  is clearly impossible.

To complete the problem we simply note that

$$C = 180^\circ - (38.4^\circ + 63.25^\circ) = 78.35^\circ$$

The remaining side  $c$  is calculated from the sine rule, using either  $a$  and  $\sin A$  or  $b$  and  $\sin B$ .



Complete this problem by finding the length of side  $c$ .

**Your solution**

$$\begin{aligned} c &= \frac{\sin A}{\sin C} \times 32 = \frac{0.6212}{\sin 78.35^\circ} \times 32 \\ &= \frac{0.6212}{32 \times 0.9794} \\ &= 50.45 \text{ cm.} \end{aligned}$$

Using, for example, we have

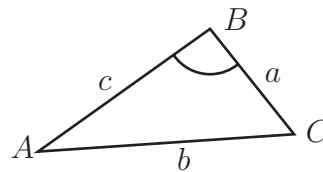
**The ambiguous case**

When, as the above example, we are given two sides and the non-included angle of a triangle, particular care is required.

Suppose, to be definite, that sides  $b$  and  $c$  and the angle  $B$  are given.

Then the angle  $C$  is given by the sine rule as

$$\sin C = c \frac{\sin B}{b}$$



Various cases can arise:

(i)  $c \sin B > b$

This implies that  $\frac{c \sin B}{b} > 1$  in which case no triangle exists since  $\sin C$  cannot be greater than 1.

(ii)  $c \sin B = b$

In this case  $\sin C = \frac{c \sin B}{b} = 1$  so  $C = 90^\circ$ .

(Clearly this case can only arise if the given angle  $B$  is less than  $90^\circ$ !)

(iii)  $c \sin B < b$

Hence  $\sin C = \frac{c \sin B}{b} < 1$ .

As mentioned earlier there are two possible values of angle  $C$  in the range  $0$  to  $180^\circ$ , one acute angle ( $< 90^\circ$ ) and one obtuse (between  $90^\circ$  and  $180^\circ$ .) These angles are  $C_1 = x$  and  $C_2 = 180 - x$ . See Figure 4.

If the given angle  $B$  is greater than  $90^\circ$  then the obtuse angle  $C_2$  is not a possible solution because, of course, a triangle cannot possess two obtuse angles.

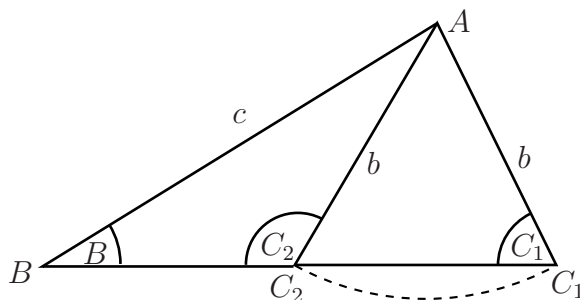


Figure 4

For  $B$  less than  $90^\circ$  there are still two possibilities.

If the given side  $b$  is greater than the given side  $c$ , the obtuse angle solution  $C_2$  is not possible because then the larger angle would be opposite the smaller side. (This was the situation in the worked example earlier.)

The final case

$$b < c, \quad B < 90^\circ$$

**does** give rise to two possible values  $C_1, C_2$  of the angle  $C$  and is referred to as the **ambiguous case**. In this case there will be two possible values  $a_1$  and  $a_2$  for the third side of the triangle corresponding to the two angle values

$$A_1 = 180^\circ - (B + C_1)$$

$$A_2 = 180^\circ - (B + C_2)$$



Show that two triangles fit the following data for a triangle  $ABC$ .

$$a = 4.5 \text{ cm} \quad b = 7 \text{ cm} \quad A = 35^\circ$$

Obtain the sides and angle of both possible triangles.

**Your solution**

You can clearly see that we have one acute angled triangle  $AB_1C_1$  and one obtuse angled  $AB_2C_2$  corresponding to the given data.

$\frac{c_1}{\sin 81.85^\circ} = \frac{4.5}{\sin 35^\circ} = \frac{4.5 \times 0.9899}{0.5736} = 7.766 \text{ cm}$ $c_1 = 7.766 \text{ cm}$	$\frac{c_2}{\sin 28.15^\circ} = \frac{4.5}{\sin 35^\circ} = \frac{4.5 \times 0.4718}{0.5736} = 3.701 \text{ cm}$ $c_2 = 3.701 \text{ cm}$
$B = B_1 = 63.15^\circ$ $C = C_1 = 81.85^\circ$ $c = c_1$ where	$B = B_2 = 116.85^\circ$ $C = C_2 = 28.15^\circ$ $c = c_2$ where

This is indeed the ambiguous case with two possible triangles.

