Content 5

functions and modelling

Learning outcomes

Needs doing

Time allocation

You are expected to spend approximately nine hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Functions and Modelling

5.1

Introduction

In this section we look at the process of modelling with mathematics as it is vitally important in engineering. Knowledge of mathematics is not much use unless it can be applied to engineering problems. After discussing the mathematical modelling process we discuss the use of linear models.

Prerequisites

Before starting this Section you should . . .

① be familiar with algebraic manipulation

② be familiar with linear functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ have some understanding of the modelling process

✓ use linear functions to model motion under constant acceleration

✓ analyse motion under gravity

1. Functions and Modelling

Engineers use mathematics to a considerable extent. Mathematical techniques offer ways of handling *mathematical models* of an engineering problem and coming up with a solution. Of course it is possible to model a problem in ways that are not mathematical e.g. by physical or scale modelling. but this learning suport programme is concerned exclusively with mathematical modelling, so we'll drop the word 'mathematical' and refer just to modelling. This section is intended to introduce some *modelling* ideas as well as to show applications of the functions and techniques introduced in Workbooks 2,3 and 6. By *modelling* we mean the process by which we set up a mathematical model of a situation or of an assumed situation, use the model to make some predictions and then interpret the results in the original context. The mathematical techniques themselves contribute only to part of the modelling procedure. The modelling procedure can be regarded as a cycle. If we don't like the outcome for some reason we can try again. Five steps of a modelling cycle can be identified as follows:

Step 1 **Specify** the **purpose** of the model.

Step 2 **Create** the mathematical **model** after making and stating relevant assumptions.

Step 3 **Do** the resulting **math**ematic**s**.

Step 4 **Interpret** the **results**.

Step 5 **Evaluate** the outcome, usually by comparing with reality and/or purpose and, if necessary, try again!.

Much of this first section of engineering exercises is concerned with steps 2 and 3 of the cycle: **creating a mathematical model** and **doing the maths**. The engineering case studies will aim to demonstrate the complete cycle.

An important part of step 2 may include choosing an *appropriate* function based on the assumptions made also as part of this step. This choice will influence the kind of mathematical activity that is involved in step 3.

So far in your engineering mathematical studies you might have had little opportunity to think about what is appropriate, since the type of function to be studied and used has been chosen for you. Sometimes, however, you may be faced with making appropriate choices of function for yourself so it is important to have some understanding of what might be appropriate in any given circumstance. A well chosen function will be appropriate in two different ways. First the function should be consistent with the purpose of the model, with known *data* or *theory* or *facts*, and with known or assumed *behaviour*. For example, the purpose might be to predict the future behaviour of a quantity which is expected to increase with time. In this case time can be identified as the independent variable since the quantity depends on time. The function chosen for mathematical activity should be one in which the value of the dependent variable increases with time. Secondly, *bearing in mind that the modelling process is a cycle and so it is possible, and usual, to go round it more than once*, the first choice of function should be as simple as allowed by the modelling context. The main reason for doing this is to avoid complication unless it is really necessary. Philosophically, an initial choice of a simple function is consistent with the fundamental belief that most phenomena may be modelled adequately by simple laws and theories. It is common engineering practice always to use the simplest model that can be used in a given situation. So, for the first trip around the cycle, the appropriate function should be the simplest that is consistent with known facts, behaviours, theory or data. If the quantity of interest is known not to be constant, this might be a linear function. If the first choice turns out to be inadequate at the stage of the cycle (step 5) where the result is interpreted or the outcome

is evaluated then it is reasonable to try something more complicated; a quadratic function might be the second choice if the first choice was linear.

It is important to realise that sophistication is not necessarily a virtue in itself. The merits of complication depend upon the purpose for which the model is being formulated. A model of the weather that enables a decision on whether or not to take an umbrella to work on any particular day will be rather less sophisticated than that required to give an accurate prediction of the amount of rainfall in the vicinity of the workplace on that day.

In the next sub-section we will look at various types of functions that have been introduced so far but in a different way, concentrating more on their graphical behaviour and their parameters. As mentioned earlier, appropriateness is determined by the extent to which the *behaviour* of the chosen function as the independent variable varies reflects the behaviour to be modelled. The behaviour of a function is determined by whether it is linear, non-linear, or periodic and its range of validity. An important task of this Workbook is to get you to think more and more in modelling terms about the forms and associated behaviours of functions. We shall also take the opportunity of deriving some generalities from specific examples.

2. Constant functions

There are two physical interpretations of constancy that are of interest here. A very common form is *constancy with time*. Motion under gravity may be modelled as motion with constant acceleration. By definition, Fixed Rate Mortgages (increasingly popular in the late 1990s) offer a constant rate of interest over a specified period. In these examples, the constancy will be limited to a certain time interval. Motion under gravity will only involve the constant acceleration due to the Earth's gravitational pull as long as the motion is close to the Earth's surface. In any case the acceleration will only be from the time the object is released to the time it stops. Unfortunately, increases in base interest rates eventually feed into mortgage rates. So mortgage lenders are only able to offer fixed rates for a certain time. A mathematical statement of these limits is a statement of the *range of validity* of the constant function model. Another type of constancy is *constancy in space*. Long stretches of Roman roads were build in a fixed direction. For at least part of their lengths, roads have constant width. In modelling the formation and movement of seismic waves in the Earth's crust it is convenient to assume that the layers from which the Earth's crust is formed have constant thickness with respect to the Earth's surface. In these cases the assumption of constancy will only be valid within certain limits in space.

