Logarithms

6.3

Introduction

In this Section we introduce the logarithm: $\log_a b$. The operation of taking a logarithm essentially reverses the operation of raising a base a to a power n. We will formulate the basic laws satisfied by all logarithms and learn how to manipulate expressions involving logarithms. We shall see that to every law of indices there is an equivalent law of logarithms. Although logarithms to any base are defined it has become common practice to employ only two kinds of logarithms: logs to base 10 and logs to base e.

Prerequisites

Before starting this Section you should . . .

① have a knowledge of exponents and of the laws of indices

Learning Outcomes

After completing this Section you should be able to . . .

✓ invert $b = a^n$ using logarithms

✓ simplify expressions involving logarithms

✓ change bases in logarithms

1. Logarithms

Logarithms are introduced to reverse the process of raising a base 'a' to a power 'n'. As with all exponentials we demand that the base should be a positive number.

If $b = a^n$ then we write $\log_a b = n$.

Of course, the reverse statement is equivalent

If $\log_a b = n$ then $b = a^n$

The expression $\log_a b = n$ is read

"the log to base a of the number b is equal to n"

The phrase "log" is short for the word **logarithm**.

Example Determine the log equivalents of
(a) $16 = 2^4$, (b) $16 = 4^2$, (c) $27 = 3^3$,
(d) $134.896 = 10^{2.13}$, (e) $8.414867 = e^{2.13}$

Solution

(a) Since $16 = 2^4$ then $\log_2 16 = 4$
(b) Since $16 = 4^2$ then $\log_4 16 = 2$
(c) Since $27 = 3^3$ then $\log_3 27 = 3$
(d) Since $134.896 = 10^{2.13}$ then $\log_{10} 134.896 = 2.13$
(e) Since $8.41467 = e^{2.13}$ then $\log_e 8.414867 = 2.13$

Key Point

If $b = a^n$ then $\log_a b = n$

If $\log_a b = n$ then $b = a^n$

Find the log equivalents of
(a) $6.06287 = 4^{1.3}$ (b) (i) $b = a^n$, (ii) $c = a^m$, (iii) $bc = a^n a^m = a^{n+m}$

(a) Here, on the right-hand side, the base is 4 so:

Your solution

$6.06287 = 4^{1.3}$ implying $1.3 =$

$1.3 = \log_4 6.06287$

(b)(i) Here the base is a so

Your solution

$b = a^n$ implying $n =$

$n = \log_a b$

(b)(ii) Here the base is a so

Your solution

$c = a^m$ implying $m =$

$m = \log_a c$

(b)(iii) Here the base is a so

Your solution

$bc = a^{n+m}$ implying $n + m =$

$n + m = \log_a(bc)$

From the last guided exercise we have found, using the property of indices, that $\log_a(bc) = n + m = \log_a b + \log_a c$. We conclude that the index law $a^n a^m = a^{n+m}$ has an equivalent logarithm law

$$\log_a(bc) = \log_a b + \log_a c$$

in words

"the log of a product is the sum of logs"

Indeed this is one of the major advantages of using logarithms. They transform *products* of numbers (which is a relatively difficult operation) to a *sum* of numbers (which is a relatively easy operation).

All of the index laws have an equivalent logarithm law which are recorded in the following keypoint:

Key Point

The laws of logarithms

- $\log_a(AB) = \log_a A + \log_a B$
- $\log_a(\dfrac{A}{B}) = \log_a A - \log_a B$
- $\log_a 1 = 0, \qquad \log_a a = 1$
- $\log_a(A^k) = k \log_a A$

2. Simplifying expressions involving logarithms

To simplify an expression involving logarithms their laws, given in the keypoint above, need to be used.

Example Simplify:

$$\log_3 2 - \log_3 4 + \log_3(4^2) + \log_3(\frac{3}{4})$$

Solution

The third term $\log_3(4^2)$ simplifies to $2\log_3 4$ and the last term $\log_3(\frac{3}{4}) = \log_3 3 - \log_3 4 = 1 - \log_3 4$

Therefore
$$\begin{aligned}\log_3 2 - \log_3 4 + \log_3(4^2) + \log_3(\tfrac{3}{4}) &= \log_3 2 - \log_3 4 + 2\log_3 4 + 1 - \log_3 4\\ &= \log_3 2 + 1\end{aligned}$$

Simplify the expression:

$$\log_4(\frac{1}{4}) - \log_4(\frac{4}{27}) - \log_4 9$$

First simplify $\log_4(\frac{1}{4})$.

Your solution

$\log_4(\frac{1}{4}) =$

$\log_4(\frac{1}{4}) = \log_4 1 - \log_4 4 = 0 - 1 = -1$

Now simplify $\log_4(\frac{4}{27})$.

