Applications of Trigonometry to Waves

4.5

Introduction

Waves and vibrations occur in many contexts. The water waves on the sea and the vibrations of a stringed musical instrument are just two everyday examples. If the vibrations are simple 'back and fro' oscillations they are referred to as 'sinusoidal' which implies that a knowledge of trigonometry, particularly of the sine and cosine functions, is a necessary pre-requisite for dealing with their analysis. In this Section we give a brief introduction to this topic.

Prerequisites

Before starting this Section you should . . .

① have a knowledge of the basics of trigonometry

② be aware of the standard trigonometric identities

Learning Outcomes

After completing this Section you should be able to . . .

✓ use simple trigonometric functions to describe waves

✓ be able to combine two waves of the same frequency as a single wave in amplitude-phase form

1. Applications of trigonometry to waves

Suppose that a wheel of radius R is rotating anticlockwise as shown in Figure 6.

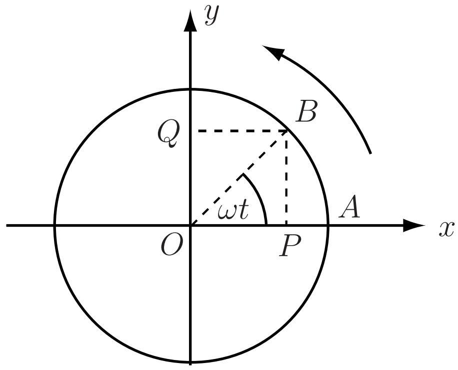

Figure 6

Assume that the wheel is rotating with an angular velocity ω radians per second so that, in a time t seconds, a point A on the rim of the wheel moves to a point B such that angle $AOB = \omega t$ radians.

Then the coordinates of the point B are

$$x = OP = R\cos\omega t \tag{4a}$$

$$y = OQ = PB = R\sin\omega t \tag{4b}$$

(from which $x^2 + y^2 = R^2(\cos^2\omega t + \sin^2\omega t) = R^2$)

We know that both the standard sine function and cosine function have period 2π. Hence the wheel will complete one complete revolution in a time t_1 such that

$$\omega t_1 = \omega t + 2\pi \qquad \text{or} \qquad t_1 = t + \frac{2\pi}{\omega}$$

The time $t_1 - t = \dfrac{2\pi}{\omega}$ for one complete revolution is called the **period** of rotation of the wheel. The number of complete revolutions per second is thus $\dfrac{1}{T} = f$ say which is called the **frequency** of revolution. Clearly $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ relates the three quantities introduced here. The angular velocity $\omega = 2\pi f$ is sometimes called the **angular frequency**.

The situation we have just outlined is a two-dimensional one. More simply we might consider **one-dimensional motion** where

$$x = R\cos\omega t \tag{5}$$

x being the co-ordinate of a moving 'particle'.

Clearly, from the known properties of the cosine function, we can deduce the following:

1. x varies periodically with t with period $T = \dfrac{2\pi}{\omega}$.
2. x will have maximum value R, minimum value $-R$,

   (This quantity R is called the **amplitude** of the motion.)

In other words (5) is a description of a common 'back and fro' motion between $-R$ and R.

Using (5) write down the values of x at the following times: $t = 0$, $t = \dfrac{\pi}{2\omega}$, $t = \dfrac{\pi}{\omega}$, $t = \dfrac{3\pi}{2\omega}$, $t = \dfrac{2\pi}{\omega}$. Describe in words the form of the motion:

Your solution

t	$x = R\cos\omega t$
0	R
$\dfrac{\pi}{2\omega}$	0
$\dfrac{\pi}{\omega}$	$-R$
$\dfrac{3\pi}{2\omega}$	0
$\dfrac{2\pi}{\omega}$	R

Using (5) this 'back and fro' or 'vibrational' or 'oscillatory' motion between R and $-R$ continues indefinitely. The technical name for this motion is **simple harmonic**. To a good approximation it is the motion exhibited (i) by the end of a pendulum pulled through a small angle and then released (ii) by the end of a hanging spring pulled down and then released. See Figure 7 (in these cases damping of the pendulum or spring is ignored.)