The rate of flow of water from a tap is denoted as r (litres per minute). The time for which it is turned on is denoted by t (minutes). In a specific example suppose that a tap is turned on and that the rate of water running out of a tap is assumed to be constant at 3 litres per minute and that it is turned off after 10 minutes.

(a) Write down a mathematical statement of the model for the flow from the tap, including its range of validity.

(b) Sketch a graph of the variation of r with t.

(c) Find an equation for the number of litres of water that have run out of the tap after t minutes.

(d) Calculate the volume of water that has run out of the tap three minutes before it is turned off.

Your solution

(a) $r = 3$ $(0 \leq t \leq 10)$

(b) Figure 5.1

(c) $V = rt$ $(0 \leq t \leq 10)$

(d) The tap will have run for seven minutes; 3 litres per minute x 7 minutes = 21 litres

Note that a more sophisticated model would allow for the variation in flow rate as the tap is turned on and turned off.
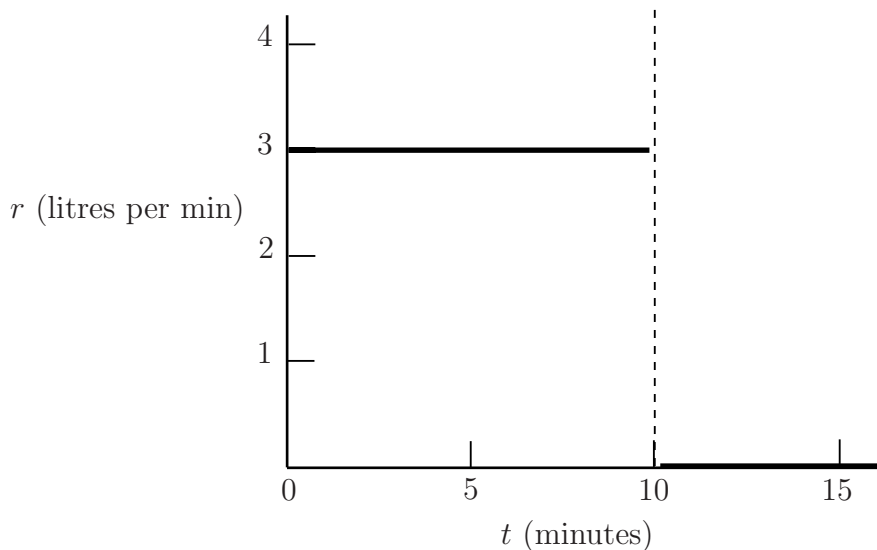
Figure 5.1   Flow from tap in Exercise 1

3. Linear functions

Workbook 2 has introduced linear functions of the form $y = ax + b$. Such functions give rise to straight-line graphs. The coefficient a is the slope. If a is positive the graph of y against x slopes upwards. If a is negative the graph slopes downwards. The coefficient b gives the intercept on the y-axis. The terms a and b may be called the *parameters* of the line. Note that this is a different use of the term 'parameter' than in the parametrisation of functions discussed in Workbook 2.

Linear models for falling rocks

In modelling it is wise to use a notation which fits in with the application. When modelling velocity under constant acceleration, we shall replace the dependent variable y by v (for velocity), and the independent variable x by t (for time). The acceleration will be denoted by the symbol a. Consider the motion of a rock dislodged from the top of a cliff (35 m high) by a villain during the filming of a thriller. The film producer might be interested in how long the rock would take to fall to the ground below the cliff and how fast it would be travelling at ground impact. The rock may be assumed to have a constant downward acceleration of 9.8 m s^{-2} which the acceleration due to gravity. The velocity (v m s^{-1}) of a rock, falling from the top of a cliff 35m high, can be modelled by the equation

$$v = 9.8t \qquad (0 \leq t \leq 2.7)$$

where t is the time in seconds after the rock starts to fall. This follows from the fact that acceleration is the rate of change of velocity with time. If the acceleration is constant and the object starts from rest, then the velocity is given simply by the product of acceleration and time. The upper limit for t is the time at which the rock hits the ground measured with a stop-watch. Figure 2 shows v as a function of t. Velocity is a linearly increasing function of time and its graph is a straight line passing through $t = 0, v = 0$. Note that various assumptions are needed to obtain the quoted result of a linear variation in speed with time. It is assumed that there is no air resistance, no spinning and no wind.
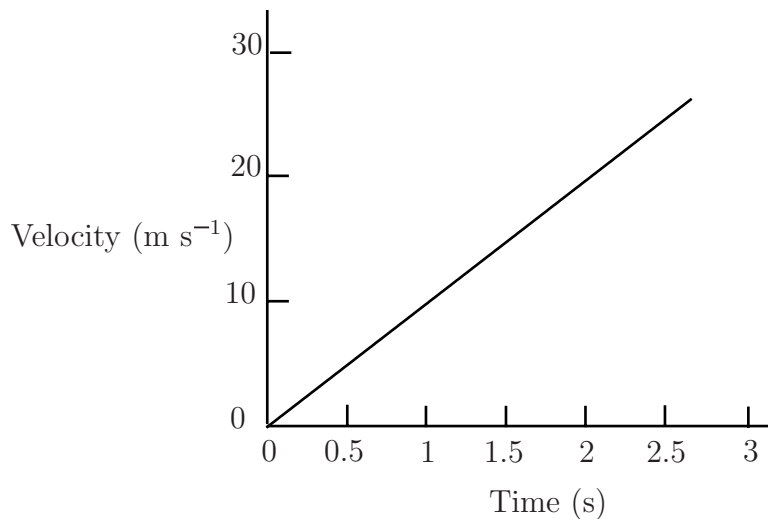
Figure 5.2   Graph of $v = 9.8t$ for the falling rock