Your solution

$\log_4(\frac{4}{27}) =$

$\log_4(\frac{4}{27}) = \log_4 4 - \log_4 27 = 1 - \log_4 27$

Finally collect all your terms together:

Your solution

$\log_4(\frac{1}{4}) - \log_4(\frac{4}{27}) + \log_4 9 = -1 - (1 - \log_4 27) - \log_4 9 =$

$-1 - (1 - \log_4 27) - \log_4 9 = -2 + \log_4 27 - \log_4 9 = -2 + \log_4(\frac{27}{9}) = -2 + \log_4 3$

3. Logs to base 10 and natural logs

In practice only two kinds of logarithms are used, those to base 10, written $\log_{10}$ (or just simply log) and those to base e, written ln (these are called **natural logarithms**). Most scientific calculators will determine the logarithm, either to base 10 or to base e. For example, using a calculator:

$$\log 13 = 1.11394 \qquad (\text{implying} \quad 10^{1.11394} = 13)$$

$$\ln 23 = 3.13549 \qquad (\text{implying} \quad e^{3.13549} = 23)$$

Use your calculator to determine
(a) $\log 10$, (b) $\log 1000000$, (c) $\ln e$,
(d) $\ln 29.42$, (e) $\log e$, (f) $\ln 10$

Your solution

(a) $\log 10 =$ (b) $\log 1000000 =$ (c) $\ln e =$

(a) 1, (b) 6, (c) 1. Each of these could be determined directly, without the use of a calculator. For example, since $\log_a a = 1$ then $\log 10 \equiv \log_{10} 10 = 1$ and $\ln e \equiv \log_e e = 1$. Finally, since $\log_a A^k = k \log_a A$ then $\log 1000000 = \log 10^6 = 6 \log 10 = 6$

Your solution

(d) $\ln 29.42 =$ (e) $\log e =$ (f) $\ln 10 =$

(d) $\ln 29.42 = 3.38167$, (e) $\log e = 0.43429$, (f) $\ln 10 = 2.30258$

4. Changing bases in logarithms

It is sometimes required to express the logarithm with respect to one base in terms of a logaritm with respect to another base.
Now

$$b = a^n \qquad \text{implies} \qquad \log_a b = n$$

where we have used logs to base a. What happens if, for some reason, we want to use another base, p say? We take logs (to base p) of both sides of $b = a^n$:

$$\log_p(b) = \log_p(a^n) = n \log_p a \qquad \text{using one of the logarithm laws}$$

So

$$n = \frac{\log_p(b)}{\log_p(a)} \qquad \text{that is} \qquad \log_a b = \frac{\log_p(b)}{\log_p(a)}$$

This is the rule to be used when converting logarithms from one base to another. In particular, for base 10 logs and for natural logs:

$$\log_a b = \frac{\log(b)}{\log(a)} \qquad \text{or} \qquad \log_a b = \frac{\ln(b)}{\ln(a)}$$

For example,

$$\log_3 7 = \frac{\log 7}{\log 3} = \frac{0.8450980}{0.4771212} = 1.7712437$$

(check, on your calculator, that $3^{1.7712437} = 7$).
Also

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1.9459101}{1.0986123} = 1.7712437$$

Of course, $\log_3 7$ cannot be determined on your calculator since logs to base 3 are not available.

Use your calculator to determine the values of
(a)$\log_{21} 7$ (b) $\log_3 4$ (c) $\log_8 17$

(a) Re-express the log using either base 10 or base e.

Your solution

$$\log_{21} 7 = \frac{\log 7}{\log 21} =$$

$$\log_{21} 7 = \frac{\log 7}{\log 21} = \frac{0.845098}{1.322193} = 0.6391511$$

(b) Repeat as for (a)

Your solution

$$\log_3 4 =$$

$$\log_3 4 = \frac{\log 4}{\log 3} = \frac{0.6020599}{0.4771212} = 1.2618597$$

(c) Repeat as for (a)

Your solution

$\log_8 17 =$

$\log_8 17 = \frac{\ln 17}{\ln 8} = \frac{2.8332133}{2.0794415} = 1.3624876$

Example Simplify the expression $10^{\log x}$.

Solution

Let $y = 10^{\log x}$ then take logs (to base 10) of both sides:

$$\log y = \log(10^{\log x}) = (\log x)\log 10$$

where we have used: $\log A^k = k \log A$. However, since we are using logs to base 10 then $\log 10 = 1$ and so

$$\log y = \log x \qquad \text{implying} \qquad y = x$$

Therefore, finally

$$10^{\log x} = x$$

This is an important result and can be generalised to logarithms of other bases:

Key Point

$$a^{\log_a x} = x$$

This is because raising to the power and taking logs are **inverse** operations.

Exercises

1. Find the values of (a) $\log_2 8$ (b) $\log_{16} 50$ (c) $\log_3 28$

2. Simplify
(a) $\log_4 1 - 3\log_4 2 + \log_4 18$.
(b) $3\log_3 x - 2\log_3 x^2$.
(c) $\ln 3 \log_3 7 - \ln 7$.
(d) $\ln(8x - 4) - \ln(4x - 2)$.

Answers 1. (a) 3 (b) 1.41096 (c) 3.033
2. (a) $\log_4 9 - 1$, (b) $-\log_3 x$, (c) 0, (d) $\ln 2$