

Log-linear Graphs

6.5



Introduction

In this Section we employ our knowledge of logarithms to simplify plotting the relation between one variable and another. In particular we consider those situations in which one of the variables requires scaling because the range of its data values is very large in comparison to the range of the other variable.

In this work we only employ logarithms to base 10. To aid the plotting process we explain how log-linear graph paper is used. Unlike ordinary graph paper, one of the axes is scaled using logarithmic values instead of the values themselves. By this process, values which range from (say) 1 to 1,000,000 are scaled to range over the values 0 to 6. We do not discuss log-log graphs, in which both data sets require scaling, as the reader will easily be able to adapt the technique described here to those situations.



Prerequisites

Before starting this Section you should ...

- ① have knowledge logarithms to base 10
- ② be able to solve equations involving logarithms
- ③ be familiar with the laws of logarithms



Learning Outcomes

After completing this Section you should be able to ...

- ✓ appreciate when to use log-linear graph paper
- ✓ be able to use log-linear graph paper to analyse functions of the form $y = ka^{px}$

1. Logarithms and scaling

In this Section we shall work entirely with logarithms to base 10. We are already familiar with a particular property of logarithms:

$$\log A^k = k \log A$$

Now, choosing $A = 10$:

$$\log 10^k = k \log 10 = k$$

The effect of taking a logarithm is to replace a power: 10^k (which could be very large) by the value of the exponent k . Thus a range of numbers extending from 1 to 1,000,000 say, can be transformed, by taking logarithms to base 10, into a range of numbers from 0 to 6. This approach is especially useful in the exercise of plotting one variable against another in which one of the variables has a wide range of values.

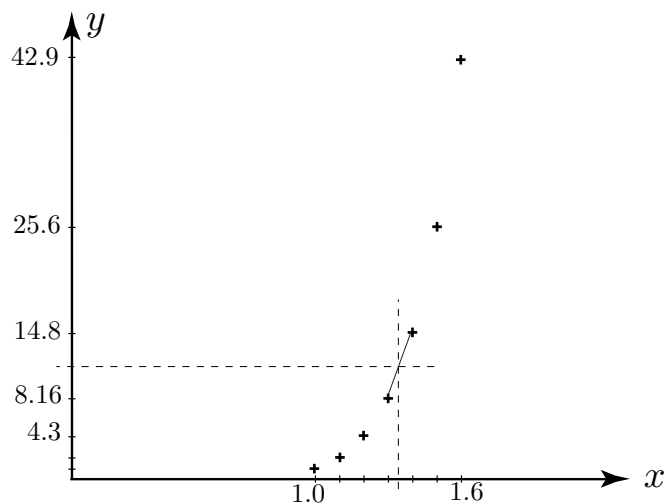
Example Plot the values (x, y) if

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	1.0	2.14	4.3	8.16	14.8	25.6	42.9

Estimate the value of y when $x = 1.35$.

Solution

If we attempt to plot these values on ordinary graph paper in which both vertical and horizontal scales are linear we find the large range in the y -values presents a problem. The values near the lower end are bunched together and interpolating a value of y when $x = 1.35$ is difficult.

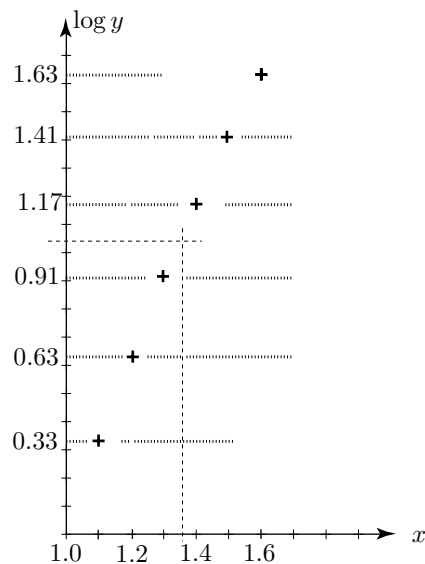


Solution (continued)

To alleviate this problem we employ logarithms to scale down the y -values. If we replace the y -values by their logarithms we obtain the amended data values:

x	1	1.1	1.2	1.3	1.4	1.5	1.6
$\log y$	0	0.33	0.63	0.97	1.17	1.41	1.63

Re-plotting, we obtain the following graph.