In what way should the equation for v be altered if the villain were able to *throw* the rock downwards at 5 m s^{-1}? Provided we are measuring position or displacement downwards, a downwards velocity is positive. Now we have that $v = 5$ when $t = 0$. So a new model for v is

$$v = 9.8t + 5 \qquad (0 \leq t \leq T_1)$$

Since they are both downwards, the initial velocity simply adds to the velocity at any time resulting from falling under gravity. Note that T_1 is being used now for the upper limit on t (instead of 2.7) because 2.7 is (approximately) the time taken to fall 35 m *from rest* rather than with an initial downwards velocity. Using the symbol T_1 saves us trying to work out its value for the moment. Note that a general form of the model for motion under constant acceleration of magnitude a ms^{-2} given an initial speed b ms^{-1} is $v = at + b$. In the case most recently considered $a = 9.8$ and $b = 5$.

(a) Will T_1 be more or less than 2.7?

(b) Sketch a graph of v for $0 \leq t \leq T_1$.

Your solution

(a) T_1 will be less than 2.7 since the rock will be moving faster throughout its descent.

(b) See Figure 5.3. The graph is still a straight line but is displaced upwards compared with Figure 5.2.
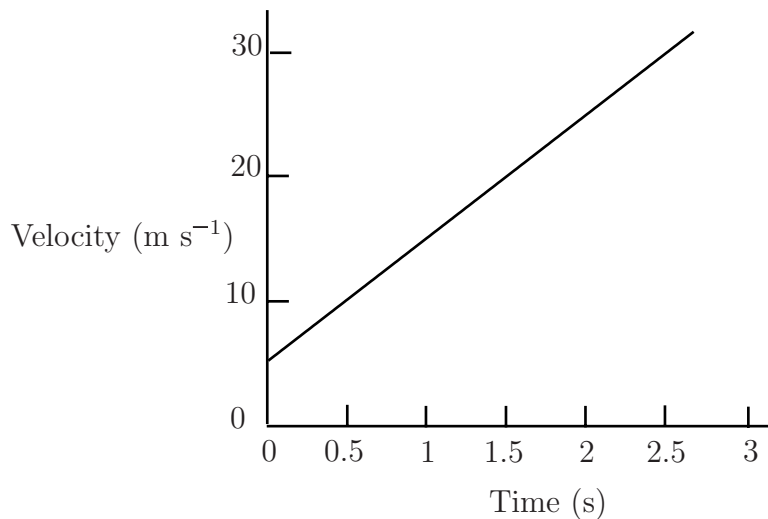
Figure 5.3   Graph of $v = 9.8t + 5$ for the falling rock

Consider now how the function for v will change if the villain is even mightier than we previously thought and throws the rock it upwards with an initial speed of 5m s^{-1} instead of simply dislodging it or throwing it downwards. In this circumstance, the initial velocity is directed upwards, and position is being measured downwards, so the initial velocity is negative. We can use the equation $v = 9.8t + b$ again. This time $v = -5$ when $t = 0$, leading to $b = -5$ and

$$v = 9.8t - 5 \qquad (0 \le t \le T_2)$$

The new time at which the rock hits the ground is denoted by T_2. The rock will rise before falling to the gound this time so T_2 will be larger than T_1.

From the modelling point of view, there is one other significant time before the rock hits the ground. Figure 5.4 shows the new graph of v against t. Notice that there is a time at which v (which starts at -5) is zero. What does this mean?
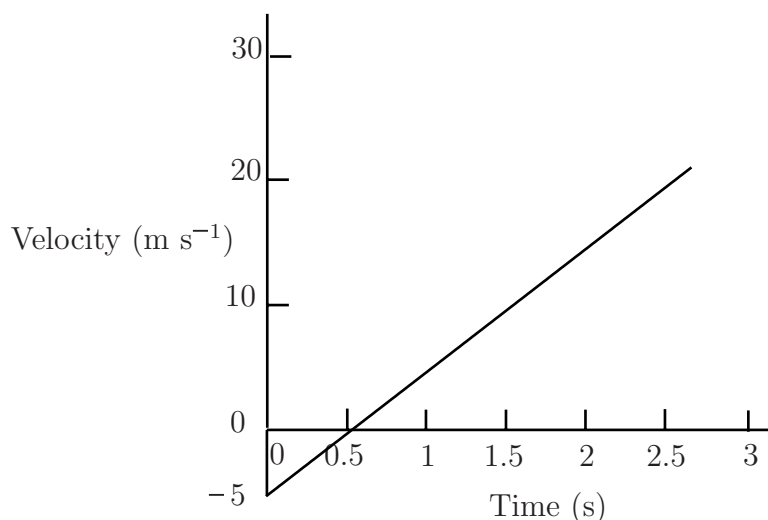

Figure 5.4   Graph of $v = 9.8t - 5$ for the falling rock

As time goes by, the fact that gravity is causing the rock to accelerate downwards means that the rock's upward motion will slow. Its velocity will decrease in magnitude until it reaches zero. At this particular instant the rock will be at its highest point and its velocity will change from

upwards to downwards passing instantaneously through zero in the process.

