Oscillating functions and Modelling

5.3

Introduction

This section describes ways in which trigonometric functions can be used to model motion involving periodic motion.

Prerequisites

Before starting this Section you should . . .

① be familiar with algebraic manipulation

② be familiar with trigonometric functions

Learning Outcomes

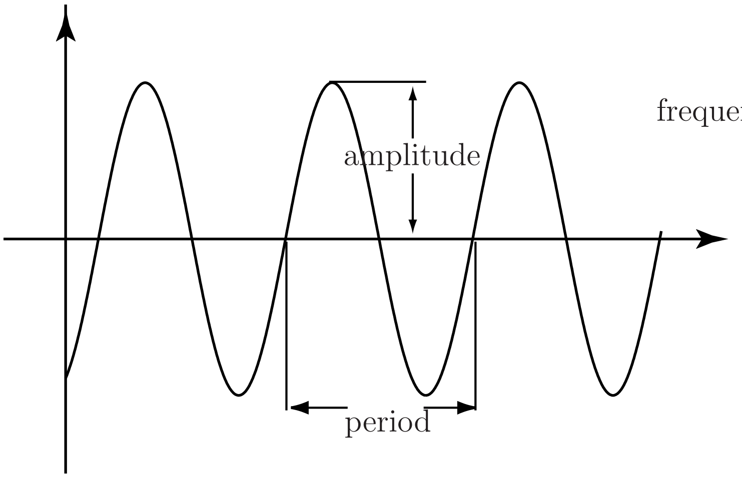
After completing this Section you should be able to . . .

✓ Use trigonometric functions to model periodic motion

✓ Define terms associated with the description of periodic motion

1. Oscillating Functions: Amplitude, period and frequency

Particular types of periodic functions (Workbook 2, section 2) that are important in engineering are the sine and cosine functions. These are possible choices when modelling behaviour that involves oscillation or motion in a circle. The usefulness of these functions is rather limited if we confine our attention only to $\sin(x)$ and $\cos(x)$. Equal acquaintance with functions such as $3\sin(2x)$, $5\cos(3x)$ and so on, and other functions made up of sums of functions of this type, enables the modelling of a great variety of situations where the quantity being modelled is known to change in a periodic way. In this engineering exercises section we will examine the behaviour of sine and cosine functions and consider a modelling context where choice of a sine function is appropriate. The following diagram shows how the terms *amplitude*, *period* and *frequency* are defined with respect to a general sinusoid.



$$\text{frequency} = \frac{1}{\text{period}}$$

Figure 12 Defining amplitude and period for a sinusoid

Sketch the sinusoids: (a) $y = \sin x$ (b) $y = \cos x$ (c) $y = 2\sin x$ (d) $y = \cos\frac{x}{2}$

Your solution

The amplitude represents the difference between the maximum or minimum values of a sinusoidal function and its mean value (which is zero for these examples). The frequency represents the number of complete cycles of the function in each unit change in x. The period is such that $f(x+T) = f(x)$ for all x, e.g. for $\sin x$, $T = 2\pi$.

(1) amplitude = 1, frequency = $1/2\pi$, period = 2π.

(2) amplitude = 1, frequency = $1/2\pi$, period = 2π.

(3) amplitude = 2, frequency = $1/2\pi$, period = 2π.

(4) amplitude = 1, frequency = $1/4\pi$, period = 4π.

See Figure 12(a) for the sine functions and 12(b) for the cosine functions. Note that function (3) has twice the amplitude and function (4) has half the frequency and twice the period of the others.

