Tests Concerning Two Samples

41.3

Introduction

So far we have dealt with situations in which we either had a single sample drawn from a population, or paired data whose differences were considered essentially as a single sample.

In this Section we shall look at the situations occurring when we have two random samples each drawn from *independent* populations. While the basic ideas involved will essentially repeat those already met, you will find that the calculations involved are more complex than those already covered. However, you will find as before that calculations do follow particular routines. Note that in general the samples will be of different sizes. Cases involving samples of the same size, while included, should be regarded as special cases.

Prerequisites

Before starting this Section you should ...

① be familiar with the normal t-, F- and chi-squared distributions

② be familiar with the contents of Section 41.1 and 41.2

Learning Outcomes

After completing this Section you should be able to ...

✓ be able to apply the ideas of hypothesis testing to a range of problems underpinned by a substantial range of statistical distributions and involving two samples of different sizes

1. Tests concerning Two Samples

Two Independent Populations each with a Known Variance

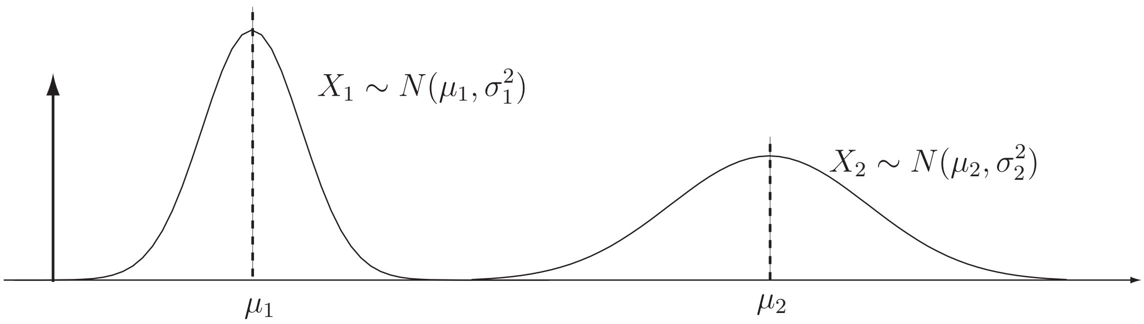
We assume that the populations are normally distributed. This may not always be true and you should note this basic assumption while studying this section of the booklet.

A standard notation often used to describe the populations and samples is:

Population	Sample
$X_1 \sim N(\mu_1, \sigma_1^2)$	$x_{11}, x_{12}, x_{13}, \cdots, x_{1n_1}$ with n_1 members.
$X_2 \sim N(\mu_2, \sigma_2^2)$	$x_{21}, x_{22}, x_{23}, \cdots, x_{2n_2}$ with n_2 members.

If you are not familiar with the double suffix notation used to represent the samples, simply remember that a random sample of size n_1 is drawn from $X_1 \sim N(\mu_1, \sigma_1^2)$ and a random sample of size n_2 is drawn from $X_1 \sim N(\mu_1, \sigma_1^2)$.

In diagrammatic form the populations may be represented as follows:



When we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$$H_0: \quad \mu_1 - \mu_2 = \text{Value}$$

As you might expect, 'Value' will often be zero and we will be trying to detect whether there is any statistically significant evidence of a difference between the means.

We know, from our previous work on continuous distributions (see Workbook 38) that:

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu_1 - \mu_2$$

and that

$$V(\bar{X}_1 - \bar{X}_2) = V(\bar{X}_1) - V(\bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

since $\bar{X}_1$ and $\bar{X}_2$ are independent. Given the assumptions made we can assert that the quantity Z defined by

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

follows the standard normal distribution $N(0, 1)$.

We are now ready to apply this formula to practical problems in which random samples of different sizes are drawn from normal populations. The conditions for the rejection of H_0 at the

5% and the 1% levels of significance are exactly the same as those previously used for single sample problems.

Example A motor manufacturer wishes to replace steel suspension components by aluminium components to save weight and thereby improve performance and fuel consumption. Tensile strength tests are carried out on randomly chosen samples of two possible components before a final choice is made. The results are:

Component Number	Sample Size	Mean Tensile Strength (kr/mm^2)	Standard Deviation (kr/mm^2)
1	15	90	2.3
2	10	88	2.2

Is there any difference between the measured tensile strengths at the 5% level of significance?