This approach has spaced-out the vertical values allowing a much easier assessment for the value of y at $x = 1.35$. From the graph we see that at $x = 1.35$ the ' $\log y$ ' value is approximately 1.05. Now, if $\log y = 1.05$ then, inverting

$$y = 10^{1.05} = 11.22$$

2. Log-linear graph paper

Ordinary graph paper has linear scales in both the horizontal (x) and vertical (y) directions. As we have seen, this can pose problems if the range of one of the variables, y say, is very large. One way round this is to take the logarithm of the y -values and re-plot on ordinary graph paper. Another common approach is to use log-linear graph paper in which the vertical scale is a non-linear logarithmic scale. Use of this special graph paper means that the original data can be plotted directly without the need to convert to logarithms.

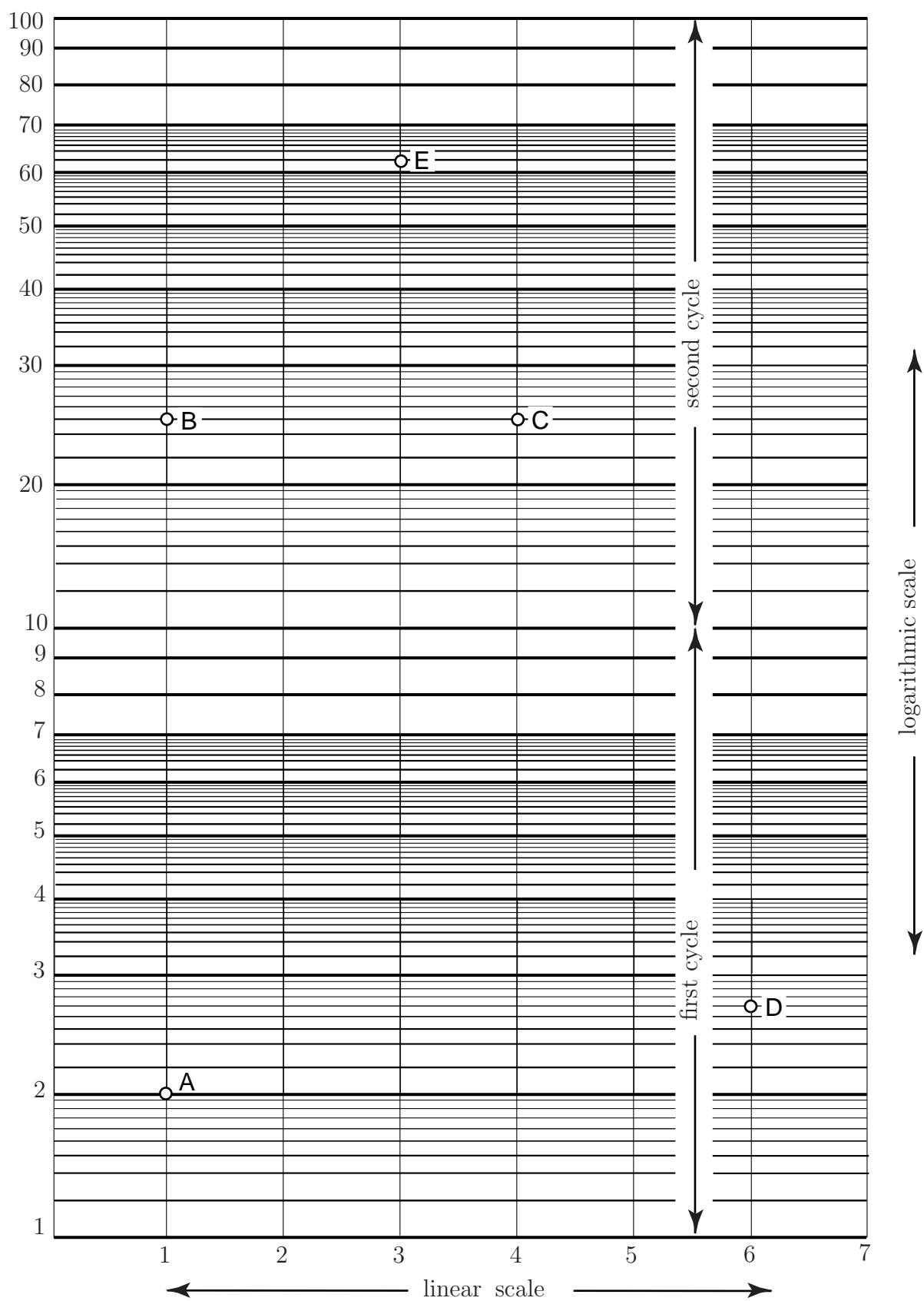
In log-linear graph paper the vertical axis is divided into a number of **cycles**. Each cycle corresponds to a jump in the data values by a factor of 10. For example, if your range of y -values extended from (say) 1 to 100 (or equivalently 10^0 to 10^2) then 2-cycle log-linear paper would be required. If your y -values extended from (say) 1,000 to 1,000,000 (or equivalently from 10^3 to 10^6) then 3-cycle log-linear paper would be used. Some other examples are given on the next page.