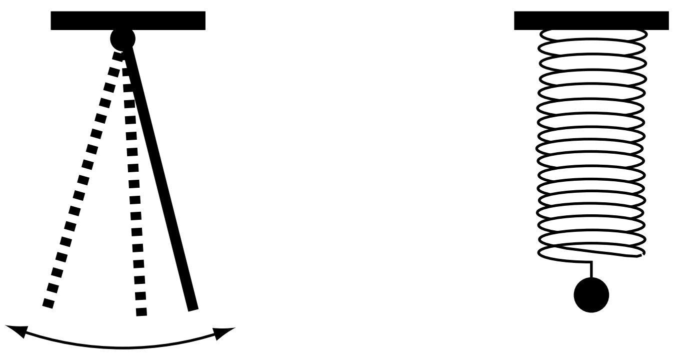

Figure 7

Using your knowledge of the cosine function and the results of the previous exercise plot the graph of x against t if

$$x = R\cos\omega t$$

for $t = 0$ to $t = \dfrac{4\pi}{\omega}$

Your solution

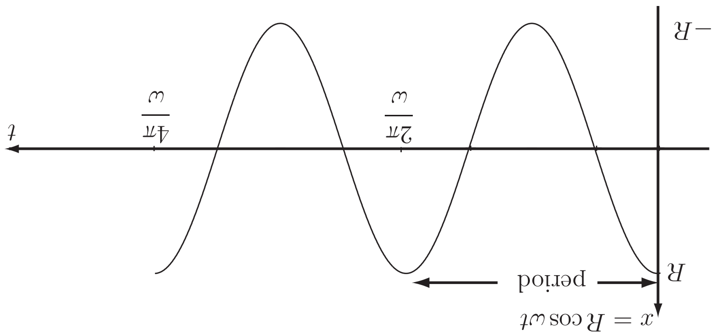

Figure 8

This graph shows part of a cosine **wave** specifically two **periods** of oscillation. The shape of the graph suggests that the term wave is indeed an appropriate description.

We know that the **shape** of the cosine graph and the sine graph are identical but just offset by $\dfrac{\pi}{2}$ radians horizontally. Bearing this in mind, attempt the following guided exercise.

Write the equation of the wave, part of which is shown in the following graph. You will need to find the period T and angular frequency ω.

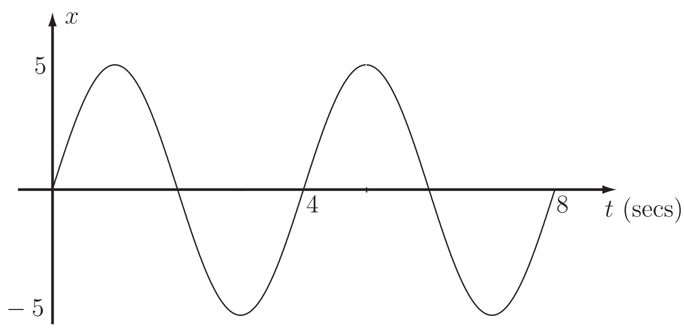

Figure 9

Your solution

From the shape of the graph we have a **sine** wave rather than a cosine wave. The amplitude is 5. The period $T = 4s$ so the angular frequency $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$. Hence the equation of the wave is

$$x = 5\sin\left(\frac{\pi t}{2}\right).$$

The quantity x, a function of t, is referred to as the **displacement** of the wave.

Phase of a wave

We recall that $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ which simply means that the graph of $x = \sin\theta$ is the same shape as that of $x = \cos\theta$ but is shifted to the right by $\frac{\pi}{2}$.
Suppose now that we consider the waves

$$x_1 = R\cos 2t \qquad\qquad x_2 = R\sin 2t$$

Both have amplitude R, angular frequency $\omega = 2$ rads s^{-1}. Also

$$x_2 = R\cos\left(2t - \frac{\pi}{2}\right) = R\cos\left[2\left(t - \frac{\pi}{4}\right)\right]$$

The graphs of x_1 against t and of x_2 against t are said to have a **phase difference** of $\frac{\pi}{4}$. Specifically x_1 is ahead of, or 'leads' x_2 by $\frac{\pi}{4}$ radians.
More generally consider the following 2 sine waves of the same amplitude and frequency:

$$\begin{aligned} x_1(t) &= R\sin\omega t \\ x_2(t) &= R\sin(\omega t - \alpha) \end{aligned}$$

Now $x_1\left(t - \frac{\alpha}{\omega}\right) = R\sin\left[\omega\left(t - \frac{\alpha}{\omega}\right)\right] = R\sin(\omega t - \alpha) = x_2(t)$
so it is clear that the waves x_1 and x_2 are out of phase by $\frac{\alpha}{\omega}$. Specifically x_1 leads x_2 by $\frac{\alpha}{\omega}$.