(Side  $b$  is longer than side  $a$ .)

In this case, **both** values of  $B$  are indeed possible since both values are larger than angle  $A$ .

So  $B = \sin^{-1} 0.8922 = 63.15^\circ$  or  $180 - 63.15^\circ = 116.85^\circ$ .

$$\sin B = \frac{b \sin A}{c} = \frac{7 \sin 35^\circ}{4.5} = 0.8922$$

We have, by the sine rule

## The Cosine Rule

The cosine rule is an alternative formula for 'solving a triangle'  $ABC$ . It is particularly useful for the case where the sine rule cannot be used, viz. when two sides of the triangle are known together with the angle **between** these two sides.

Consider the two triangles  $ABC$  shown in Figure 5.

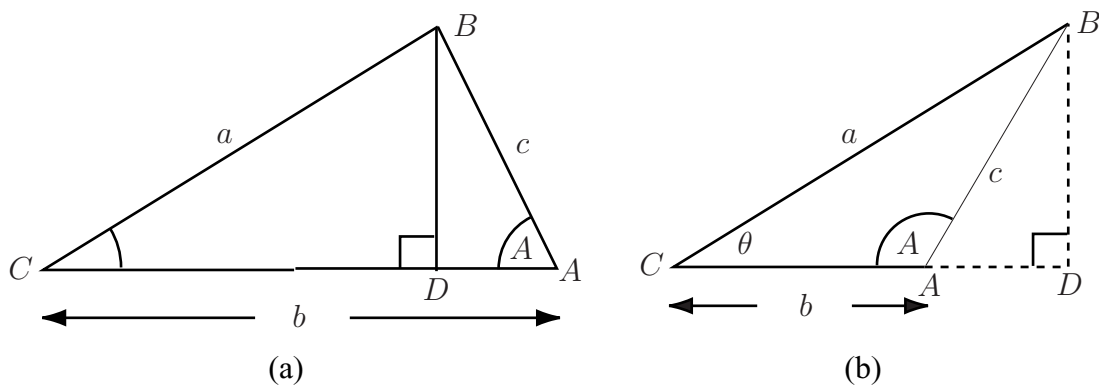


Figure 5

In both cases, using the right-angled triangle  $ABD$

$$BD = c \sin A$$

In Figure 5(a)

$$\begin{aligned} DA &= c \cos A \\ \therefore CD &= b - c \cos A \end{aligned}$$

In Figure 5(b)

$$\begin{aligned} DA &= c \cos(180 - A) = -c \cos A \\ \therefore CD &= b + AD = b - c \cos A \end{aligned}$$

In both cases, in the right-angled triangle  $BDC$

$$(BC)^2 = (CD)^2 + (BD)^2$$

So, using the above results,

$$\begin{aligned} a^2 &= (b - c \cos A)^2 + c^2(\sin A)^2 \\ &= b^2 - 2bc \cos A + c^2(\cos^2 A + \sin^2 A) \end{aligned}$$

giving

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos A} \quad (3a)$$

Equation (3a) is one form of the cosine rule. Clearly it can be used, as we stated above, to calculate the side  $a$  if the sides  $b$  and  $c$  and the **included** angle  $A$  are known.

Note that if  $A = 90^\circ$ ,  $\cos A = 0$  and (3a) reduces to Pythagoras' Theorem.

Two similar formulae to (3a) for the cosine rule can be similarly derived. One is

$$b^2 = c^2 + a^2 - 2ca \cos B \quad (3b)$$



Write down the third form of the cosine rule, with  $c^2$  as the subject.

**Your solution**

(3c)

$$c^2 = a^2 + b^2 - 2ab \cos C$$



**Example** Solve the triangle where  $b = 7.00$  cm,  $c = 3.59$  cm,  $A = 47^\circ$ .

### Solution

Since two sides and the angle  $A$  between these sides is given we must first use the cosine rule in the form (3a):

$$\begin{aligned}a^2 &= (7.00)^2 + (3.59)^2 - 2(7.00)(3.59) \cos 47^\circ \\ &= 49 + 12.888 - 34.277 \\ &= 27.610 \\ \text{so } a &= \sqrt{27.610} = 5.255 \text{ cm.}\end{aligned}$$

We can now most easily use the sine rule to solve one of the remaining angles:

$$\begin{aligned}\frac{7.00}{\sin B} &= \frac{5.255}{\sin 47^\circ} \\ \text{so } \sin B &= \frac{7.00 \sin 47^\circ}{5.255} = 0.9742\end{aligned}$$

from which  $B = B_1 = 76.96^\circ$  or  $B = B_2 = 103.04^\circ$ .