Solution

The null and alternative hypotheses are:

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The null hypothesis represent the statement 'there is no difference in the tensile strengths of the two components.' The test statistic Z is calculated as:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(90 - 88) - (0)}{\sqrt{\frac{2.3^2}{15} + \frac{2.2^2}{10}}} = \frac{2}{\sqrt{0.3527 + 0.484}} = 2.186$$

Since $2.186 > 1.96$ we conclude that, on the basis of the (limited) evidence available, there is a difference in tensile strength between the components tested. The manufacturer should carry out more comprehensive tests before making a final decision as to which component to use. The decision is a serious one with safety implications as well as economic implications. As well as carrying out more tests the manufacturer should consider the level of rejection of the null hypothesis, perhaps using 1% instead of 5%. Component 1 appears to be stronger but this may not be the case after more tests are carried out.

A motor manufacturer is considering whether or not a new fuel formulation will improve the maximum power output of a particular type of engine. Tests are carried out on randomly chosen samples of the two fuels in order to inform a decision. The results are:

Fuel Type	Sample Size	Mean Maximum Power Output (bhp)	Standard Deviation (bhp)
1	20	1350	10
2	16	131	8

Is there any difference between the measured power outputs at the 5% level of significance?

Your solution

The null and alternative hypotheses are:

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The null hypothesis represent the statement 'there is no difference in the measured maximum power outputs.' The test statistic Z is calculated as:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(135 - 131) - (0)}{\sqrt{\frac{10^2}{20} + \frac{8^2}{16}}} = \frac{4}{\sqrt{5+4}} = 1.33$$

Since $1.33 < 1.96$ we conclude that, on the basis of the (limited) evidence available, there is insufficient evidence to conclude that there is a difference in the maximum power output of the engines tested when run on the different types of fuel.

Two Independent Populations each with an Unknown Variance

Again we assume that the populations are normally distributed and use the same standard notation used previously to describe the populations and samples, namely:

Population	Sample
$X_1 \sim N(\mu_1, \sigma_1^2)$	$x_{11}, x_{12}, x_{13}, \cdots, x_{1n_1}$ with n_1 members.
$X_2 \sim N(\mu_2, \sigma_2^2)$	$x_{21}, x_{22}, x_{23}, \cdots, x_{2n_2}$ with n_2 members.

There are two distinct cases to consider. Firstly, we will assume that although the variances are unknown, they are in fact equal. Secondly, we will assume that the unknown variances are not necessarily equal.

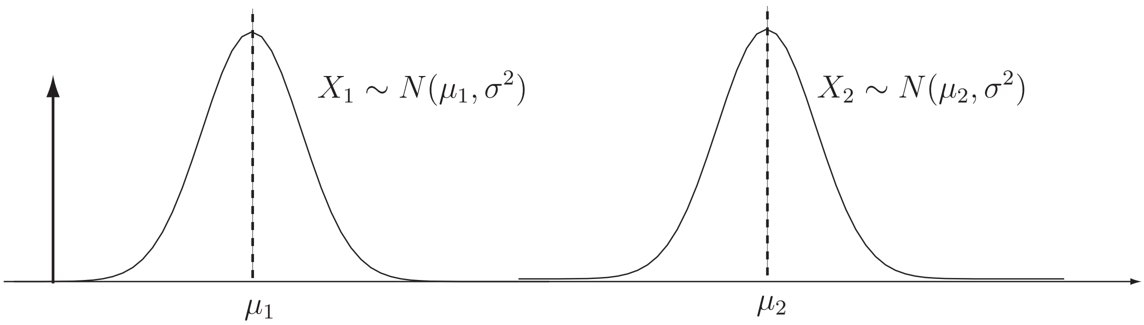
Case (i) - Unknown but Equal Variances

Again, when we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$$H_0: \quad \mu_1 - \mu_2 = \text{Value}$$

and again Value will often be zero and we will be trying to detect whether there is any statistically significant difference between the means.

We will take $\sigma_1^2 = \sigma_2^2 = \sigma^2$ so that in diagrammatic form the populations are:



The results from our work on continuous distributions (see Workbook *.*) tell us that:

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu_1 - \mu_2$$

as before, and that

$$V(\bar{X}_1 - \bar{X}_2) = V(\bar{X}_1) - V(\bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Given that we do not know the value of σ, we must estimate it. This is done by combining (or pooling) the sample variances say S_1^2 and S_2^2 for samples 1 and 2 respectively according to the formula:

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Notice that

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{(n_1 - 1)S_1^2}{n_1 + n_2 - 2} + \frac{(n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

so that you can see that S_c^2 is a weighted average of S_1^2 and S_2^2. In fact, each sample variance is weighted according to the number of degrees of freedom available. Notice also that the first sample contributes degrees of freedom and the second sample contributes degrees of freedom so that has degrees of freedom.