Calculate the phase difference between the waves

$$\begin{aligned} x_1 &= 3\cos(10\pi t) \\ x_2 &= 3\cos\left(10\pi t + \frac{\pi}{4}\right) \end{aligned}$$

The time t is in seconds.

Your solution

Note firstly that the waves have the same amplitude 3 and angular frequency 10π (corresponding to a common period $\frac{2\pi}{10\pi} = \frac{1}{5}$ s)

Now $\cos\left(10\pi t + \frac{\pi}{4}\right) = \cos\left(10\pi\left(t + \frac{1}{40}\right)\right)$

so $x_1\left(t + \frac{1}{40}\right) = x_2(t)$.

In other words the phase difference is $\frac{1}{40}$ s, the wave x_2 **leads** the wave x_1 by this amount.

Alternatively we could say that x_1 **lags** x_2 by $\frac{1}{40}$ s.

Key Point

The equations

$$x = R\cos\omega t \qquad\qquad x = R\sin\omega t$$

both represent waves of amplitude R and period $\dfrac{2\pi}{\omega}$.

The **phase difference** between these waves is $\dfrac{\pi}{2\omega}$ because $\cos\left[\omega\left(t - \dfrac{\pi}{2\omega}\right)\right] = \sin\omega t$.

Combining two wave equations

A situation that arises in some applications is the need to combine two trigonometric terms such as

$$A\cos\theta + B\sin\theta.$$

For example this sort of situation might arise if we wish to combine two waves of the same frequency but not necessarily the same amplitude and with a phase difference. In particular we wish to be able to deal with an expression of the form

$$R_1\cos\omega t + R_2\sin\omega t$$

where the individual waves have, as we have seen, a phase difference of $\dfrac{\pi}{2\omega}$.

General Theory

Consider an expression

$$A\cos\theta + B\sin\theta.$$

We seek to transform this into the single form

$$C\cos(\theta - \alpha)$$

(or $C\sin(\theta-\alpha)$), where C and α have to be determined. The problem is easily solved with the aid of trigonometric identities.
We know that

$$C\cos(\theta-\alpha) = C(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

Hence if

$$A\cos\theta + B\sin\theta = C\cos(\theta-\alpha)$$

then

$$A\cos\theta + B\sin\theta = (C\cos\alpha)\cos\theta + (C\sin\alpha)\sin\theta$$

For this to be an identity (true for all values of θ) we must be able to equate the coefficients of $\cos\theta$ and $\sin\theta$ on each side.
Hence

$$A = C\cos\alpha \qquad\qquad B = C\sin\alpha$$

By squaring and adding these equations, obtain C in terms of A and B. Also, by eliminating C, obtain α in terms of A and B.

Your solution

$A = C\cos\alpha \qquad B = C\sin\alpha \quad$ gives

$$\begin{aligned} A^2 + B^2 &= C^2\cos^2\alpha + C^2\sin^2\alpha \\ &= C^2(\cos^2\alpha + \sin^2\alpha) \\ &= C^2 \end{aligned}$$

$\therefore \quad C = \sqrt{A^2+B^2}$ (We take the positive square root.)

Also, by division, $\dfrac{B}{A} = \dfrac{C\sin\alpha}{C\cos\alpha} = \tan\alpha$

so α is obtained by solving $\tan\alpha = \dfrac{B}{A}$. However, care must be taken to obtain the correct quadrant for α.

Key Point

If $A\cos\theta + B\sin\theta = C\cos(\theta-\alpha)$ then $C = \sqrt{A^2+B^2}$ and $\tan\alpha = \dfrac{B}{A}$

In terms of waves we have

$$R_1\cos\omega t + R_2\sin\omega t = R\cos(\omega t - \alpha)$$

where $R = \sqrt{R_1^2+R_2^2}$ $\qquad \tan\alpha = \dfrac{R_2}{R_1}$

The form $R\cos(\omega t-\alpha)$ is said to be the **amplitude/phase** form of the wave.