At this stage it is not obvious which value is correct or whether this is the ambiguous case and both values of  $B$  are possible.

The two possible values for the remaining angle  $C$  are

$$\begin{aligned}C_1 &= 180^\circ - (47^\circ + 76.96^\circ) = 56.04^\circ \\ C_2 &= 180^\circ - (47^\circ + 103.04^\circ) = 29.96^\circ\end{aligned}$$

Since for the sides of this triangle

$$b > a > c$$

then similarly for the angles we must have

$$B > A > C$$

so the value  $C_2 = 29.96^\circ$  is the correct one for the third side.

The cosine rule can also be applied to some triangles where the lengths  $a$ ,  $b$  and  $c$  of the three sides are known and the only calculations needed are finding the angles.



A triangle  $ABC$  has sides

$$a = 7 \text{ cm} \quad b = 11 \text{ cm} \quad c = 12 \text{ cm.}$$

Obtain the values of all the angles of the triangle. (You will need to transpose one of the cosine rule formulae (3) to obtain the cosine of an angle as the subject.)

**Your solution**

after which  $C = 80.3^\circ$

$$\begin{aligned} \cos B &= \frac{12^2 + 7^2 - 11^2}{2 \times 12 \times 7} \\ &= 0.429 \\ B &= \cos^{-1}(0.429) \\ &= 64.6^\circ \end{aligned}$$

so here

$$\cos B = \frac{2ca}{c^2 + a^2 - b^2}$$

We shall use the latter to obtain  $B$ . Transposing (3b).

Another angle  $B$  or  $C$  could now be obtained using the sine rule or the cosine rule.

(There is no other possibility between  $0^\circ$  and  $180^\circ$  for  $A$ . No 'ambiguous case' arises using the cosine rule!)

$$\begin{aligned} \cos A &= \frac{11^2 + 12^2 - 7^2}{2 \times 11 \times 12} \\ &= 0.818 \\ A &= \cos^{-1}(0.818) \\ &= 35.1^\circ \end{aligned}$$

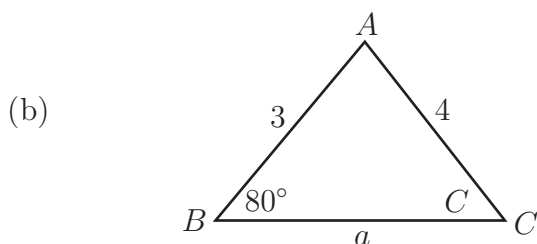
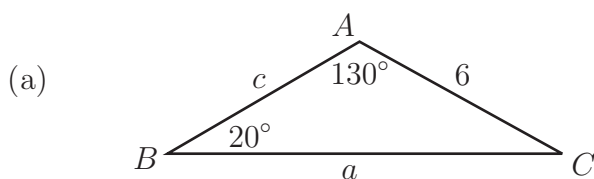
In this example

$$\therefore \cos A = \frac{2bc}{b^2 + c^2 - a^2} \quad (\text{a general result})$$

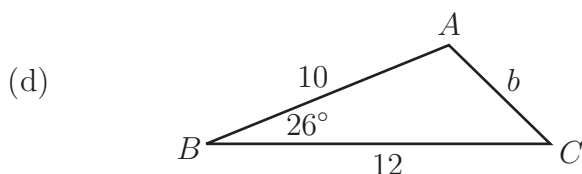
Suppose, for the sake of example, we find angle  $A$  first using the transposed form of (3a):

## Exercises

1. Determine the remaining angles and sides for the following triangles:



- (c) The triangles  $ABC$  with  $B = 50^\circ$ ,  $b = 5$ ,  $c = 6$ . (Take special care here!)



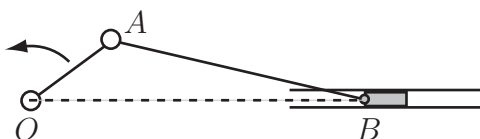
2. Determine all the angles of the triangles  $ABC$  where the sides have lengths  $a = 7$ ,  $b = 66$  and  $c = 9$
3. (a) Use the cosine rule and half angle formulae to show that, for a triangle with sides of length  $a$ ,  $b$ ,  $c$

$$\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \quad \sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

where  $s = \frac{a+b+c}{2}$  (the semi-perimeter of the triangle.)