Since we are estimating unknown variances, the quantity T defined by

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

will follow Students t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

We are now ready to apply this formula to practical problems in which random samples of different sizes with unknown but equal variances are drawn from independent normal populations. The conditions for the rejection of H_0 at the 5% and the 1% levels of significance are found from tables of the t-distribution, a copy of which is attached to the end of this booklet.

Example A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the following results are obtained.

Test Number	Circuit 1 - Current (mA)	Circuit 2 - Current (mA)
1	80.1	80.7
2	82.3	81.3
3	84.1	84.6
4	82.6	81.7
5	85.3	86.3
6	81.3	84.3
7	83.2	83.7
8	81.7	84.7
9	82.2	82.8
10	81.4	84.4
11		85.2
12		84.9

On the assumption that the populations from which the samples are drawn have equal variances, should the manufacturer replace the old circuit design by the new one? Use the 5% level of significance.

Solution

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The pooled estimate of the variance is

$$S_c^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2} = \frac{9\times 2.00+11\times 2.72}{20} = 2.396$$

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{2.396}\sqrt{\frac{1}{10}+\frac{1}{12}}} = -1.267$$

From $t$tables, the critical values with 20 degrees of freedom and a two-tailed test are . Since $-2.086 < -1.267 < 2.086$ we conclude that we cannot reject the null hypothesis in favour of the alternative. A 95% confidence interval for the variance is given by where . The confidence interval is

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The following results were obtained, the failure tensions being given in kg$\times 10^3$.

Test Number	New Cable	Original Cable
1	92.7	90.2
2	91.6	92.4
3	94.7	94.7
4	93.7	92.1
5	96.5	95.9
6	94.3	91.1
7	93.7	93.2
8	96.8	91.5
9	98.9	
10	99.9	

The cable manufacturer, on looking at health and safety legislation, decides that a 1% level of significance should be used in any statistical testing procedure adopted to distinguish between the cables. On the basis of the results given, should the manufacturer replace the old cable by the new one? You may assume that the populations from which the samples are drawn have equal variances.

Your solution

If the average tensions are represented by μ_1 (new cable) and μ_2 (old cable) we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 > 0$$

in order to test the hypothesis that the new cable is stronger on average than the old cable.

The sample means are $\bar{X}_1 = 95.28$ and $\bar{X}_2 = 92.64$.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The pooled estimate of the variance is $\quad S_c^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2} = \frac{9\times 6.47+7\times 3.14}{16} = 5.013$

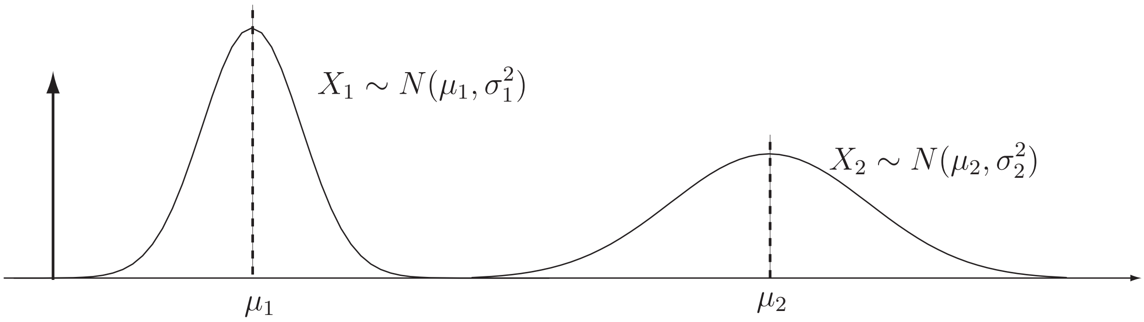
The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{95.28 - 92.64}{\sqrt{2.239}\sqrt{\frac{1}{10}+\frac{1}{8}}} = \frac{2.64}{2.239 \times \sqrt{0.225}} = 2.486$$

Using t-distribution tables with 16 degrees of freedom, we see that the critical value at the 1% level of significance is 2.583. Since $2.486 < 2.583$ we conclude that we cannot reject the null hypothesis in favour of the alternative. However, the close result indicates that more tests should be carried out before making a final decision. At this stage the cable manufacturer should not replace the old cable by the new one on the basis of the evidence available.

Case (ii) - Unknown and Unequal Variances

In this case we will take $\sigma_1^2 \neq \sigma_2^2$ so that in diagrammatic form the populations may be represented as shown below.