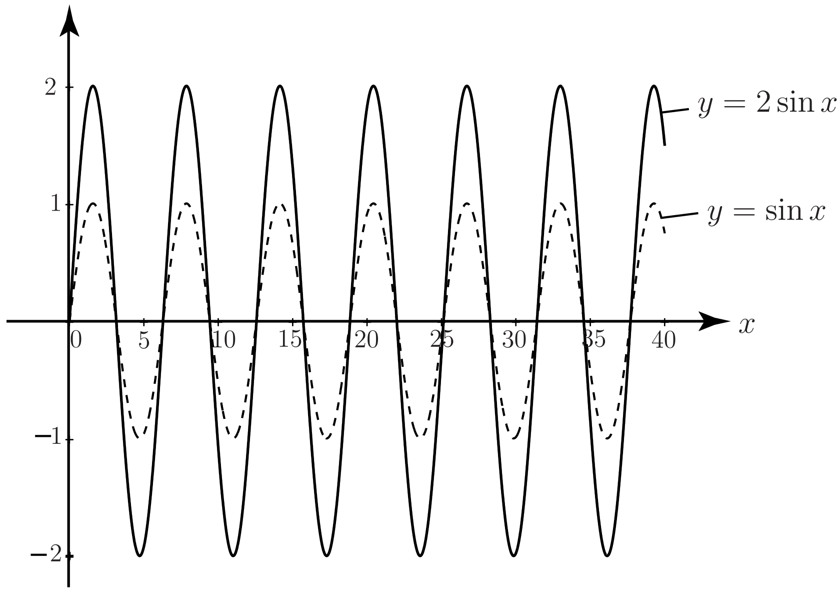


Figure 12 (a)

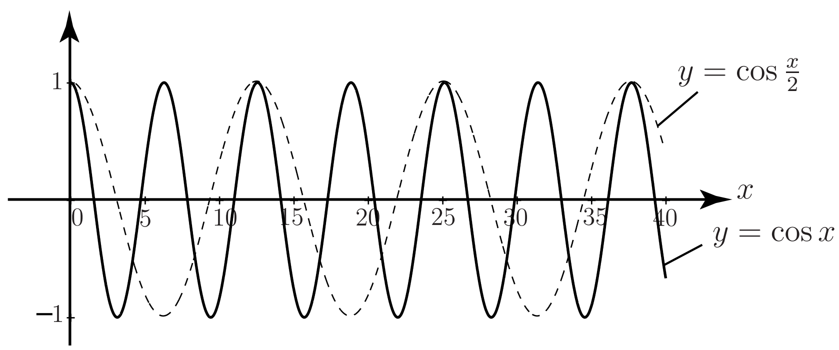


Figure 12(b)

Note that cosine functions have the same shape as the sine functions but, at $x = 0$, the cosine functions have a peak, whereas sine functions have the value zero, which is the mean value for both of these functions. Indeed the graph of $y = \cos x$ is like that for $y = \sin x$ with all the x values displaced by $\pi/2$.

More general forms of sine and cosine function are given by $y = a\sin(bx)$, and $y = a\cos(bx)$ where a and b are arbitary constants. These are functions with frequency $b/2\pi$, period $\frac{2\pi}{b}$ and amplitude a. The peak values of the sine functions occur at x values equal to $\frac{\pi}{2}$, $\frac{5\pi}{2}$, $\frac{9\pi}{2}$ etc. The minimum values occur at x values equal to $\frac{3\pi}{2}$, $\frac{7\pi}{2}$, $\frac{11\pi}{2}$ etc.
When period is measured in seconds, frequency is measured in cycles per second or Hz which has units of 1/time.

Exercises

Figure 12(a) shows on the same axes the graphs of $y = \sin x$ and $y = 2\sin x$.

1. (a) State in words how the graph of $y = 2\sin x$ relates to the graph of $y = \sin x$

   (b) Sketch the graphs of (i) $y = \frac{1}{2}\sin x$, (ii) $y = \frac{1}{2}\cos x + \frac{1}{2}$

2. Figure 12(b) shows on the same axes the graph $y = \cos x$ and $y = \cos\frac{x}{2}$

   (a) State in words how the graph of $y = \cos x$ relates to the graph of $y = \cos\frac{x}{2}$

   (b) Sketch graphs of (i) $y = \sin 2x$, (ii) $y = \sin\frac{x}{2}$

Answers need supplying by HULL team

2. Oscillating Functions: Modelling tides

Let's consider how the function

$$h = 3.2\sin(2.7t + 8.5)$$

might be used to model the rise and fall of the tide in a harbour.
Figure 13 shows a graph of this function for $(0 \geq t \geq 5)$.