- (b) Hence show that the area of a triangle is given by  $\sqrt{s(s-a)(s-b)(s-c)}$

4. Two ships leave a port at 8.00 am, one travelling at 12 knots (nautical miles per hour) the other at 10 knots. The faster ship maintains a bearing of  $N47^\circ W$ , the slower one a bearing  $S20^\circ W$ . Calculate the separation of the ships at midday. (Hint: Draw an appropriate diagram.)
5. The crank mechanism shown below has an arm  $OA$  of length 30 mm rotating anticlockwise about  $O$  and a connecting rod  $AB$  of length 60 mm.  $B$  moves along the horizontal line  $OB$ . What is the length  $OB$  when  $OA$  has rotated by  $\frac{1}{8}$  of a complete revolution from the horizontal?



(a) Using the sine rule  $\frac{\sin 130^\circ}{a} = \frac{\sin 20^\circ}{6} = \frac{\sin C}{c}$ . From the two left-hand equations  $a = 6 \frac{\sin 130^\circ}{\sin 20^\circ} \approx 13.44$ .

Then, since  $C = 30^\circ$ , the right hand pair of equations give  $c = 6 \frac{\sin 30^\circ}{\sin 20^\circ} \approx 8.77$ .

(b) Again using the sine rule  $\frac{\sin A}{a} = \frac{\sin 80^\circ}{4} = \frac{\sin C}{3}$  so  $\sin C = \frac{3}{4} \sin 80^\circ = 0.7386$  there are two possible angles satisfying  $\sin C = 0.7386$  or  $C = \sin^{-1}(0.7386)$ .

These are  $47.61^\circ$  and  $180^\circ - 47.61^\circ = 132.39^\circ$ . However the obtuse angle value is impossible here because the angle  $B$  is  $80^\circ$  and the sum of the angles would then exceed  $180^\circ$ . Hence  $c = 47.01^\circ$  so  $A = 180^\circ - (80^\circ + 47.61^\circ) = 52.39^\circ$ .

Then,  $\frac{a}{4} = \frac{\sin 52.39^\circ}{\sin 80^\circ}$  so  $a = 4 \frac{\sin 52.39^\circ}{\sin 80^\circ} \approx 3.22$

(c) By the sine rule

$$\frac{\sin A}{a} = \frac{\sin 50^\circ}{5} = \frac{\sin C}{6} \quad \therefore \quad \sin C = 6 \frac{\sin 50^\circ}{5} = 0.9193$$

Then  $C = \sin^{-1}(0.9193) = 66.82^\circ$  (calculator) or  $180^\circ - 66.82^\circ = 113.18^\circ$ . In this case both values of  $C$  say  $C_1 = 66.82^\circ$  and  $C_2 = 113.18^\circ$  are possible and there are two possible triangles satisfying the given data. Continued use of the sine rule produces

i. with  $C_1 = 66.82^\circ$  (acute angle triangle)  $A = A_1 = 180^\circ - (66.82^\circ + 50^\circ) = 63.18^\circ$   
 $a = a_1 = 5.83$   
 ii. with  $C_2 = 113.18^\circ$   $A = A_2 = 16.82^\circ$   $a = a_2 = 1.89$

(d) In this case since two sides and the included angle are given we must use the cosine rule. The appropriate form is

$$b^2 = c^2 + a^2 - 2ca \cos B = 10^2 + 12^2 - (2)(10)(12) \cos 26^\circ = 28.2894$$

so  $b = \sqrt{28.2894} = 5.32$

Continuing we use the cosine rule again to determine say angle  $C$  where

$$c^2 = a^2 + b^2 - 2ab \cos C \quad \text{that is} \quad 10^2 = 12^2 + (5.32)^2 - 2(12)(5.32) \cos C$$

from which  $\cos C = 0.5663$  and  $C = 55.51^\circ$  (There is no other possibility for  $C$  between  $0^\circ$  and  $180^\circ$ . Recall that the cosine of an angle between  $90^\circ$  and  $180^\circ$  is negative.) Finally,  $A = 180^\circ - (26^\circ + 55.51^\circ) = 98.49^\circ$ .