Again, when we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$$H_0: \quad \mu_1 - \mu_2 = \text{Value}$$

and again Value will often be zero and we will be trying to detect whether there is any statistically significant difference between the means.

In the case where we assume unequal variances, there is no exact statistic which we can use to test the validity or otherwise of the null hypothesis $H_0: \quad \mu_1 - \mu_2 = \text{Value}$. However, the following approximation allows us to overcome this problem.

Key Point

Provided that the null hypothesis is true, the statistic

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

will *approximately* follow Students distribution with the number of degrees of freedom given by the expression:

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1+1} + \frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2+1}} - 2$$

Essentially, this means that the actual test procedure is similar to that used previously but with T and the number of degrees of freedom ν calculated using the above formulae.

We are now ready to apply these formulae to practical problems in which random samples of different sizes with unknown and unequal variances are drawn from independent normal populations. We will illustrate the test procedure by reworking the examples done previously but we will assume unequal rather than equal variances.

Example A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the results are obtained re those given in the Worked Example on Page 6. On the assumption that the populations from which the samples are drawn do not have equal variances, should the manufacturer replace the old circuit design by the new one? Use the 5% level of significance.

Solution

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $\S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{\frac{2.00}{10} + \frac{2.72}{12}}} = -\frac{1.3}{\sqrt{0.427}} = -1.990$$

The number of degrees of freedom is given by

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1+1} + \frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2+1}} - 2 = \frac{\left(\frac{2.00}{10} + \frac{2.72}{12}\right)^2}{\frac{(2.00/10)^2}{11} + \frac{(2.72/12)^2}{13}} - 2 = \frac{0.182}{0.004 + 0.004} - 2 \approx 21$$

From t-tables, the critical values (two-tailed test, 5% level of significance) are ± 2.080. Since $-2.080 < -1.990 < 2.080$ we conclude that there is insufficient evidence to reject the null hypothesis in favour of the alternative at the 5% level of significance.

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The results obtained are those given in the Worked Example on Page 23, the failure tensions being given in kg$\times 10^3$.

Test Number	New Cable	Original Cable
1	92.7	90.2
2	91.6	92.4
3	94.7	94.7
4	93.7	92.1
5	96.5	95.9
6	94.3	91.1
7	93.7	93.2
8	96.8	91.5
9	98.9	
10	99.9	

The cable manufacturer, on looking at health and safety legislation, decides that a 1% level of significance should be used in any statistical testing procedure adopted to distinguish between the cables. On the basis of the results given and assuming that the populations from which the samples are drawn do not have equal variances, should the manufacturer replace the old cable by the new one?

Your solution

If the average tensions are represented by μ_1 (new cable) and μ_2 (old cable) we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 > 0$$

in order to test the hypothesis that the new cable is stronger on average than the old cable.

The sample means are $\bar{X}_1 = 95.28$ and $\bar{X}_2 = 92.64$.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{95.28 - 92.64}{\sqrt{\frac{6.47}{10} + \frac{3.14}{8}}} = \frac{2.64}{\sqrt{1.017}} = 2.589$$

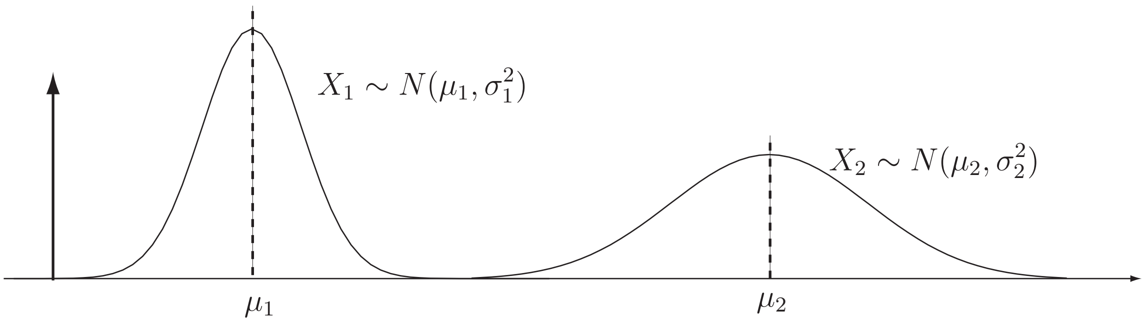
The number of degrees of freedom is given by

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1+1} + frac\left(\frac{S_2^2}{n_2}\right)^2 n_2 + 1} - 2 = \frac{\left(\frac{6.47}{10} + \frac{3.14}{8}\right)^2}{\frac{(6.47/10)^2}{11} + \frac{(3.14/8)^2}{9}} - 2 = \frac{1.081}{0.038 + 0.017} - 2 \approx 4$$

Using t-distribution tables with 18 degrees of freedom, we see that the critical value at the 1% level of significance is 2.552. Since $2.589 < 2.552$ we conclude that we reject the null hypothesis in favour of the alternative. Notice that the result could still be considered marginal. The cable manufacturer should exercise caution if the old cable is replaced by the new one on the basis of the evidence available.