We can work out this time. We substitute zero for v and work out the corresponding t.

$$0 = 9.8t - 5, \quad \text{so} \quad t = \frac{5}{9.8} = 0.51.$$

This means that the rock is stationary about a half second after being thrown upwards. Subsequently the rock will fall until it hits the ground. But there is yet one more time that may be significant in the modelling context chosen here. During its journey to the ground 35m beneath the cliff-top, the rock will pass the top of the cliff again. Note that we are modelling the motion of a particular point, say the lowest point, on the rock. A real rock, with appreciable size, will only pass the top of the cliff, without landing on it or hitting it, if it is thrown a little to the side as well as up. Anyway, in principle we could use the function that we started with, representing the velocity of an object falling from rest under gravity, to work out how long the rock will tak to pass the top of the cliff having reached the highest point in its path. A simpler method is to argue that, as long as the rock is thrown from the cliff top level (this requires the villain to be lying down!), the rock should take exactly the same time (approximately 0.5 s) to return to the level of the cliff top as it took to rise above the cliff top to the highest point in its path. So we simply double 0.5 s to deduce that the rock passes the cliff top again about 1 s after being thrown.

(a) Add lines to your sketch version of Figure 4 to represent velocity as a function of time if the rock is (i) dislodged (ii) thrown with (downwards) velocity 3 m s^{-1} (iii) thrown with velocity -2 m s^{-1}

(b) What do you deduce about the effect of the initial velocity on the graph of velocity against time?

(c) Imagine that the filming was on the moon with roughly one-sixth the gravitational pull of Earth. Find a linear function that would describe the velocity of a dislodged rock.

(d) What do you deduce about the effect of changing the acceleration due to gravity on the graph of velocity against time?
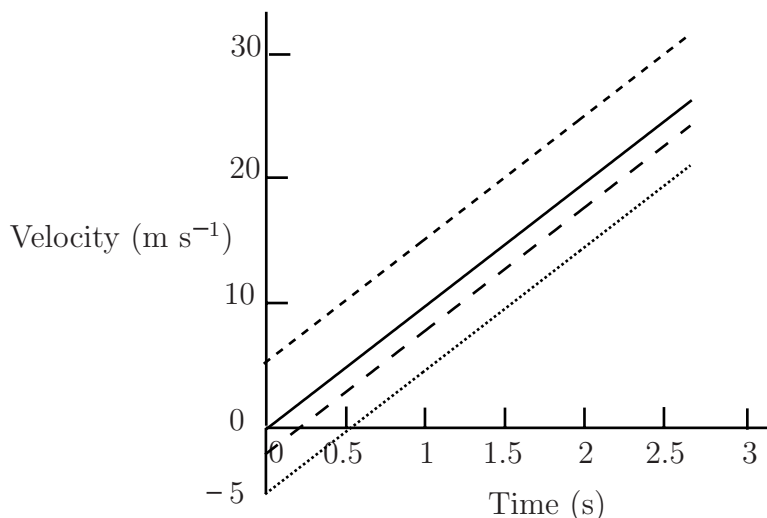
Your solution

Figure 5.4a   Answer to Exercise 3(a)

So in the context of modelling motion under gravity, the **initial velocity** determines the vertical displacement of the line, its *intercept* on the v-axis, and the **acceleration** determines the *slope*. Again, given the modelling context, both of these influence the range of validity of the model since they alter the time taken for the rock to reach the ground and this fixes the upper limit on time.

Like velocity, acceleration has direction as well as magnitude. As long as position is being measured downwards, and only gravity is considered to act, falling objects do not provide any examples of negative accelerations but rocket motion does. Where downwards accelerations are represented as positive, an upwards acceleration will be negative. So a model of the motion of a rocket accelerating away from the Earth could include a constant negative acceleration. Horizontal acceleration, say of a road vehicle, in the same direction as position as being measured, is represented as positive. Deceleration, for example when this vehicle is being braked, implies that velocity is decreasing with time, and is represented as negative. In mathematical modelling, it is usual to refer to *acceleration*, whether it represents positive acceleration or deceleration.

Suppose that we are describing the motion of a rocket taking off vertically during its initial booster stage of 10 s. We might model the acceleration as a constant -20 m s^{-2}.

The negative sign arises because downwards is positive but the acceleration is upwards. Since the rocket is starting from rest an appropriate function is

$$v = -20t \qquad (0 \leq t \leq 10)$$

This should describe the variation of its velocity with time until the end of the initial booster stage of its flight. Figure 5.5 shows the corresponding graph of velocity against time. Note the

way in which the graph slopes downwards to the right. This function describes an increasingly *negative* velocity as time passes. This consistent with an increasing *upwards* velocity. The corresponding graph for a positive acceleration of the same magnitude would slope upwards towards the right.
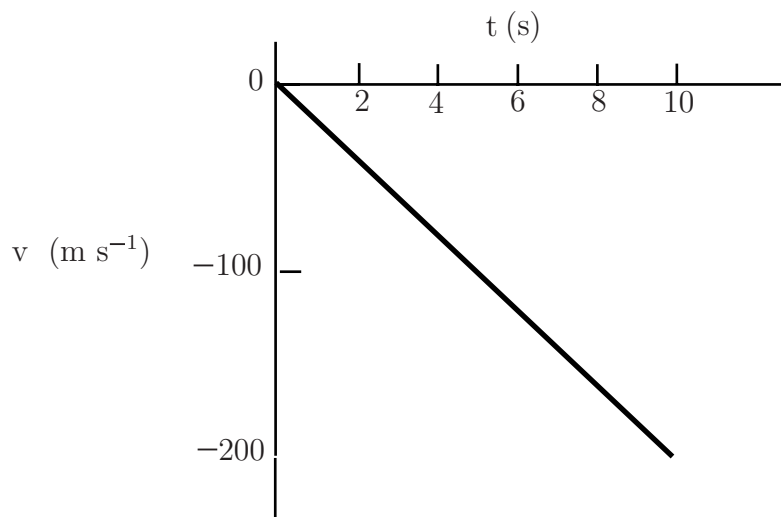

Figure 5.5   Variation of velocity of rocket during the initial booster stage.

