Contents 6

exponential and **logarithmic functions**

Learning **outcomes**

In this workbook you will learn about one of the most important functions in mathematics, science and engineering -- the exponential function. You will learn how to combine exponential functions to produce other important functions -- the hyperbolic functions which, as you will note from later workbooks, are related to the trigonometric functions. You will also learn about logarithms and the logarithm function which is the function inverse to the exponential function. Finally you will learn what a log-linear graph is and how they can be used to simplify the presentation of certain kinds of data.

Time **allocation**

You are expected to spend approximately nine hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

The Exponential Function

6.1

Introduction

In this Section we revisit the use of exponents. We consider how the expression a^x is defined when a is a positive number and x is *irrational*. Previously we have only considered examples in which x is a *rational* number. We consider these exponential functions $f(x) = a^x$ in more depth and in particular consider the special case when the base a is the so-called exponential constant:

$$e = 2.7182818\ldots$$

We then examine the behaviour of e^x as $x \to \infty$, called **exponential growth** and of e^{-x} as $x \to \infty$ called **exponential decay**.

Prerequisites

Before starting this Section you should . . .

① have a good knowledge of indices and their laws

② have knowledge of rational and irrational numbers

③ know how to take limits

Learning Outcomes

After completing this Section you should be able to . . .

✓ approximate a^x when x is irrational

✓ assess the behaviour of a^x: in particular the exponential function e^x

✓ understand the terms *exponential growth* and *exponential decay*

1. Exponents revisited

We have seen in Workbook 1 (Section 2) the meaning to be assigned to the expression a^p where a is a positive number. We remind the reader that 'a' is called the **base** and 'p' is called the **exponent**. There are various cases to consider:

If m, n are positive integers

- $a^n = a \times a \times \cdots \times a$ with n factors
- $a^{1/n}$ means the n^{th} root of a. That is, $a^{1/n}$ is that positive number which satisfies

$$(a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n}) = a$$

  where there are n factors on the left hand side.
- $a^{m/n} = (a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n})$ where there are m factors.
- $a^{-n} = \dfrac{1}{a^n}$

For convenience we again list the basic laws of exponents:

Key Point

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^1 = a, \qquad \text{and if} \quad a \neq 0 \qquad a^0 = 1$$

Simplify the expressions (a) $\dfrac{b^{m-n}b^3}{b^{2m}}$ (b) $\dfrac{(5a^m)^2a^2}{(a^3)^2}$

(a) First simplify the numerator

Your solution

$b^{m-n}b^3 =$

$b^{m-n}b^3 = b^{m+3-n}$

Now include the denominator

Your solution

$$\frac{b^{m-n}b^3}{b^{2m}} = \frac{b^{m+3-n}}{b^{2m}} =$$

$$\frac{b^{m+3-n}}{b^{2m}} = b^{m+3-n-2m} = b^{3-m-n}$$

(b) Again simplify the numerator

Your solution

$$(5a^m)^2a^2 =$$

$$(5a^m)^2a^2 = 25a^{2m}a^2 = 25a^{2m+2}$$

Now include the denominator

Your solution

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} =$$

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} = 25a^{2m+2-6} = 25a^{2m-4}$$

What is a^x if x is a real number?

So far we have given the meaning of a^p where p is, at worst, a rational number, that is, one which can be written as a quotient of integers. So, if p is rational, then

$$p = \frac{m}{n} \qquad \text{where } m, n \text{ are integers}$$

Now consider x as a real number which cannot be written as a rational number. Two common examples of these **irrational** numbers are $x = \sqrt{2}$ and $x = \pi$. What we shall do is *approximate* x by a rational by working to a fixed number of decimal places. For example if

$$x = 3.914712334317\ldots$$

then, if we are working to 3 d.p. we would write

$$x \approx 3.915$$

and **this** number can certainly be expressed as a rational number:

$$x \approx 3.915 = \frac{3915}{1000}$$

so, in this case

$$a^x = a^{3.914712\ldots} \approx a^{3.915} = a^{\frac{3915}{1000}}$$

and the final term: $a^{\frac{3915}{1000}}$ can be determined in the usual way by using your calculator. From henceforth we shall therefore assume that the expression a^x is defined for all positive values of a and for **all** values of x.

By working to 3 d.p. find, using your calculator, the value of $3^{\pi/2}$.

First, approximate the value of $\frac{\pi}{2}$

Your solution

$\dfrac{\pi}{2} \approx$ to 3 d.p.