y -values	$\log y$ values	no. of cycles
$1 \rightarrow 10$	$0 \rightarrow 1$	1
$1 \rightarrow 100$	$0 \rightarrow 2$	2
$10 \rightarrow 10,000$	$1 \rightarrow 4$	3
$\frac{1}{10} \rightarrow 100$	$-1 \rightarrow 2$	3

An example of 2-cycle log-linear graph paper is shown on the next page. We see the horizontal scale is linear. The vertical scale is divided by lines denoted by 1,2,3,...,10,20,30,...,100. In the first cycle each of the horizontal blocks (separated by a slightly thicker line) is also divided according to a log-linear scale; so, for example, in the range $1 \rightarrow 2$ we have 9 horizontal lines representing the values 1.1, 1.2, ..., 1.9. These subdivisions have been repeated (appropriately scaled) in blocks 2-3, 3-4, 4-5, 5-6, 6-7. The subdivisions have been omitted from blocks 7-8, 8-9, 9-10 for reasons of clarity. On this graph paper, we have noted the positions of $A : (1, 2)$, $B : (1, 23)$, $C : (4, 23)$, $D : (6, 2.5)$, $E : (3, 61)$.



On the 2-cycle log-linear graph paper locate the positions of the points $F : (2, 21)$, $G : (2, 51)$, $H : (5, 3.5)$.



Example It is thought that the relationship between two variables x, y is *exponential*

$$y = ka^x$$

An experiment is performed and the following pairs of data values (x, y) were obtained

x	1	2	3	4	5
y	5.9	12	26	49	96

Verify that the relation $y = ka^x$ is valid by plotting values on log-linear paper to obtain a set of points lying on a straight line and estimate the values of k, a .

Solution

First we rearrange the relation $y = ka^x$ by taking logarithms (to base 10).

$$\therefore \log y = \log(ka^x) = \log k + x \log a$$

So, if we define a new variable $Y \equiv \log y$ then the relationship between Y and x will be linear – its graph (on log-linear paper) should be a straight line. The vertical intercept of this line is $\log k$ and the gradient of the line is $\log a$. Each of these can be obtained from the graph and the values of a, k inferred.

When using log-linear graphs, the reader should keep in mind that, on the vertical axis, the values are not as written but the logarithms of those values.

We have plotted the points and drawn a straight line (as best we can) through them (see next page). (We will see in later workbooks how we might improve on this subjective approach to fitting straight lines to data points). The line intercepts the vertical axis at a value $\log(3.13)$ and the gradient of the line is

$$\frac{\log 96 - \log 3.13}{5 - 0} = \frac{\log(96/3.13)}{5} = \frac{\log 30.67}{5} = 0.297$$