(a) Imagine that a satellite is falling towards Earth at 5 m s^{-1} when a booster rocket is fired for 5 s accelerating it away from the Earth at 10m s^{-2}. Write down a corresponding linear function that would describe its velocity during the booster stage.

(b) Sketch the corresponding graph of velocity against time (a) if position is measured downwards towards Earth and (b) if position is measured upwards away from the Earth.

(c) At what time would the velocity of the satellite be zero?

(d) What is the value of velocity at the end of the booster stage?

Your solution

(a) If position is measured downwards, acceleration away from the Earth may be written as -10m s^{-2}. The initial velocity toward the Earth may be denoted by $(+)5$m s^{-1} $v = -10t + 5$ $(0 \leq t \leq 5)$.   If position is measured upwards $v = 10t - 5$ $(0 \leq t \leq 5)$.

(b) Figure 6 (a) or (b)

(c) When v is 0, $0 = -10t + 5$, so $t = 0.5$ and the satellite has zero velocity towards the Earth after 0.5 s.

(d) When $t = 1$, either $v = -10 \times 1 + 5 = -5$, so the velocity is 5m s^{-1} away from the Earth, or, using the second equation in (a), $v = 10 - 5 = 5$, leading to the same conclusion.
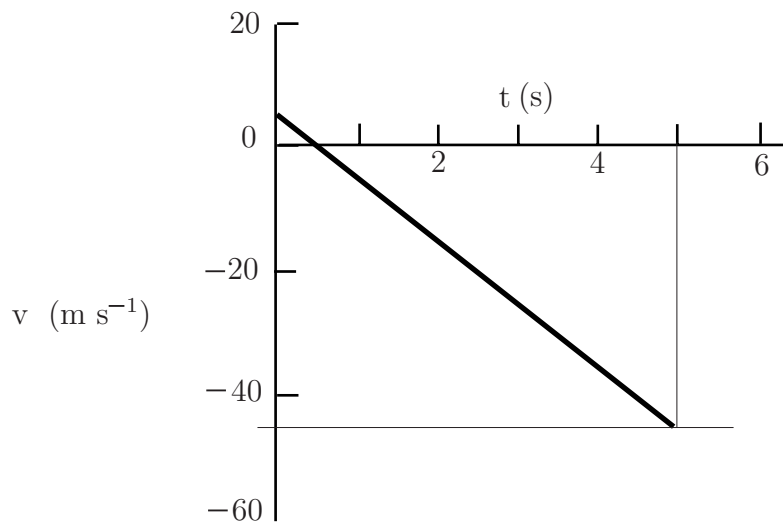

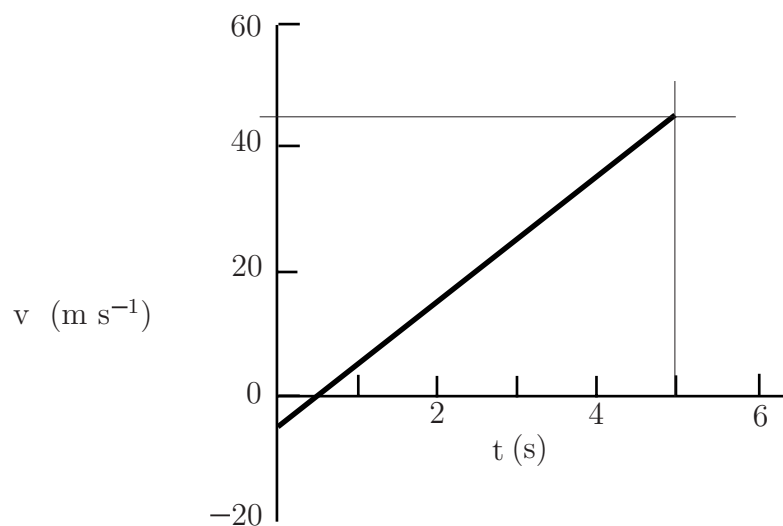Figure 5.6a   Satellite velocity (position measured downwards towards Earth)

Figure 5.6b   Satellite velocity (position measured upwards away from Earth)

Other contexts for linear models

Linear functions may arise in contexts other than when modelling motion under constant acceleration. In each of these situations, the slope and intercept values will have some modelling significance. Indeed the behaviour and hence the suitability of a linear function, of the form $y = ax + b$, when modelling any given situation (not just motion under constant acceleration) will be determined by the values of a and b.

During 20 minutes of rain, a cylindrical rain barrel that is initially empty is filled to a depth of 1.5 cm.

(a) Choose variables to represent the level of water in the barrel and time. Sketch a graph representing the level of water in the barrel if the intensity of rainfall remains constant over the 20 minute period.

(b) Write down a linear function that represents the level of water in the vessel together with its range of validity.

(c) State any assumptions that you have made.

(d) Write down the amended form of your answer to (b), if the vessel contains 2 cm of water initially.

Your solution

(a) In this answer h cm is used for the level of water measured from the bottom of the barrel and t minutes for time. See Figure 5.

(b) The intensity of rainfall is stated to be constant, so the rate at which the barrel fills may be taken as constant. The gradient of an appropriate linear function relating level of water (h cm) measured from the bottom of the vessel and time (t, minutes) would be $\frac{1.5}{20} = 0.075$ and an appropriate linear function would be $h = 0.075t + c$. Since the barrel is empty to start with, $h = 0$ when $t = 0$, implying that $c = 0$. So the appropriate linear function and its range of validity are expressed by $h = 0.075t$ $(0 \leq t \leq 20)$.