Note that we can write

$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\left\{\frac{A}{\sqrt{A^2+B^2}}\cos\theta + \frac{B}{\sqrt{A^2+B^2}}\sin\theta\right\}$$

Since $\dfrac{A}{\sqrt{A^2+B^2}}$ and $\dfrac{B}{\sqrt{A^2+B^2}}$ cannot be greater than 1 in magnitude we can write

$$\frac{A}{\sqrt{A^2+B^2}} = \cos\alpha \qquad \frac{B}{\sqrt{A^2+B^2}} = \sin\alpha \tag{6a, b}$$

So

$$\begin{aligned} A\cos\theta + B\sin\theta &= \sqrt{A^2+B^2}(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= C\cos(\theta-\alpha) \end{aligned}$$

where $C = \sqrt{A^2+B^2}$ as before and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{B}{\sqrt{A^2+B^2}}}{\frac{A}{\sqrt{A^2+B^2}}} = \frac{B}{A} \qquad \text{again as before.}$$

1. $A > 0,\ B > 0:$ 1st quadrant

2. $A < 0,\ B > 0:$ 2nd quadrant

3. $A < 0,\ B < 0:$ 3rd quadrant

4. $A > 0,\ B < 0:$ 4th quadrant

Example Express in the form $C\cos(\theta-\alpha)$ each of the following:

(a) $3\cos\theta+3\sin\theta$

(b) $-3\cos\theta+3\sin\theta$

(c) $-3\cos\theta-3\sin\theta$

(d) $3\cos\theta-3\sin\theta$

Solution

In each case : $C=\sqrt{A^2+B^2}=\sqrt{9+9}=\sqrt{18}$

(a) $\tan\alpha=\dfrac{B}{A}=\dfrac{3}{3}=1$ gives $\alpha=45°$ (A and B are both positive so the first quadrant is the correct one.) Hence

$$\begin{aligned} 3\cos\theta+2\sin\theta &= \sqrt{18}\cos(\theta-45°) \\ &= \sqrt{18}\cos\left(\theta-\frac{\pi}{4}\right) \end{aligned}$$

(b) The angle α must be in the second quadrant as $A=-3<0$, $B=+3>0$. By calculator : $\tan\alpha=-1$ gives $\alpha=-45°$ but this is in the 4th quadrant. Remembering that $\tan\alpha$ has period π or $180°$ we must therefore add $180°$ to the calculator value to obtain the correct α value of $135°$. Hence

$$-3\cos\theta+3\sin\theta=\sqrt{18}\cos(\theta-135°)$$

(c) Here $A=-3$, $B=-3$ so α must be in the 3rd quadrant. $\tan\alpha=\dfrac{-3}{-3}=1$ giving $\alpha=45°$ by calculator. Hence adding $180°$ to this tells us that

$$-3\cos\theta-3\sin\theta=\sqrt{18}\cos(\theta-225°)$$

(d) Here $A=3$ $B=-3$ so α is in the 4th quadrant. $\tan\alpha=-1$ gives us (correctly) $\alpha=-45°$ so

$$3\cos\theta-3\sin\theta=\sqrt{18}\cos(\theta+45°).$$

Note that in the amplitude/phase form the angle may be expressed in degrees or radians.

Write the wave form

$$x=3\cos\omega t+4\sin\omega t$$

in amplitude/phase form. Express the phase in radians.

Your solution

We have $x = R\cos(\omega t - \alpha)$ where

$$\begin{aligned} R &= \sqrt{3^2 + 4^2} = 5 \\ \tan\alpha &= \frac{4}{3} \end{aligned}$$

from which, using the calculator in radian mode, $\alpha = 0.927$ radians. This is in the first quadrant $\left(0 < \alpha < \dfrac{\pi}{2}\right)$ which is correct since $A = 3$ and $B = 4$ are both positive. Hence

$$x = 5\cos(\omega t - 0.927).$$

Exercises

1. Write down the amplitude and the period of $y = \frac{5}{2}\sin 2\pi t$.

2. Write down the amplitude, frequency and phase of

   (a) $y = 3\sin\left(2t - \dfrac{\pi}{3}\right)$

   (b) $y = 15\cos\left(5t - \dfrac{3\pi}{2}\right)$

3. The current in an a/c circuit is

$$i(t) = 30\sin 120\pi t \text{ amps}$$

   Here t is measured in seconds. What is the maximum current and at what times does it occur?

4. The depth y of water at the entrance to a small harbour at time t is

$$y = a\sin b\left(t - \frac{\pi}{2}\right) + k$$

   where k is the average depth. If the tidal period is 12 hours, the depths at high tide and low tide are 18 metres and 6 metres respectively, obtain a, b, k and sketch two cycles of the graph of y.

5. The Fahrenheit temperature at a certain location over 1 day is modelled by

$$F(t) = 60 + 10\sin\frac{\pi}{12}(t-8) \qquad 0 \leq t \leq 24$$

   where t is in the time in hours after midnight.

   (a) What are the temperatures at 8.00 am and 12.00 noon?