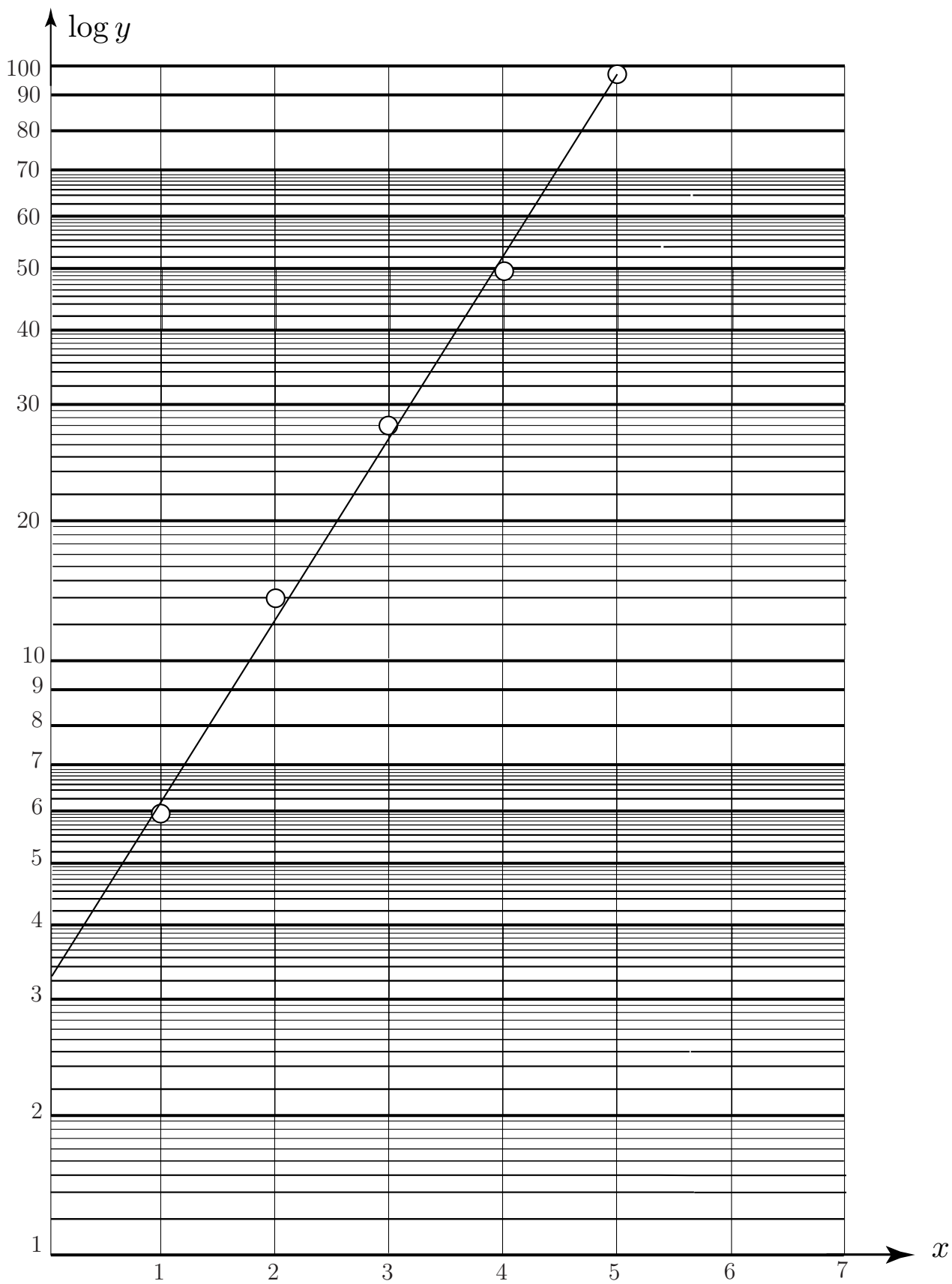
But the intercept is $\log k$ so

$$\log k = \log 3.13 \quad \text{implying} \quad k = 3.13$$

and the gradient is $\log a$ so

$$\log a = 0.297 \quad \text{implying} \quad a = 10^{0.297} = 1.98$$

We conclude that the relation between the x, y variables is well modelled by the relation $y = 3.13(1.98)^x$. If the points did not lie more-or-less on a straight line then we would conclude that the relationship was *not* of the form $y = ka^x$.





Using a log-linear graph estimate the values of k, a if it is assumed that $y = ka^{-2x}$ and the data values connecting x, y are:

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
y	190	155	123	100	80	63	52

First take logs of the relation $y = ka^{-2x}$ and introduce appropriate new variables.

Your solution

$$y = ka^{-2x} \text{ implies } \log y = \log(ka^{-2x}) =$$

introduce $Y =$

$\log y = \log k - 2x \log a$. Let $Y = \log y$ then $Y = \log k + x(-2 \log a)$. We therefore expect a linear relation between Y and x (i.e. on log-linear paper).

Now determine how many cycles are required in your log-linear paper.

Your solution

The range of values of y is 140; from 5.2×10 to 1.9×10^2 . So 2-cycle log-linear paper is needed.