(c) It is assumed that the barrel has a uniformaly cylindrical cross-section, that no water is removed during the rainfall and there are no holes or leaks up to 1.5 cm depth.

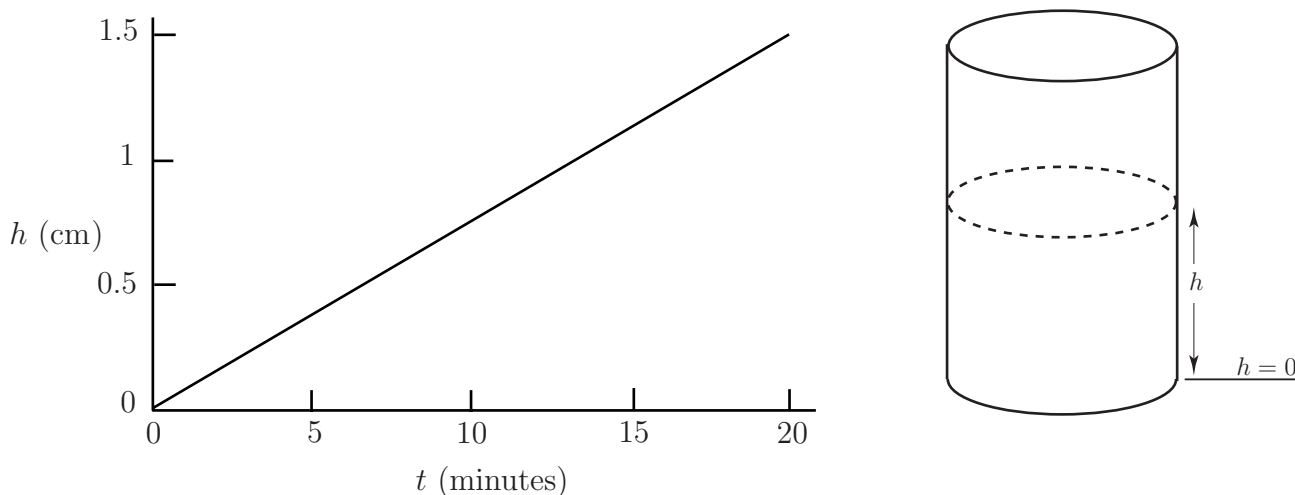(d) $h = 0.075t + 2$ $(0 \leq t \leq 20)$

Figure 5.7   Height (depth) of rainwater in a barrel. Answer to Exercise 5(a)

Suppose that you travel often from Nottingham to Milton Keynes which is a distance of 87 miles almost all of which is along the M1 motorway. Usually it takes 1.5 hours. Suppose also that, on one occasion, you have agreed to pick someone up at the Leicester junction (21) of the M1. This is 25 miles from the start of your journey in Nottingham. If you start your journey at 8 a.m., what time should you advise for the pick-up?

Your solution

These values can be plotted in a graph of distance agains time.
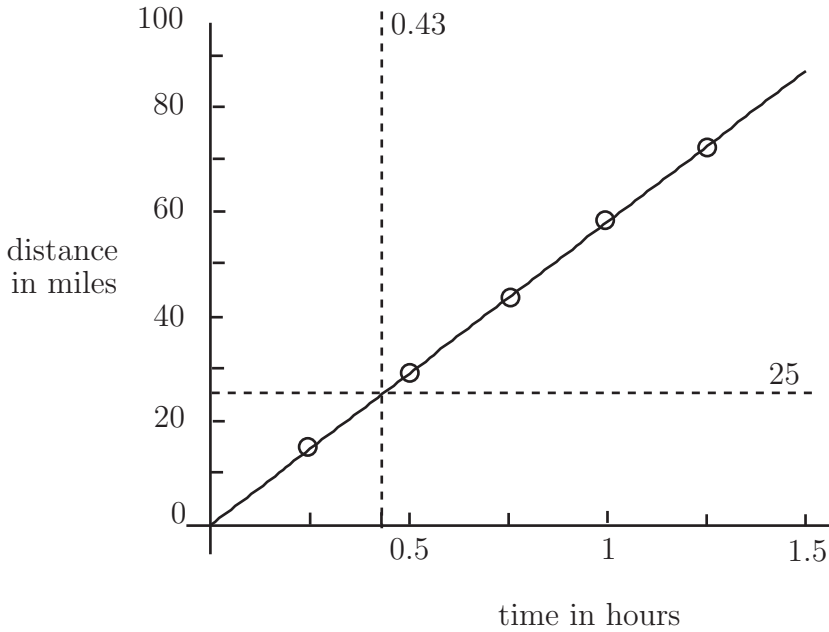
A distance of 25 miles will be covered in 0.43 hours

The average speed is $\frac{87}{1.5} = 58$ mph.

This is also the *gradient* of the graph.

Figure 5.8

4. Further Exercises on linear models

Exercises

1. A cross channel ferry usually takes 2 hours to make the 40 km crossing between England and France.

   (a) What is the boat's average speed?

   (b) Use a linear model to determine when the boat will be 15 km and 35 km from England.

   (c) According to this model, what will be the location of the boat after 15 minutes and after 1 hour 10 minutes?

2. During one winter, the roads in the country district were completely free from snow when it started snowing at midnight and it snowed steadily all day. At 10 a.m. it was 19 cm deep. To save money, the local practice was to wait until the snow was 30cm deep before ploughing the roads. Forecast when ploughing would start, stating any assumptions.

3. A major river is rising fast and the authorities are worried about the possibility of flooding. Several readings of the depth have been taken. The results are shown in the following table. The banks are 26 m high. How much time is likely to be available to evacuate people living on low-lying ground near the river? State any assumptions that you have made.

