The Hyperbolic Functions

6.2

Introduction

The hyperbolic functions $\cosh x$, $\sinh x$, $\tanh x$ etc are certain combinations of the exponential functions e^x and e^{-x}. The notation implies a close relationship between these functions and the trigonometric functions $\cos x$, $\sin x$, $\tan x$ etc. The close relationship is algebraic rather than geometrical. For example, the functions $\cosh x$ and $\sinh x$ satisfy the relation

$$\cosh^2 x - \sinh^2 x = 1$$

which is very similar to the trigonometric identity $\cos^2 x + \sin^2 x = 1$. (In fact any trigonometric identity has an equivalent hyperbolic function identity).

The hyperbolic functions are not introduced because they are a mathematical nicety. These combinations of exponentials do arise naturally and sufficiently often to warrant sustained study. For example, the shape of a chain hanging under gravity is well described by $\cosh x$ and the deformation of uniform beams can be expressed in terms of hyperbolic tangents.

Prerequisites

Before starting this Section you should . . .

① have a good knowledge of the exponential function

② have knowledge of odd and even functions

③ have familiarity with the definitions of $\tan x$, $\sec x$, $\text{cosec } x$ and of trigonometric identities

Learning Outcomes

After completing this Section you should be able to . . .

✓ understand how hyperbolic functions are defined in terms of exponential functions

✓ be able to obtain hyperbolic function identities and manipulate expressions involving hyperbolic functions

1. Constructing even and odd functions

A given function $f(x)$ can always be split into two parts, one of which is even and one of which is odd. To do this write $f(x)$ as $\frac{1}{2}[f(x) + f(x)]$ and then simply add and subtract $\frac{1}{2}f(-x)$ to this to give

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

The term $\frac{1}{2}[f(x) + f(-x)]$ is **even** because when x is replaced by $-x$ we have $\frac{1}{2}[f(-x) + f(x)]$ which is the same as the original. However, the term $\frac{1}{2}[f(x) - f(-x)]$ is **odd** since, on replacing x by $-x$ we have $\frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)]$ which is the negative of the original.

Separate the function $x^2 - 3^x$ into odd and even parts.

First, define $f(x)$ and find $f(-x)$.

Your solution	
$f(x) =$	$f(-x) =$

$f(x) = x^2 - 3^x, \quad f(-x) = x^2 - 3^{-x}$

Now construct $\frac{1}{2}[f(x) + f(-x)], \quad \frac{1}{2}[f(x) - f(-x)]$

Your solution	
$\frac{1}{2}[f(x) + f(-x)] =$	$\frac{1}{2}[f(x) - f(-x)] =$

$\frac{1}{2}[f(x) + f(-x)] = \frac{1}{2}(x^2 - 3^x + x^2 - 3^{-x}) = x^2 - \frac{1}{2}(3^x + 3^{-x})$. This is the even part of $f(x)$.

$\frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}(x^2 - 3^x - x^2 + 3^{-x}) = \frac{1}{2}(3^{-x} - 3^x)$. This is the odd part of $f(x)$.

The odd and even parts of the exponential function

Using the approach outlined above we see that the even part of e^x is

$$\frac{1}{2}(e^x + e^{-x})$$

and the odd part of e^x is

$$\frac{1}{2}(e^x - e^{-x})$$

We give these new functions special names: $\cosh x$ (pronounced 'cosh' x) and $\sinh x$ (pronounced 'shine' x)

Key Point

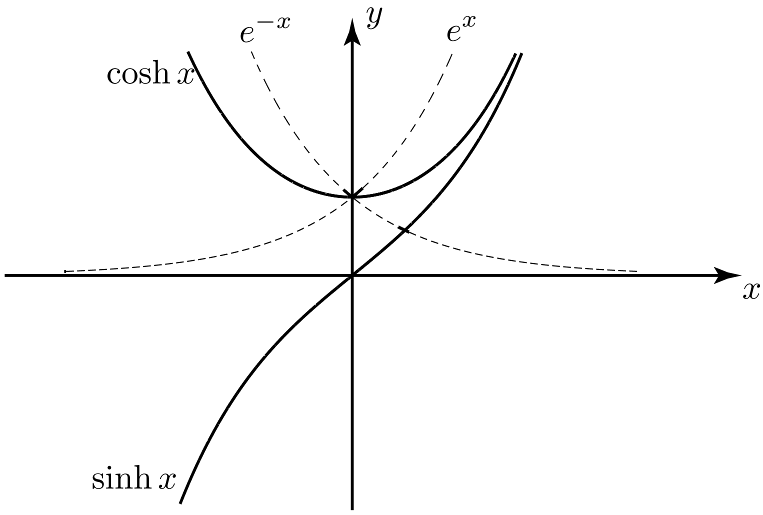
$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$\cosh x$ and $\sinh x$ are called **hyperbolic functions**