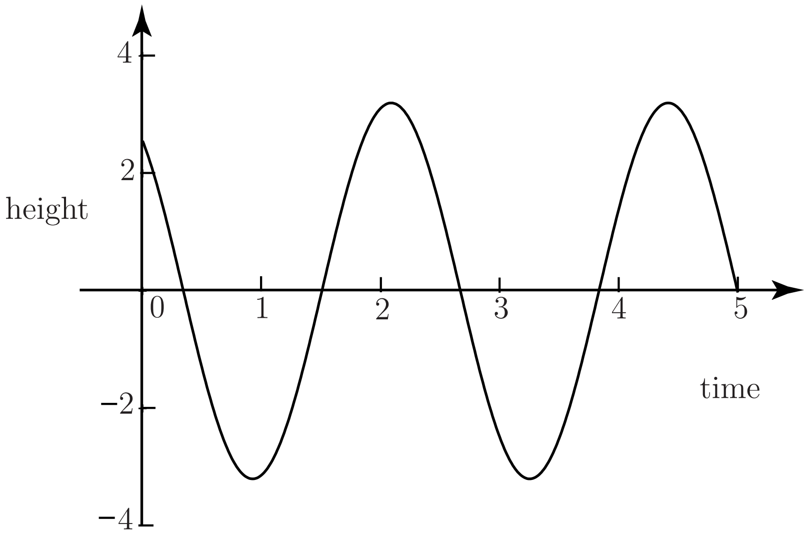


Figure 13

Let's consider some aspects of this graph and model. It seems reasonable to suppose that the tide creates an oscillation of the water level in the harbour of hm about some mean value represented on the graph by $h = 0$. There seems to be a low tide near $t = 1$ and another just after $t = 3$. Since we expect intervals of 12 to 14 hours between low tides around the U.K., this suggests that time is specified in 6-hour intervals.

Write down the amplitude, period and frequency of the function.

Your solution

The amplitude of the change in water level in the harbour is 3.2m. The period of the function is given by $2\pi/2.7 = 2.3271$ (corresponding to 2.3271×6 hours) between high or low tides and $2.7/2\pi = 0.4297$ is the frequency of the function.

The peak levels of the graph correspond to times when the sine function has the value 1. The lowest points correspond to times when the sine function is -1. At these times the arguments of the sine function (ie $2.7t + 8.5$) are an odd number of $\pi/2$ starting at $3\pi/2$ for the first low tide.

So far all of this may be deduced from the general form $y = a\sin(bx)$ and from the modelling context. However there is an additional term in the function being considered here. This is a constant 8.5 within the sine function. When $t = 0$ the presence of this constant means that the intercept on the height axis is 3.2sin(8.5) = 2.56, implying that the water level is 2.56 m above the mean value at the start of non-zero positive constant effectively has displaced the sine curve to the right. It is known as the *phase* of the function. Phase is measured in radians as it is an angle.

As remarked earlier, at $t = 0$, this function has the value $3.2\sin(8.5)$. Since $\sin(8.5) = \sin(8.5 - 2\pi) \approx \sin(2.2168)$, we can replace the constant 8.5 by 2.2168 without altering the values on the graph. This means that the function

$$h = 3.2\sin(2.74t + 2.2168)$$

does just as well as the original function in representing the tidal variation in the harbour. Let's rewrite this latest form of the function, representing the variation of water level in the harbour, so that time is measured in hours rather than in six-hourly invervals. The effect of changing the units of time to hours from 6-hours is to decrease the coefficient of t in the sine function by a factor of 6, so that the new function is

$$h = 3.2\sin(0.45t + 2.168).$$

See Figure 14.