The F-test

In the tests above, we distinguished between the cases of equal and unequal variances of samples chosen from independent normal populations. As you have seen, the analysis changes according to the assumptions made, conclusions reached and recommendations made - accepting or rejecting a null hypothesis for example - may also change. In view of this, we may wish to test in order to decide whether the assumption that the variances σ_1^2 and σ_2^2 of the independent normal populations shown in the diagram below, may be regarded as equal.



Essentially, we will test the null hypothesis

$$H_0: \quad \sigma_1^2 = \sigma_2^2$$

against one of the alternatives

$$H_1: \quad \sigma_1^2 \neq \sigma_2^2 \qquad H_1: \quad \sigma_1^2 > \sigma_2^2 \qquad H_1: \quad \sigma_1^2 < \sigma_2^2$$

In order to do this, we use the F distribution. The hypothesis test for the equality of two variances σ_1^2 and σ_2^2 is encapsulated in the following Key Point.

Key Point

Consider a random sample of size n_1 taken from a normal population with mean μ_1 and variance σ_1^2 and a random sample of size n_1 taken from a second normal population with mean μ_2 and variance σ_2^2. Denote the respective sample variances by S_1^2 and S_2^2 and assume that the populations are independent. The ratio

$$F = \frac{S_1^2}{\sigma_1^2} / \frac{S_2^2}{\sigma_2^2}$$

follows an F distribution in which the numerator has $n_1 - 1$ degrees of freedom and the denominator has $n_2 - 1$ degrees of freedom. Note that if the null hypothesis $H_0: \quad \sigma_1^2 = \sigma_2^2$ is true, then the value of F reduces to the ratio of the sample variances and that in this case

$$F = \frac{S_1^2}{S_2^2}$$

Note

Recall that if a random sample of size n_1 is taken from a normal population with mean μ_1 and variance σ_1^2 and if the sample variance is denoted by S_1^2, the random variable

$$X_1^2 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2}$$

has a χ^2 distribution with $n_1 - 1$ degrees of freedom. Similarly, if a random sample of size n_2 is taken from a normal population with mean μ_2 and variance σ_2^2 and if the sample variance is denoted by S_2^2, the random variable

$$X_2^2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2}$$

has a χ^2 distribution with $n_2 - 1$ degrees of freedom. This means that the ratio

$$F = \frac{S_1^2}{\sigma_1^2} / \frac{S_2^2}{\sigma_2^2}$$

is a ratio of χ^2 random variables with $n_1 - 1$ degrees of freedom in the numerator and $n_2 - 1$ degrees of freedom in the denominator. Under the null hypothesis

$$H_0: \quad \sigma_1^2 = \sigma_2^2$$

we know that the expression for F reduces to

$$F = \frac{S_1^2}{S_2^2}$$

and we say that F has an F-distribution with $n_1 - 1$ degrees of freedom in the numerator and $n_2 - 1$ degrees of freedom in the denominator. This distribution is denoted by

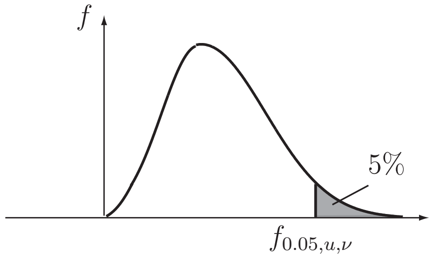
$$F_{n_1-1,n_2-1}$$

and some tabulated values are given in tables at the end of this booklet. If you check the tables, you will find that only right-tail values are given. The left-tail values are calculated by using the following formula:

$$F_{\text{left-tail value},\ n_1-1,\ n_2-1} = \frac{1}{F_{\text{right-tail value},\ n_2-1,\ n_1-1}}$$

Note the reversal in the order in which the expressions for the number of degrees of freedom occur.

Example The following is an extract from the F distribution tables (5% tail) given at the end of this booklet.