4. In a drought, the population of a particular water beetle in a pond is observed to have halved when the volume of water in the pond has fallen by half. Make a simple assumption about the relationship between the beetle population and the volume of water in the pond and express this in symbols as an equation. What would your model predict for the population when the water volume is only one third of what it was originally.

5. A firm produces a specialised instrument and, although it has the facilities to produce 100 instruments per week, it rarely produces more than 50. It is finding it difficult to assess the cost of producing the instruments and to set realistic prices. The firm's accountant estimates that the firm pays out £5000 per week on fixed costs (overheads, saleries etc.) and that once costs have been incurred, the cost of producing each instrument is 50.

   (a) Derive and use a linear model for the variation in total costs with the quantity of instruments produced. State any limitation of this model.

   (b) What is the model's prediction for the cost of producing 80 instruments per week?

Answers to exercises on linear models

1. (i) A distance of 40 km is covered in 2 hours. So the average speed is $40/2 = 20$.

   (ii) A linear model assumes that the boat is a point moving at constant speed and will only be valid for 2 hours (or 40 km) while the boat is travelling from England to France (or vice versa). It does not allow for variations in speed. If the distance from the English port at any time t hours is denoted by d km, then $d = 20t$.

   (iii) When the boat is 15 km from England, so $t = 15/20 = 0.75$, so the boat is 0.75 hour (45 minutes) from port. When the boat is 35 km from England, $35 = 20t$, so $t = 35/20 = 1.75$, so the boat is 1.75 hour (1 hour 45 minutes) from port.

2. Assume that there is no snow at midnight and that the rate of accumulation of snow is constant. Then, if the snow is 110 cm deep at 10 a.m. then the rate of accumulation is 1.9 cm per hour. A linear model for the depth (d cm) of snow t hours after midnight is $d = 1.9t$. d will be 30 when $30 = 1.9t$, i.e. $t = 30/1.9 = 15.79$. This corresponds to about 3.45 p.m. So ploughing should start at about 3.45 p.m.

3. The depth of the river is increasing at 4 m every two hours i.e. a rate of 2 m per hour. At this rate the depth d m of the river at time t hours after 6.00 a.m. may be modelled by $d = 5 + 2t$. So the depth will be 26 m when $26 = 5 + 2t$, or $t = 21/2 = 10.5$, i.e. at 4.30 p.m.

4. Denote population by P and volume of pond by V. Then P is proportional to V so $P = kV$ where k is a constant of proportionality. When V becomes $V/3$, then P becomes $P/3$.

5. Denote the number of instruments made per week by N and the corresponding cost by C. Assume that C increases at a constant rate with N (i.e. C is proportional to N). Then a linear model for total costs (T) is $T = 5000 + 50N$. This will be valid only for $0 < N < 100$. If $N = 80$, then $T = 5000 + 50 \times 80 = 9000$. So the predicted total cost is 9000.

Methods for calculating gradient

Occasionally you may be faced with two different pairs of values or *coordinates* with which to determine the parameters of a linear function. Put another way, two pairs of values are needed to determine the two (unknown) parameters. Perhaps, unconsciously, you might have used this result already when carrying out the rain barrel exercise (5). The gradient, written as $\frac{1.5}{20}$ in the answer, may be expressed also as $\frac{1.5-0}{20-0}$ since the line connects the (time, level of water) coordinates (20, 1.5) with (0, 0). In general the gradient is given by

$$\frac{\text{the change in the dependent variable}}{\text{the corresponding change in the independent variable}}$$

Once the gradient of the line has been calculated, it can be used with one of the known points to determine the intercept. If one of the points is (0, 0) the intercept is zero.

Suppose that a new type of automatic car is being road tested. The measuring team wants to know the maximum acceleration between 0 and 30 m s^{-1}. It plans to calculate this by assuming that it is constant and measuring the time taken from rest to achieve a speed of 30 m s^{-1} at

maximum acceleration. In their first test the Speedometer reading is 30 m s^{-1} after 12 s from start of timing and motion. We can think of these values in terms of (time, velocity) coordinates. At the start of timing the coordinates are (0, 0). When the speedometer reads 30 m s^{-1} the coordinates are (12, 30). If the acceleration is constant then its magnitude will be given by the gradient of the line joining these two points. Using the 'change in variable idea', the gradient is $\frac{30-0}{12-0} = 2.5$, and so the magnitude of the acceleration is 2.5m s^{-2}. The'change in variable' route to calculating the gradient is an abridged version of a more general method. The two pairs of coordinates may be used with the general equation of a line to work out the parameters of the particular line that passes through these two points. The assumption of constant acceleration leads to a linear relationship between the velocity (v m s^{-1}) and time (t s) of the form $v = at + b$ where a and b are the parameters corresponding to gradient and intercept respectively. The road test gives $v = 0$ when $t = 12$. These may be substituted into the general form to give $0 = 0 + b$ and $30 = 12a + b$.

You may recognise that these are simultaneous equations. The first gives $b = 0$ which may be substituted into the second to give $a = 2.5$, corresponding to an acceleration of 2.5 m s^{-2} as before.