$\frac{\pi}{2} \approx \frac{3.1415927...}{2} = 1.5707963\cdots \approx 1.571$

Now determine $3^{\pi/2}$

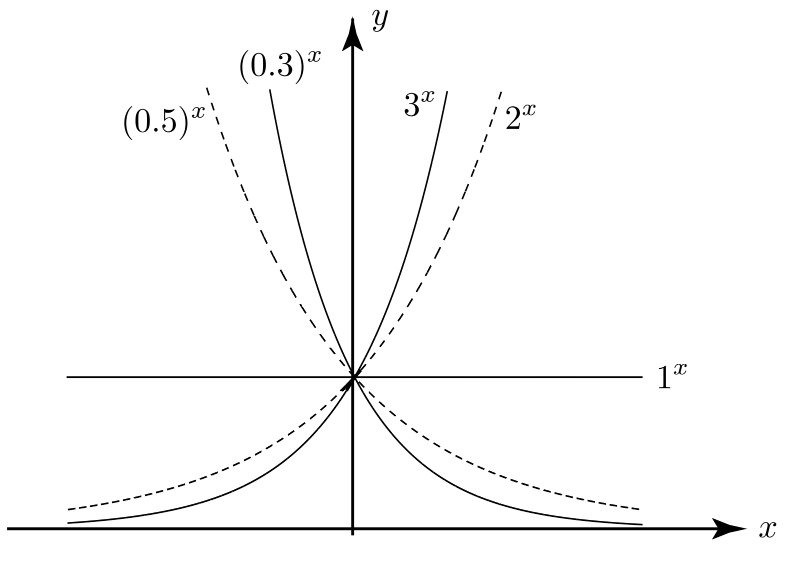
Your solution

$3^{\pi/2} \approx$

$3^{\pi/2} \approx 3^{1.571} = 5.618$ to 3 d.p.

2. Exponential Functions

For a fixed value of the base a the expression a^x clearly varies with the value of x: it is a function of x. We draw, in the diagram below, the graphs of $(0.5)^x, (0.3)^x, 1^x, 2^x$ and 3^x.



The functions a^x (as different values are chosen for a) are called **exponential functions**. From the graphs we see (and these are true for *all* exponential functions):

If $a > b > 0$ then

$$a^x > b^x \quad \text{if } x > 0 \qquad \text{and} \qquad a^x < b^x \quad \text{if } x < 0$$

The most important and widely used exponential function has the particular base 2.7182818 . . . , a number always denoted by the single letter e:

$$e = 2.7182818\ldots$$

It will not be clear to the reader why this particular value is so important. However, its importance will become clear as your knowledge of mathematics increases. The number e is as important as the number π and, like π, is also irrational. That is, e cannot be written as the quotient of two integers. The value of e is stored in most calculators. There are numerous ways of *calculating* the value of e. For example, it can be shown that the value of e is the end-point of the sequence of numbers:

$$\left(\frac{2}{1}\right)^1, \quad \left(\frac{3}{2}\right)^2, \quad \left(\frac{4}{3}\right)^3, \quad \ldots, \qquad \left(\frac{17}{16}\right)^{16}, \quad \ldots, \quad \left(\frac{65}{64}\right)^{64}, \ldots$$

which, in decimal form (each to 6 d.p.) are

$$2.000000, \quad 2.250000, \quad 2.370370, \quad \ldots, \quad 2.637929, \quad \ldots, 2.697345, \quad \ldots$$

This is a slowly converging sequence. However, it does lead to a precise definition for the value of e:

$$e = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

An alternative way of calculating e is to use the (infinite) series:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \ldots$$

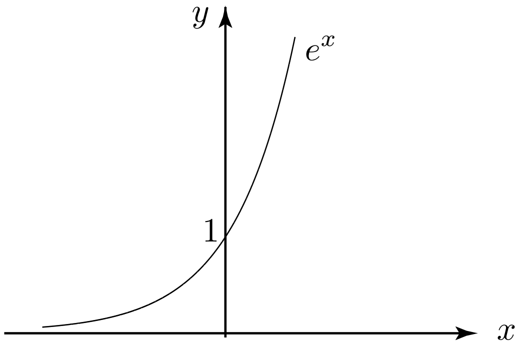
where, we remember,

$$n! = n \times (n-1) \times (n-2) \times \ldots (3) \times (2) \times (1)$$

It can be shown that the first eleven terms of this series provide a value of e with an error of less than 3×10^{-8}. (The reader is encouraged to carry out this calculation).
Although all functions of the form a^x are called exponential functions we usually refer to e^x as *the* exponential function.

Key Point

e^x is **the** exponential function



The exponential function (and its variants) appear in various areas of mathematics and engineering. For example, the shape of a hanging chain or rope, under the effect of gravity, is well described by a combination of the exponential curves e^{kx}, e^{-kx}. The function e^{-x^2} plays a major role in statistics; it being fundamental in the important **Normal distribution** which describes the variability in many naturally occurring phenomena. The exponential function e^{-kx} appears directly, again in the area of statistics, in the Poisson distribution which (amongst other things) is used to predict the number of events (which occur randomly) in a given time interval.
From now on, when we refer to an exponential function, it will be to the function e^x that we will refer.

Use a calculator to determine the values (to 2 d.p.)
(a) $e^{1.5}$, (b) e^{-2}, (c) e^{17}.