These two relations, when added and subtracted, give

$$e^x = \cosh x + \sinh x \qquad \text{and} \qquad e^{-x} = \cosh x - \sinh x$$

The hyperbolic functions are closely related to the trigonometric functions $\cos x$ and $\sin x$. Indeed, this explains the notation that we use. The hyperbolic cosine is written 'cos' with a 'h' to get cosh and the hyperbolic sine is written 'sin' with a 'h' to get sinh. The graphs of $\cosh x$ and $\sinh x$ are shown in the following diagram.



Note that $\cosh x > 0$ for all values of x and that $\sinh x$ only vanishes when $x = 0$.

2. Hyperbolic identities

The hyperbolic functions $\cosh x$, $\sinh x$ satisfy similar (but not identical) identities to those satisfied by $\cos x$, $\sin x$. We note first, some basic notation similar to that employed with trigonometric functions:

$$\cosh^n x \quad \text{means} \quad (\cosh x)^n \qquad \sinh^n x \quad \text{means} \quad (\sinh x)^n \qquad n \neq -1$$

In the special case that $n = -1$ we **do not** use $\cosh^{-1} x$ and $\sinh^{-1} x$ to mean $\frac{1}{\cosh x}$ and $\frac{1}{\sinh x}$ respectively. (The notation $\cosh^{-1} x$ and $\sinh^{-1} x$ is reserved for the **inverse functions** of $\cosh x$ and $\sinh x$ respectively).

Show that $\cosh^2 x - \sinh^2 x = 1$ for all x.

First find an expression for $\cosh^2 x$ in terms of the exponential functions $e^x,\ e^{-x}$.

Your solution

$$\cosh^2 x = \left[\tfrac{1}{2}(e^x + e^{-x})\right]^2 =$$

$$\cosh^2 x = \tfrac{1}{4}(e^x + e^{-x})^2 = \tfrac{1}{4}[(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2] = \tfrac{1}{4}[e^{2x} + 2e^x e^{-x} + e^{-2x}] = \tfrac{1}{4}[e^{2x} + 2 + e^{-2x}]$$

Similarly, find an expression for $\sinh^2 x$ in terms of $e^x,\ e^{-x}$

Your solution

$$\sinh^2 x = \left[\tfrac{1}{2}(e^x - e^{-x})\right]^2 =$$

$$\sinh^2 x = \tfrac{1}{4}(e^x - e^{-x})^2 = \tfrac{1}{4}[(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2] = \tfrac{1}{4}[e^{2x} - 2e^x e^{-x} + e^{-2x}] = \tfrac{1}{4}[e^{2x} - 2 + e^{-2x}]$$

Finally determine $\cosh^2 x - \sinh^2 x$.

Your solution

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}[e^{2x} + 2 + e^{-2x}] - \tfrac{1}{4}[e^{2x} - 2 + e^{-2x}] =$$

$$\cosh^2 x - \sinh^2 x = 1$$

As an alternative to the calculation in this guided exercise we could, instead, use the relations

$$e^x = \cosh x + \sinh x \qquad\qquad e^{-x} = \cosh x - \sinh x$$

and so, remembering the algebraic identity: $(a+b)(a-b) = a^2 - b^2$ we see that

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} = 1 \qquad \text{that is} \qquad \cosh^2 x - \sinh^2 x = 1$$

Key Point

The fundamental identity relating hyperbolic functions is:

$$\cosh^2 x - \sinh^2 x = 1$$

This is the hyperbolic function equivalent of the trigonometric identity: $\cos^2 x + \sin^2 x = 1$

Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

First, find $\cosh x \cosh y$ in terms of exponentials.