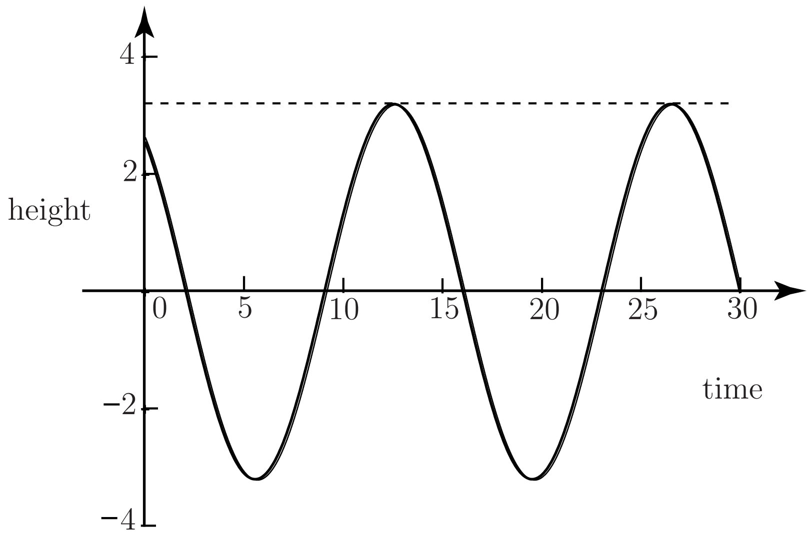


Figure 14

We can use the latest form of the function to calculate the time of the first low tide assuming that $t = 0$ corresponds to midnight. At the first low tide, $h = -3.2$ and

$$sin(0.45t + 2.2168) = -1,$$

or

$$(0.45t + 2.2168) = 3\pi/2$$

$$0.45t = 3\pi 2 - 2.2168 = 2.4956$$

$$t = 5.545777777778 = 5.55 \text{ to 2d.p.}$$

so the first low tide is a little before 6a.m.

Assume that $t = 0$ corresponds to midnight. Calculate (i) the time of the first high tide after midnight (ii) the times either side of midnight at which the water level is at the mean level.

Your solution

(i) At the first high tide, $h = 3.2$ and $\sin(0.45t + 2.2168) = 1$, or

$$(0.45t + 2.2168) = 5\pi/2$$

$$0.45t = 5\pi/2 - 2.2168$$

$$t = 12.5271$$

so the first high tide is a little before 1 p.m.

(ii) When the water level is at the mean value,

$$sin(0.45t + 2.2168) = 0.$$

At the mean level before midnight,

$$0.45t = -2.2168$$

$$t = -4.926222222222 = -4.93 \text{ to 2d.p.}$$

So this mean level occurs a little under 5 hours before midnight, i.e. about 7 p.m. the previous day. The next mean level will occur one whole period, or 13.963 hours, later, at approximately 9 a.m.

There are various rules connected with sine and cosine functions that can be summarised at this point.

(1) Placing a multiplier before $\sin x$ or $\cos x$ changes the amplitude without changing the period.

(2) Placing a multiplier before x in $\sin x$ or $\cos x$, i.e. *inside* a sine or cosine function, changes the period or frequency without changing the amplitude.

(3) As with any function, the addition of a constant raises or lowers the whole graph of the sine or cosine function. It alters the mean value without changing the amplitude.

(4) Changing the sign within a cosine function has no effect, i.e. $\cos(-x) = \cos x$.

(6) Placing a constant or altering the constant b in $\sin(ax + b)$ or $\cos(ax + b)$ changes the phase and shifts the sine or cosine function along the x-axis.

(a) Write down the amplitude and period of $y = \sin(3x)$

(b) Write down the amplitude and frequency of $y = 3\sin(2x)$

(c) Write down the amplitude, period and frequency of $y = a\sin(bx)$

(d) Write down the amplitude, period, frequency and phase of

$$y = 4\sin(2x + 7).$$

(e) Write down an equivalent expression to that in (d) but with the phase of less than 2π.

Your solution

(a) amplitude $= 1$ period $= 2\pi/3$

(b) amplitude $= 3$ frequency $= \frac{2}{2\pi} = \frac{1}{\pi}$

(c) amplitude $= a$ period $= 2\pi/b$ frequency $= b/2\pi$

(d) amplitude $= 4$ period $= \frac{2\pi}{2} = \pi$ frequency $= \frac{1}{\pi}$ phase $= 7$

(e) $y = 4\sin(2x + 7 - 2\pi) = 4\sin(2x + 0.7168)$

Write down a function relating water level (L m) in a harbour to time (T hours), starting when the level is equal to the mean level of 5m, that has an amplitude of 2m and a period of twelve hours.

Your solution

In the general form $y = a\sin(bx + c) + d$, the phase $c = 0$, the period $\frac{2\pi}{b} = 12$, so $b + \frac{\pi}{6}$ amplitude $a = 2$, mean value $d = 5$. $L = 2\sin(-T)\ 5 \qquad (T \geq 0)$