Your solution

$e^{1.5} =$ $e^{-2} =$ $e^{17} =$

$e^{1.5} = 4.48$, $e^{-2} = 0.14$, $e^{17} = 2.4 \times 10^7$

Simplify the expression $\dfrac{e^{2.7}e^{-3(1.2)}}{e^2}$ and determine its numerical value to 3 d.p.

First simplify the expression

Your solution

$$\frac{e^{2.7}e^{-3(1.2)}}{e^2} =$$

$$\frac{e^{2.7}e^{-3(1.2)}}{e^2} = e^{2.7}e^{-3.6}e^{-2} = e^{2.7-3.6-2} = e^{-2.9} = 0.055$$

3. Exponential growth

If $a > 1$ then it can be shown that, no matter how large K is:

$$\frac{a^x}{x^K} \to \infty \quad \text{as} \quad x \to \infty$$

That is, if K is fixed (though chosen as large as desired) then eventually, as x increases, a^x will overtake (and remain ahead of) the value x^K as long as $a > 1$. The growth of a^x as x increases is called **exponential growth**, though it is common practice to refer to the growth of the particular function e^x as exponential growth.

A function $f(x)$ grows exponentially and is such that $f(0) = 1$ and $f(2) = 4$. Find the exponential curve that fits through these points.

First: assume the function is $f(x) = e^{kx}$ where k is to be determined from the given information. Clearly, $f(0) = 1$ (irrespective of the value of k). Now find the value of k.

Your solution

when $x = 2$, $f(2) = 4$ so $e^{2k} = 4$

By **trying** values of k: say $0.6, 0.7, 0.8, \ldots$ find the value such that $e^{2k} = 4$.

Your solution

$e^{2(0.6)} =$ $\quad$ $e^{2(0.7)} =$ $\quad$ $e^{2(0.8)} =$

$e^{2(0.6)} = 3.32$ (too low) $\quad$ $e^{2(0.7)} = 4.055$ (too high)

Now try values of k closer to 0.7. Try $k = 0.67,\ 0.68,\ 0.69, \ldots$

Your solution

$e^{2(0.67)} =$ $\quad$ $e^{2(0.68)} =$ $\quad$ $e^{2(0.69)} =$

$e^{2(0.67)} = 3.819$ (low) $\quad$ $e^{2(0.68)} = 3.896$ (low) $\quad$ $e^{2(0.69)} = 3.975$ (low)

Finally try values closer to 0.69; say $k = 0.691,\ 0.692, \ldots$

Your solution

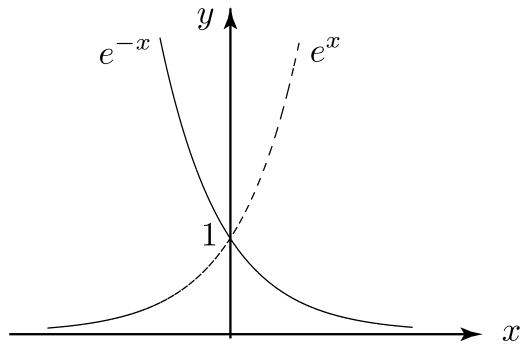
$e^{2(0.691)} =$ $\quad$ $e^{2(0.692)} =$ $\quad$ $e^{2(0.693)} =$

$e^{2(0.691)} = 3.983$, (low) $\quad$ $e^{2(0.692)} = 3.991$ (low) $\quad$ $e^{2(0.693)} = 3.999$ (low)

We conclude that the exponential function we seek is $e^{0.693x}$. We shall see, in Section 4, that a more efficient way of finding the value of k can be used.

4. Exponential decay

As we have noted, the behaviour of e^x as $x \to \infty$ is called exponential growth. In a similar manner we characterise the behaviour of the function e^{-x} as $x \to \infty$ as **exponential decay**. The graphs of e^x and e^{-x} are shown in the following diagram.



In fact e^{-x} tends to zero so quickly as $x \to \infty$ that, no matter how large the fixed number K is,

$$x^K e^{-x} \to 0 \quad \text{as} \quad x \to \infty$$

Choose $K = 10$ in the expression $x^K e^{-x}$ and calculate $x^K e^{-x}$ using your calculator for $x = 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35$.

Your solution

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$							

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$	6.5×10^4	4.5×10^5	1.7×10^5	2.1×10^4	1324	55	1.7

Exercises

1. Find, to 3 d.p., the values of
(a) 2^{-8} (b) $(5.1)^4$ (c) $(2/10)^{-3}$ (d) $(0.111)^6$ (e) $2^{1/2}$ (f) π^π (g) $e^{\pi/4}$ (h) $(1.71)^{-1.71}$

2. Plot $y = x^3$ and $y = e^x$ for $0 < x < 7$. For which integer values of x is $e^x > x^3$?

Answers 1. (a) 0.004 (b) 676.520 (c) 125 (d) 0.0 (e) 1.414 (f) 36.462 (g) 2.193 (h) 0.400

2. For integer values of x, $e^x > x^3$ if $x \geq 5$