   (b) At what time is the temperature 60°F?

   (c) Obtain the maximum and minimum temperatures and the times at which they occur.

6. In each of the following write down expressions for shifted sine and shifted cosine functions that satisfy the given conditions:

   (a) Amplitude 3, Period $\dfrac{2\pi}{3}$, Phase shift $\dfrac{\pi}{3}$

   (b) Amplitude 0.7, Period 0.5, Phase shift 4.

7. Write the a/c current

$$i = 3\cos 5t + 4\sin 5t$$

   in the form $i = C\cos(5\pi - \alpha)$.

Exercises

8. Show that if

$$A \cos \omega t + B \sin \omega t = C \sin(\omega t + \alpha)$$

then

$$C = \sqrt{A^2 + B^2}, \qquad \cos \alpha = \frac{B}{C}, \qquad \sin \alpha = \frac{A}{C}.$$

9. Using Question 8 express the following in the amplitude/phase form $C \sin(\omega t + \alpha)$

   (a) $y = -\sqrt{3} \sin 2t + \cos 2t$

   (b) $y = \cos 2t + \sqrt{3} \sin 2t$

10. The motion of a weight on a spring is given by

$$y = \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t.$$

Obtain C and α such that

$$y = C \sin(8t + \alpha)$$

11. Show that for the two a/c currents

$$i_1 = \sin\left(\omega t + \frac{\pi}{3}\right) \qquad i_2 = 3\cos\left(\omega t - \frac{\pi}{6}\right)$$

then

$$i_1 + i_2 = 4\cos\left(\omega t - \frac{\pi}{6}\right).$$

12. Show that the power $P = \dfrac{v^2}{R}$ in an electrical circuit where $v = V_0 \cos\left(\omega t + \frac{\pi}{4}\right)$ is $P = \dfrac{V_0^2}{2R}(1 - \sin 2\omega t)$

13. Show that the product of the two signals

$$f_1(t) = A_1 \sin \omega t \qquad f_2(t) = A_2 \sin[\omega(t + \tau) + \phi]$$

is given by

$$f_1(t) f_2(t) = \frac{A_1 A_2}{2}[\cos(\omega\tau + \phi) - \cos(2\omega t + \omega\tau + \phi)].$$

1. $y = \frac{5}{2}\sin 2\pi t$ has amplitude $\frac{5}{2}$. The period is $\frac{2\pi}{2\pi} = 1$.

   Check: $y(t+1) = \frac{5}{2}\sin(2\pi(t+1)) = \frac{5}{2}\sin(2\pi t + 2\pi) = \frac{5}{2}\sin 2\pi t = y(t)$

2. (a) Amplitude 3, Period $\frac{2\pi}{2} = \pi$. Writing $y = 3\sin 2\left(t - \frac{\pi}{0}\right)$ we see that there is a phase shift of $\frac{\pi}{6}$ radians in this wave compared with $y = 3\sin 2t$.

   (b) Amplitude 15, Period $\frac{2\pi}{5}$. Clearly $y = 15\cos 5\left(t - \frac{3\pi}{10}\right)$ so there is a phase shift of $\frac{3\pi}{10}$ compared with $y = 15\cos 5t$.

3. Maximum current = 30 amps at a time t such that $120\pi t = \frac{\pi}{2}$. i.e. $t = \frac{1}{240}$ s.

   This maximum will occur again at $\left(\frac{1}{240} + \frac{n}{60}\right) s, \quad n = 1, 2, 3, \ldots$

4. $y = a\sin\left[b\left(t - \frac{\pi}{2}\right)\right] + h$. The period is $\frac{2\pi}{b} = 12$ hr $\quad \therefore \quad b = \frac{\pi}{6}$ hr^{-1}.

   Also since $y_{\max} = a + k \quad y_{\min} = -a + k$ we have $a + k = 18 \quad -a + k = 6$ so $k = 12$ m, $a = 6$ m. i.e. $y = 6\sin\left[\frac{\pi}{6}\left(t - \frac{\pi}{2}\right)\right] + 12$.

5. $F(t) = 60 + 10\sin\frac{\pi}{12}(t - 8) \qquad 0 \le t < 24$

   (a) At $t = 8$: temp = 60°F. At $t = 12$: temp $= 60 + 10\sin\frac{\pi}{3} = 68.7$°F

   (b) $F(t) = 60$ when $\frac{\pi}{12}(t-8) = 0, \ \pi, \ 2\pi, \ldots$ giving $t - 8 = 0, 12, 24, \ldots$ hours so $t = 8, 20, 32, \ldots$ hours i.e. in 1 day at $t = 8$ (8.00 am) and $t = 20$ (8.00 pm)

   (c) Maximum temperature is 70° F when $\frac{\pi}{12}(t = 8) = \frac{\pi}{2}$ i.e. at $t = 14$ (2.00 pm).