3.

from which  $\sin B = 0.6659$  giving  $B = 41.75^\circ$ . (The obtuse case for  $B$  is not possible, as explained above.) Finally  $A = 180^\circ - (41.75^\circ + 87.27^\circ) = 50.98^\circ$ .

$$\frac{\sin B}{6} = \frac{\sin 87.27^\circ}{9}$$

from which  $C = 87.27^\circ$ . So there is no obtuse angle in this triangle and we can use the sine rule knowing that there is only one possible triangle fitting the data. (We could continue to use the cosine rule if we wished of course.) Choosing to find the angle  $B$  we have

$$84 \cos C = 4 \quad \cos C = \frac{4}{84}$$

from which

$$c^2 = a^2 + b^2 - 2ab \cos C \quad 81 = 49 + 36 - 84 \cos C$$

2. We use the cosine rule firstly to find the angle opposite the longest side. This will tell us whether the triangle contains an obtuse angle. Hence we solve for  $c$  using

(a) By the cosine rule

$$b^2 + c^2 - a^2 = 2bc \cos A = 2bc \cos^2 \left( \frac{A}{2} \right) - 1 \quad (*)$$

from which

$$4bc \cos^2 \left( \frac{A}{2} \right) = b^2 + c^2 - a^2 + 2bc = (b + c)^2 - a^2 = (b + c + a)(b + c - a)$$

$$\text{If } s = \frac{a + b + c}{2}, \text{ then } b + c - a = a + b + c - 2a = 2(s - a)$$

$$\therefore 4bc \cos^2 \left( \frac{A}{2} \right) = (2s)(2(s - a)) \quad \text{from which } \cos^2 \left( \frac{A}{2} \right) = \frac{bc}{s(s - a)}$$

Similarly using the first equation of (\*) but putting  $\cos A = 1 - 2 \sin^2 \left( \frac{A}{2} \right)$  we obtain

$$4bc \sin^2 \left( \frac{A}{2} \right) = 2bc - b^2 - c^2 + a^2 = a^2 - (b - c)^2 = (a + b - c)(a + b + c)$$

$$= 2(s - b)(2(s - c)) = \frac{bc}{(s - b)(s - c)} \sin^2 \left( \frac{A}{2} \right) \quad \therefore$$

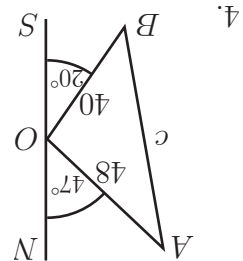
(b) We showed at the beginning of this section that the area of a triangle is given by

$$\text{area} = \frac{1}{2}bc \sin A = \frac{1}{2}bc(2) \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)$$

Using the above results we obtain

$$\text{area} = bc \sqrt{\frac{(s-b)(s-c)}{(s-a)(s-a)}} = \frac{bc}{\sqrt{s(s-a)}} = \sqrt{s(s-a)(s-b)(s-c)}$$

as required.



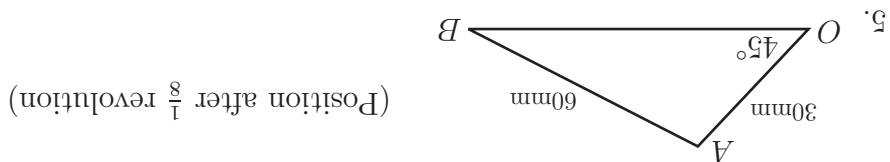
4. At midday (4 hours travelling) ships A and B are respectively 48 and 40 nautical miles from the port O. In triangle AOB we have

$$\angle AOB = 180^\circ - (47^\circ + 20^\circ) = 113^\circ.$$

We must use the cosine rule to obtain the required distance apart of the ships. Denoting the distance AB by c, as usual,

$$c^2 = 48^2 + 40^2 - 2(48)(40) \cos 113^\circ$$

from which  $c^2 = 5404.41$  and  $c = 73.5$  nautical miles.



5. By the sine rule  $\frac{\sin B}{30} = \frac{\sin 45^\circ}{60}$ .  $\therefore \sin B = \frac{60}{30} \sin 45^\circ = 0.3533$  so  $B = 20.704^\circ$ . (The obtuse value of  $\sin^{-1}(0.3533)$  is impossible. Hence,

$$A = 180^\circ - (45^\circ + 20.704^\circ) = 114.296^\circ. \text{ Using the sine rule again}$$

$$\frac{0.3533}{OB} = \frac{30}{\sin 114.296^\circ}$$

from which  $OB = 77.5$  mm.