	Degrees of Freedom for the Numerator (u)													
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	248.0	250.1	251.1	252.2	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.46	19.47	19.48	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.62	8.59	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.75	5.72	5.69	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.53	4.46	4.43	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.81	3.77	3.74	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.38	3.34	3.30	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	3.08	3.04	3.01	2.93

Write down or calculate as appropriate, the following values of F from the table

Right-tail	Values Left-tail Values
$F_{5\%,4,3}$	$F_{5\%,4,3}$
$F_{5\%,8,2}$	$F_{5\%,8,2}$
$F_{5\%,7,8}$	$F_{5\%,7,8}$

Solution

The right-tail values are read directly from the tables. The left-tail values are calculated using the formula given above.

Right-tail	Values Left-tail Values
$F_{5\%,4,3} = 9.12$	$F_{5\%,4,3} = \frac{1}{F_{5\%,3,4}} = \frac{1}{6.59} = 0.152$
$F_{5\%,8,2} = 19.37$	$F_{5\%,8,2}$
$F_{5\%,7,8} = 3.50$	$F_{5\%,7,8}$

Write down or calculate as appropriate, the following values of F from the tables given at the end of this booklet.

Right-tail	Values Left-tail Values
$F_{5\%,10,20}$	$F_{5\%,10,20}$
$F_{5\%,5,30}$	$F_{5\%,5,30}$
$F_{5\%,20,7}$	$F_{5\%,20,7}$
$F_{2.5\%,10,10}$	$F_{2.5\%,10,10}$
$F_{2.5\%,8,30}$	$F_{2.5\%,8,30}$
$F_{2.5\%,20,30}$	$F_{2.5\%,20,30}$

Your solution

The right-tail values are read directly from the tables. The left-tail values are calculated using the formula

$$F_{\text{left-tail value},\ n_1-1,n_2-1} = \frac{1}{F_{\text{right-tail value},\ n_2-1,n_1-1}}$$

Right-tail	Values Left-tail Values
$F_{5\%,10,20} = 2.35$	$F_{5\%,10,20} = \frac{1}{F_{5\%,20,10}} = \frac{1}{2.77} = 0.361$
$F_{5\%,5,30} = 2.53$	$F_{5\%,5,30} = \frac{1}{F_{5\%,30,5}} = \frac{1}{4.53} = 0.221$
$F_{5\%,20,7} = 3.44$	$F_{5\%,20,7} = \frac{1}{F_{5\%,7,20}} = \frac{1}{2.51} = 0.398$
$F_{2.5\%,10,10} = 3.72$	$F_{2.5\%,10,10} = \frac{1}{F_{5\%,10,10}} = \frac{1}{3.72} = 0.269$
$F_{2.5\%,8,30} = 2.65$	$F_{2.5\%,8,30} = \frac{1}{F_{5\%,30,8}} = \frac{1}{3.89} = 0.257$
$F_{2.5\%,20,30} = 2.20$	$F_{2.5\%,20,30} = \frac{1}{F_{5\%,30,20}} = \frac{1}{2.35} = 0.426$

We are now in a position to use the F-test to solve engineering problems. The application of the F-test will be illustrated by using the data given in a previous worked example in order to determine whether the assumption of equal variability in the samples used is realistic.

Example A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the results obtained are those given on Page 6.

Last time, we worked on the assumption that the populations from which the samples are drawn do not have equal variances. Is this assumption valid at the 5% level of significance?

Note that the manufacturer may also be interested in knowing whether the variances are equal as well as the means. We shall not address the problem here but it can be argued that equality of variances will facilitate consistent performance from the components.

Solution

We form the hypotheses

$$H_0: \quad \sigma_1^2 = \sigma_2^2 \qquad H_1: \quad \sigma_1^2 \neq \sigma_2^2$$

and perform a two-tailed test.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{2.00}{2.72} = 0.735$$

which has an F-distribution with 9 degrees of freedom in the numerator and 11 degrees of freedom in the denominator.

We require two 2.5% tails, that is we require right-tail $F_{2.5\%,9,11} = 3.59$ and left-tail $F_{2.5\%,9,11}$. The latter may be roughly estimated as

$$F_{2.5\%,9,11} = \frac{1}{F_{2.5\%,11,9}} = \frac{1}{F_{2.5\%,10,9} - \frac{F_{2.5\%,10,9} - F_{2.5\%,20,9}}{10}} = \frac{1}{3.96 - \frac{3.96-3.67}{10}} \approx 0.254$$

Since $0.254 < 0.735 < 3.59$ we conclude that we cannot reject the null hypothesis in favour of the alternative at the 5% level of significance. The evidence supports the conclusion that the samples have equal variability.