Your solution

$$\cosh x \cosh y = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) =$$

$$\cosh x \cosh y = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) = \frac{1}{4}[e^x e^y + e^{-x}e^y + e^x e^{-y} + e^{-x}e^{-y}]$$
$$= \frac{1}{4}(e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y})$$

Now find $\sinh x \sinh y$

Your solution

$$\sinh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) =$$

$$\sinh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) = \frac{1}{4}(e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y})$$

Now find $\cosh x \cosh y + \sinh x \sinh y$ and express the result in terms of a hyperbolic function.

Your solution

$$\cosh x \cosh y + \sinh x \sinh y =$$

$\cosh x \cosh y + \sinh x \sinh y = \frac{1}{2}(e^{x+y} + e^{-(x+y)})$ which we recognise as $\cosh(x+y)$

Other hyperbolic function identities can be found in a similar way. The most commonly used hyperbolic identities are listed in the following keypoint.

Key Point

- $\cosh^2 - \sinh^2 = 1$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\sinh 2x = 2 \sinh x \cosh y$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$ or $\cosh 2x = 2\cosh^2 -1$ or $\cosh 2x = 1 + 2\sinh^2 x$

3. Related hyperbolic functions

Once the trigonometric functions $\cos x$, $\sin x$ are introduced then related functions are also introduced; $\tan x$, $\sec x$, cosec x through the relations:

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x} \qquad \text{cosec } x = \frac{1}{\sin x}$$

In an exactly similar way we introduce hyperbolic functions $\tanh x$, sech x and cosech x (again the notation is obvious: take the 'trigonometric' name and append the letter 'h'). These functions are defined in the following keypoint

Key Point

Related hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{sech } x = \frac{1}{\cosh x} \qquad \text{cosech } x = \frac{1}{\sinh x}$$

Show that

(a) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

(b) $1 - \tanh^2 x = \text{sech}^2 x$

(a) Use the identity $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ and replace y by $-y$.

Your solution

$\sinh(x-y) =$

$\sinh(x-y) = \sinh x \cosh(-y) + \cosh x \sinh(-y)$.

Now obtain expressions for $\cosh(-y)$ and $\sinh(-y)$.

Your solution

$\cosh(-y) =$ $\qquad\qquad$ $\sinh(-y) =$

$\cosh(-y) = \cosh y$ since cosh is even. Also $\sinh(-y) = -\sinh y$ since sinh is odd.

Now complete the problem

Your solution

$\sinh(x-y) = \sinh x \cosh(-y) + \cosh x \sinh(-y) =$

$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

(b) Use the identity $\cosh^2 x - \sinh^2 x = 1$.

Your solution

$\cosh^2 x - \sinh^2 x = 1$ so

Dividing both sides by $\cosh^2 x$ gives

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \qquad \text{implying (see last keypoint)} \qquad 1 - \tanh^2 x = \text{sech}^2 x$$

Exercises

1. Express
(a) $2\sinh x + 3\cosh x$ in terms of e^x and e^{-x}.
(b) $2\sinh 4x - 7\cosh 4x$ in terms of e^{4x} and e^{-4x}.

2. Express
(a) $2e^x - e^{-x}$ in terms of $\sinh x$ and $\cosh x$.
(b) $\frac{7e^x}{(e^x - e^{-x})}$ in terms of $\sinh x$ and $\cosh x$, and then in terms of $\coth x$.
(c) $4e^{-3x} - 3e^{3x}$ in terms of $\sinh 3x$ and $\cosh 3x$.

3. Using only the cosh and sinh keys on your calculator find the values of
(a) $\tanh 0.35$, (b) cosech2, (c) sech(0.6).

Answers 1. (a) $\frac{5}{2}e^x - \frac{1}{2}e^{-x}$ (b) $-\frac{5}{2}e^{4x} - \frac{9}{2}e^{-4x}$
2. (a) $\cosh x + 3\sinh x$, (b) $\frac{7(\cosh x + \sinh x)}{2\sinh x}$, $\frac{7}{2}(\coth x + 1)$ (c) $\cosh 3x - 7\sinh 3x$
3. (a) 0.3364, (b) 0.2757 (c) 0.8436