   Minimum temperature is 50°F when $\frac{\pi}{12}(t - 8) = \frac{3\pi}{2}$ i.e. at $t = 26$ (2.00 am).

6. (a) $y = 3\sin(3t - \pi) \qquad y = 3\cos(3t - \pi)$

   (b) $y = 0.7\sin(4\pi t - 16) \qquad y = 0.7\cos(4\pi t - 16)$

7. $C = \sqrt{3^2 + 4^2} = 5 \ \tan\alpha = \frac{4}{3}$ and α must be in the first quadrant (since $A = 3, \ B = 4$ are both positive.) $\quad \therefore \quad \alpha = \tan^{-1}\frac{4}{3} = 0.9273$ rads $\qquad \therefore \quad i = 5\cos(5t - 0.9273)$

8. Since $\sin(\omega t + \alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$ then $A = C\sin\alpha$ (coefficients of $\cos\omega t$) $B = C\cos\alpha$ (coefficients of $\sin\omega t$) from which $C^2 = A^2 + B^2$, $\sin\alpha = \frac{A}{C}$, $\cos\alpha = \frac{B}{C}$

9.

(a) $C = \sqrt{3+1} = 2;\quad \cos\alpha = -\dfrac{\sqrt{3}}{2} \qquad \sin\alpha - \dfrac{1}{2}$ so α is in the second quadrant,

$\alpha = \dfrac{5\pi}{6} \quad \therefore \quad y = 2\sin\left(2t + \dfrac{5\pi}{6}\right)$

(b) $y = 2\sin\left(2t + \dfrac{\pi}{6}\right)$

10. $C^2 = \dfrac{4}{9} + \dfrac{1}{36} = \dfrac{17}{36}$ so $\quad C = \dfrac{\sqrt{17}}{6} \quad \cos\alpha = \dfrac{-\frac{1}{6}}{\frac{\sqrt{17}}{6}} = -\dfrac{1}{\sqrt{17}} \qquad \sin\alpha = \dfrac{\frac{2}{3}}{\frac{\sqrt{17}}{6}} = \dfrac{4}{\sqrt{17}}$

so α is in the second quadrant. $\alpha = 1.8158$ radians.

11. Since $\sin x = \cos\left(x - \dfrac{\pi}{2}\right)$ $\sin\left(\omega t + \dfrac{\pi}{3}\right) = \cos\left(\omega t + \dfrac{\pi}{3} - \dfrac{\pi}{2}\right) = \cos\left(\omega t - \dfrac{\pi}{6}\right)$

$\therefore \quad i_1 + i_2 = \cos\left(\omega t - \dfrac{\pi}{6}\right) + 3\cos\left(\omega t - \dfrac{\pi}{6}\right) = 4\cos\left(\omega t - \dfrac{\pi}{6}\right)$

12. $\qquad v = V_0\cos\left(\omega t + \dfrac{\pi}{4}\right) = V_0\left(\cos\omega t\cos\frac{\pi}{4} - \sin\omega t\sin\frac{\pi}{4}\right) = \frac{V_0}{\sqrt{2}}\left(\cos\omega t - \sin\omega t\right)$

$\therefore \qquad v^2 = \frac{V_0^2}{2}(\cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t) = \frac{V_0^2}{2}(1 - \sin 2\omega t)$

and hence

$$P = \frac{v^2}{R} = \frac{V_0^2}{2R}(1 - \sin 2\omega t.)$$

13. Since the required answer involves the difference of two cosine functions we use the identity

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Hence with $\dfrac{A+B}{2} = \omega t, \quad \dfrac{B-A}{2} - \omega t + \omega\tau + \phi.$

We find, by adding these equations $\quad B = 2\omega t + \omega\tau + \phi\;$ and by subtracting $\quad A = -\omega\tau - \phi.$

Hence $\qquad \sin(\omega t)\sin(\omega t + \omega\tau + \phi) = \dfrac{1}{2}\left[\cos(\omega\tau + \phi) - \cos(2\omega t + \omega\tau + \phi)\right].$

(Recall that $\cos(-x) = \cos x$.) The required result then follows immediately.