Note that we can adopt the rule (many statisticians do this) of always dividing the larger S^2 value by the smaller S^2 value so that you only need to look up right tail values.

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The results obtained are those given on Page 8, the failure tensions being given in kg$\times 10^3$. Last time we assumed that the populations from which the samples are drawn did not have equal variances. Is this assumption valid at the 5% level of significance?

Your solution

We form the hypotheses

$$H_0: \ \sigma_1^2 = \sigma_2^2 \qquad H_1: \ \sigma_1^2 \neq \sigma_2^2$$

and perform a two-tailed test.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{6.47}{3.14} = 2.061$$

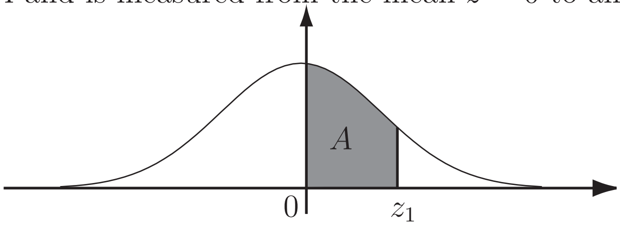
which has an F-distribution with 9 degrees of freedom in the numerator and 7 degrees of freedom in the denominator. We require two 2.5% tails, that is we require right-tail $F_{2.5\%,9,7} = 4.42$ and left-tail $F_{2.5\%,9,7}$ which may be calculated as

$$F_{2.5\%,9,7} = \frac{1}{F_{2.5\%,7,9}} = \frac{1}{4.20} = 0.238$$

Since $0.238 < 2.061 < 4.82$ we conclude that we cannot reject the null hypothesis in favour of the alternative at the 5% level of significance. The evidence does not support the conclusion that the populations have unequal variances.

The Normal Probability Integral

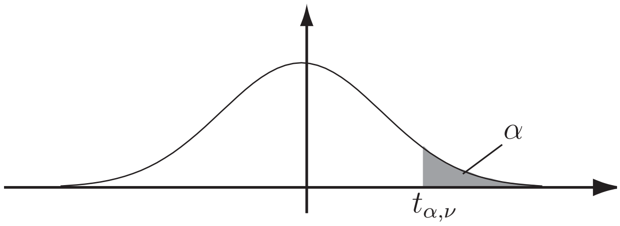
The area is denoted by A and is measured from the mean $z = 0$ to any ordinate $z = z_1$.



$Z = \frac{X-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	0040	0080	0120	0159	0199	0239	0279	0319	0359
0.1	.0398	0438	0478	0517	0557	0596	0636	0657	0714	0753
0.2	.0793	0832	0871	0910	0948	0987	1026	1064	1103	1141
0.3	.1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
0.4	.1554	1591	1628	1664	1700	1736	1772	1808	1844	1879
0.5	.1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
0.6	.2257	2291	2324	2357	2389	2422	2454	2486	2518	2549
0.7	.2530	2611	2642	2673	2704	2734	2764	2794	2823	2852
0.8	.2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
0.9	.3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	.3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	.3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	.3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	.4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	.4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	.4332	4345	4357	4370	4382	4394	4406	4418	4430	4441
1.6	.4452	4463	4474	4485	4495	4505	4515	4525	4535	4545
1.7	.4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	.4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	.4713	4719	4726	4732	4738	4744	4750	4756	4762	4767
2.0	.4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	.4621	4826	4830	4835	4838	4842	4846	4850	4854	4857
2.2	.4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	.4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952
2.6	.4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	.4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	.4974	4975	4976	4977	4977	4978	4979	4980	4980	4981
2.9	.4981	4982	4983	4983	4984	4984	4985	4985	4986	4986
3.0	.4986	4987	4987	4988	4988	4989	4989	4989	4990	4990
3.1	.4990	4991	4991	4991	4992	4992	4992	4992	4993	4993
3.2	.4993	4994	4994	4994	4994	4994	4994	4995	4995	4995
3.3	.4995	4995	4995	4996	4996	4996	4996	4996	4996	4997
3.4	.4997	4997	4997	4997	4997	4997	4997	4997	4997	4998
3.5	.4998	4998	4998	4998	4998	4998	4998	4998	4998	4998
3.6	.4998	4998	4999	4999	4999	4999	4999	4999	4999	4999
3.7	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999
3.8	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999
3.9	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999