Suppose that the test team carry out a second test. In this test they note when speeds of 15m s^{-1} and 27 m s^{-1} are reached and assume constant acceleration between these times and speeds. The speedometer reads 15 m s^{-1}, after 4 seconds from the start of motion and 27 ms^{-1} after 9 s from the start of motion. Let's use the general method of the data from this test. The (time, velocity) coordinates corresponding to the readings are (4,15) and (9,27). The equations resulting from substitutions in the general form are

$$15 = 4a + b$$

$$27 = 9a + b$$

It is necessary to use the method of solving simultaneous equations by elimination (Workbook 3). The first of these equtions may be subtracted from the second to eliminate b.

$$27 - 15 = (9 - 4)a$$

or

$$a = 2.4.$$

The resulting value of a may be substituted into either of the equations expressing the data to calculate b. In the first $15 = 4 \times 2.5 + b$, or $b = 5$. The resulting model is

$$v = 2.4t + 5 \qquad (4 \leq t \leq 9).$$

This model predicts an acceleration of 2.4 m s^{-2}, which is fairly close to the previous result of 2.5 m s^{-2} but if we try to use this model at $t = 0$, what do we predict? The model predicts that $v = 5$ when $t = 0$. This is not consistent with $t = 0$ being the time at which the vehicle starts to move! so, even if the acceleration is constant between 15 and 27 m s^{-1}, it does not have the same values between 0m s^{-1} and 15 m s^{-1} as either between 15m s^{-1} and 27 m s^{-1} and 30 m s^{-1}. A more general principle is illustrated by this example. It may be dangerous to use a model based on certain data at points other than those given by these data! The business of using a model outside the range of data for which is is known to be valid is called *extrapolation*. Use of the model between the data points on which it is based is called *interpolation*. So the general principle may also be stated as that *it is risky to extrapolate or interpolate*. Nevertheless

extrapolation or interpolation may be part of the purpose for a mathematical model in the first place. The method of finding gradient and intercept just exemplified may be generalised.

Suppose that we are specifying a linear function $y = ax + b$ where the dependent variable is y and the independent variable is x. We'll represent two known points by (p, q) and (r, s). The gradient, a, for the straight line, may be calculated either from $\frac{p-r}{q-s}$ or by substituting $y = q$ when $x = r$ in $y = ax + b$ to obtain two simultaneous equations. Subtraction of these eliminates b and allows a to be calculated. The intercept of the line on the y-axis, b, may be found by substitution in $y = ax + b$, of either p, q and a or r, s and a.

Use the general method to deduce the different accelerations (assuming that they are constant) between the start of motion and 15 m s^{-1} and between velocities of 27 m s^{-1} and 30 m s^{-1}.

Your solution

For the (time, velocity) coordinates (0,0) and (4,15),

$$0 = 0a + b$$

$$15 = 4a + b$$

From the first of these b = 0 and hence, in the second, $a = \frac{15}{4} = 3.75$. So the acceleration up to 15 m s^{-1} is 3.75 ms^{-2}. For the (time, velocity) coordinates (9,27) and (12,30),

$$27 = 9a + b$$

$$30 = 12a + b$$

Subtracting the first from the second gives

$$3 = 3a \text{ so } a = 1,$$

so the acceleration between 27m s^{-1} and 30 m s^{-1} is 1 m s^{-2}

Linear functions may be useful in economics. A lot of attention is paid to the way in which *demand* for a product varies with its price. A measure of demand is the number of items sold, if available, in a given period. For example, the purpose might be to determine the best price for a product given certain details about costs and with certain assumptions about the way the number of items sold per month varies with price. The price affects the profit and hence, in turn, the number manufactured in response to the demand. The number of items manufactured

in a given period is known as the *supply*. Information about the variation of demand or supply with price may be obtained from market surveys. Constant functions are not appropriate in this context since both demand and supply vary with price. In the absence of other information the simplest way to model the variation of either demand or supply with price is to use a linear function.

When the price of a luxury consumer item is £1000, a market survey reveals that the demand is 100000 items per year. However another survey was shown that at a price of £600, the demand for the item is 200000 items per year. Assuming that both surveys are valid, find a linear function that relates demand Q to price P. What demand would be predicted by the linear function at a price of £750? Comment on the validity of both predictions.

Your solution

The linear function will be of the form

$$Q = aP + b \qquad (600 \leq P \leq 1000)$$

The limits on P represent the given range of data on price. Substituting the first pair of values of Q and P:

$$100000 = 100a + b$$

Substituting the second pair of values:

$$200000 = 600a + b$$

Subtracting the first expression from the second:

$$100000 = -400a$$

$$a = -250$$

Note that the negative gradient is consistent with the fact that demand falls as price increases.

Check that the 'Change in variable' definition for finding a works. Change in dependent variable $(Q) = 200000 - 100000 = 100000$. Corresponding change in independent variable $(P) = 600 - 1000 = -400$. The ratio of these changes is $\frac{100000}{-250} = -250$. This value of a may be used with the first pair of values,

$$100000 = -250000 + b$$

so

$$b = 350000$$

and the linear function relating demand and price is

$$Q = 350000 - 250P.$$

[A precautionary check is to make sure that this result is consistent with the other pairs of values. When $P = 600$, $Q = 350000 - 250 \times 600 = 350000 - 150000 = 200000$, as required.]

When $P = 750$:

$$Q = 350000 - 250 \times 750 = 350000 - 187500 = 162500.$$

So a linear relationship between demand and the price for this luxury suggests a demand of 162500 items per year when the price per item is £750. At a price of £500, $P = 500$, and the model predicts that

$$Q = -250 \times 500 + 350000 = 225000.$$

So the linear model suggests a demand of 225000 items per year when the price per item is £500. Such a price however is outside the range of given data. Consequently the corresponding demand prediction represents an extrapolation and this might not be reliable. On the other hand, the price of £750 lies within the given range of data and the corresponding demand prediction is an interpolation. If the given data points are close to each other then interpolation between these points is more reliable than extrapolation to points further away.