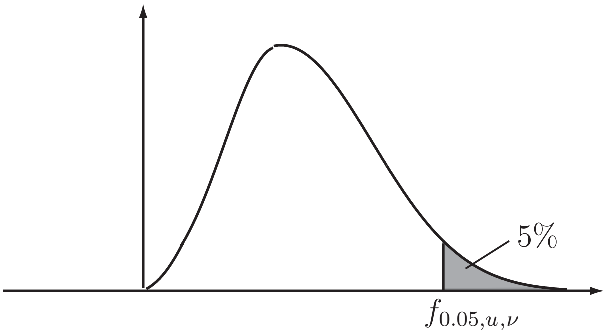
Note that some text books give the final line entries as 0.5 rather than 0.4999.
In these workbooks we shall use 0.4999.

Percentage Points of the Students t-distribution



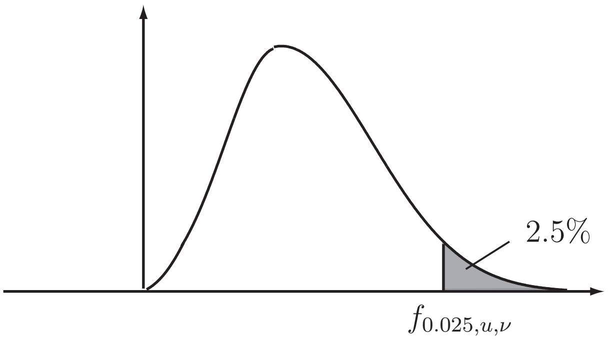
α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
ν										
1	.325	1.000	3.078	6.314	12.706	31.825	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.902	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.514	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.487
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Percentage Points of the *F* Distribution (5% tail)



	Degrees of Freedom for the Numerator (u)														
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	248.0	250.1	251.1	252.2	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.46	19.47	19.48	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.62	8.59	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.75	5.72	5.69	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.53	4.46	4.43	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.81	3.77	3.74	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.38	3.34	3.30	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	3.08	3.04	3.01	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	2.94	2.86	2.83	2.79	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.77	2.70	2.66	2.62	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.65	2.57	2.53	2.49	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.54	2.47	2.43	2.38	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.46	2.38	2.34	2.30	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.39	2.31	2.27	2.22	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.33	2.25	2.20	2.16	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.28	2.19	2.15	2.11	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.23	2.15	2.10	2.06	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.19	2.11	2.06	2.02	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.16	2.07	2.03	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.12	2.04	1.99	1.95	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.10	2.01	1.96	1.92	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.07	1.98	1.94	1.89	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.05	1.96	1.91	1.86	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.03	1.94	1.89	1.84	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.01	1.92	1.87	1.82	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	1.99	1.90	1.85	1.80	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	1.97	1.88	1.84	1.79	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	1.96	1.87	1.82	1.77	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	1.94	1.85	1.81	1.75	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	1.93	1.84	1.79	1.74	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	1.84	1.74	1.69	1.64	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.75	1.65	1.59	1.53	1.39
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.57	1.46	1.39	3.32	1.00

Percentage Points of the F Distribution (2.5% tail)



	Degrees of Freedom for the Numerator (u)														
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	993.1	1001	1006	1010	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.45	39.46	39.47	39.48	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.17	14.08	14.04	13.99	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.56	8.46	8.41	8.36	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.33	6.23	6.18	6.12	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.17	5.07	5.01	4.96	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.75	4.47	4.36	4.31	4.25	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.00	3.89	3.84	3.78	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.67	3.56	3.51	3.45	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.42	3.31	3.26	3.20	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.23	3.12	3.06	3.00	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.07	2.96	2.91	2.85	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	2.95	2.84	2.78	2.72	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	2.84	2.73	2.67	2.61	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.76	2.64	2.59	2.52	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.32	3.12	3.05	2.99	2.68	2.57	2.51	2.45	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.62	2.50	2.44	2.38	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.56	2.44	2.38	2.32	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.51	2.39	2.33	2.27	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.46	2.35	2.29	2.22	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.42	2.31	2.25	2.18	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.39	2.27	2.21	2.14	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.36	2.24	2.18	2.11	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.33	2.21	2.15	2.08	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.30	2.18	2.12	2.05	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.28	2.16	2.09	2.03	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.25	2.13	2.07	2.00	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.23	2.11	2.05	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.21	2.09	2.03	1.96	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.20	2.07	2.01	1.94	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.07	1.94	1.88	1.80	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	1.94	1.82	1.74	1.67	1.48
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.71	1.57	1.48	1.39	1.00