Now plot the data values directly onto your log-linear paper (supplied on the next page). What deductions can you make? Do the plot now, before proceeding.

Your solution

Is the relation $y = ka^{-2x}$ acceptable? yes/no

yes. On plotting the points a straight line fits the data well which is what we expect from $Y = \log k + x(-2 \log a)$

Now, using knowledge of the intercept and the gradient find the values of k, a .

Your solution

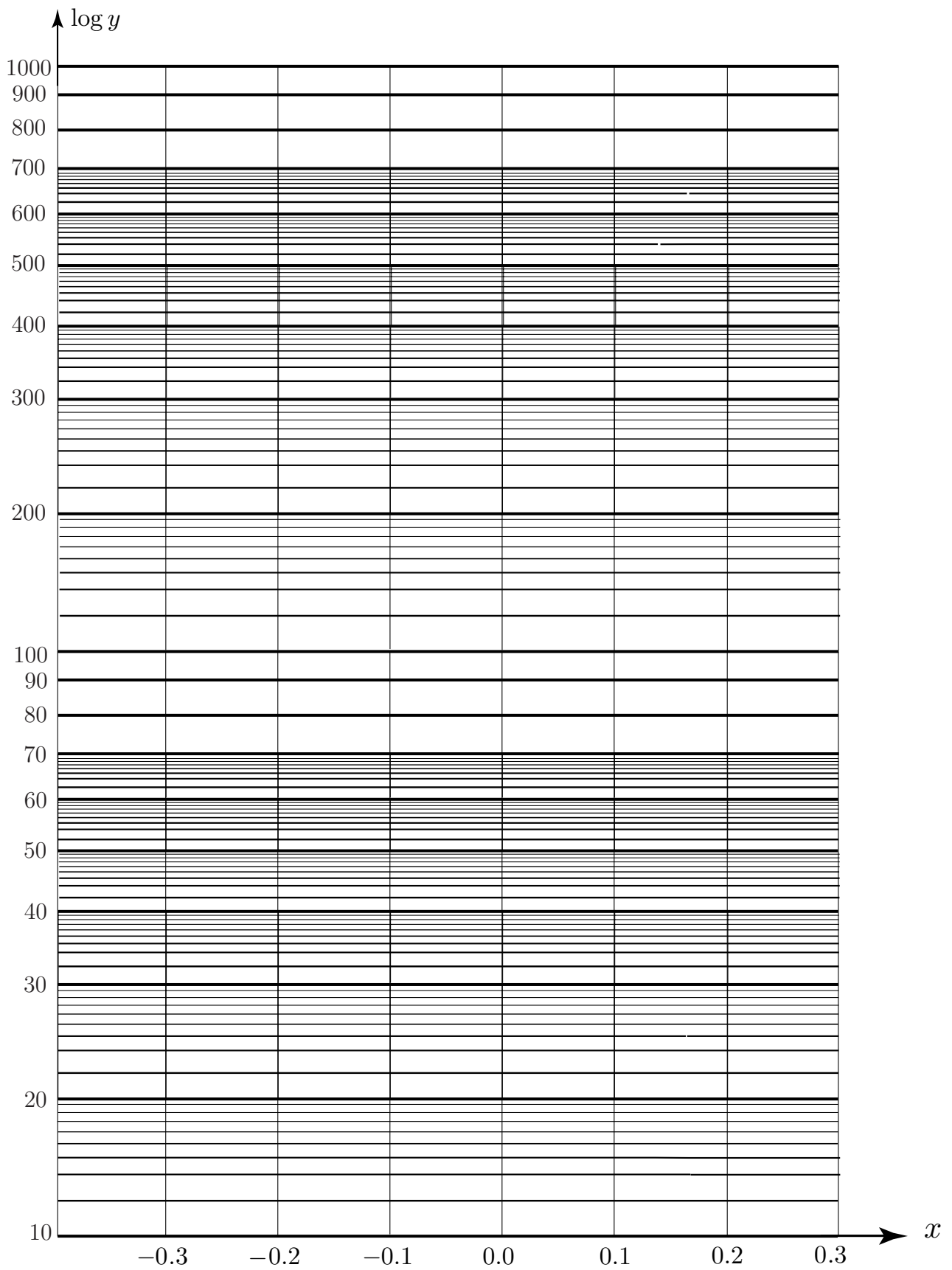
$$k = \quad \quad \quad -2 \log a =$$

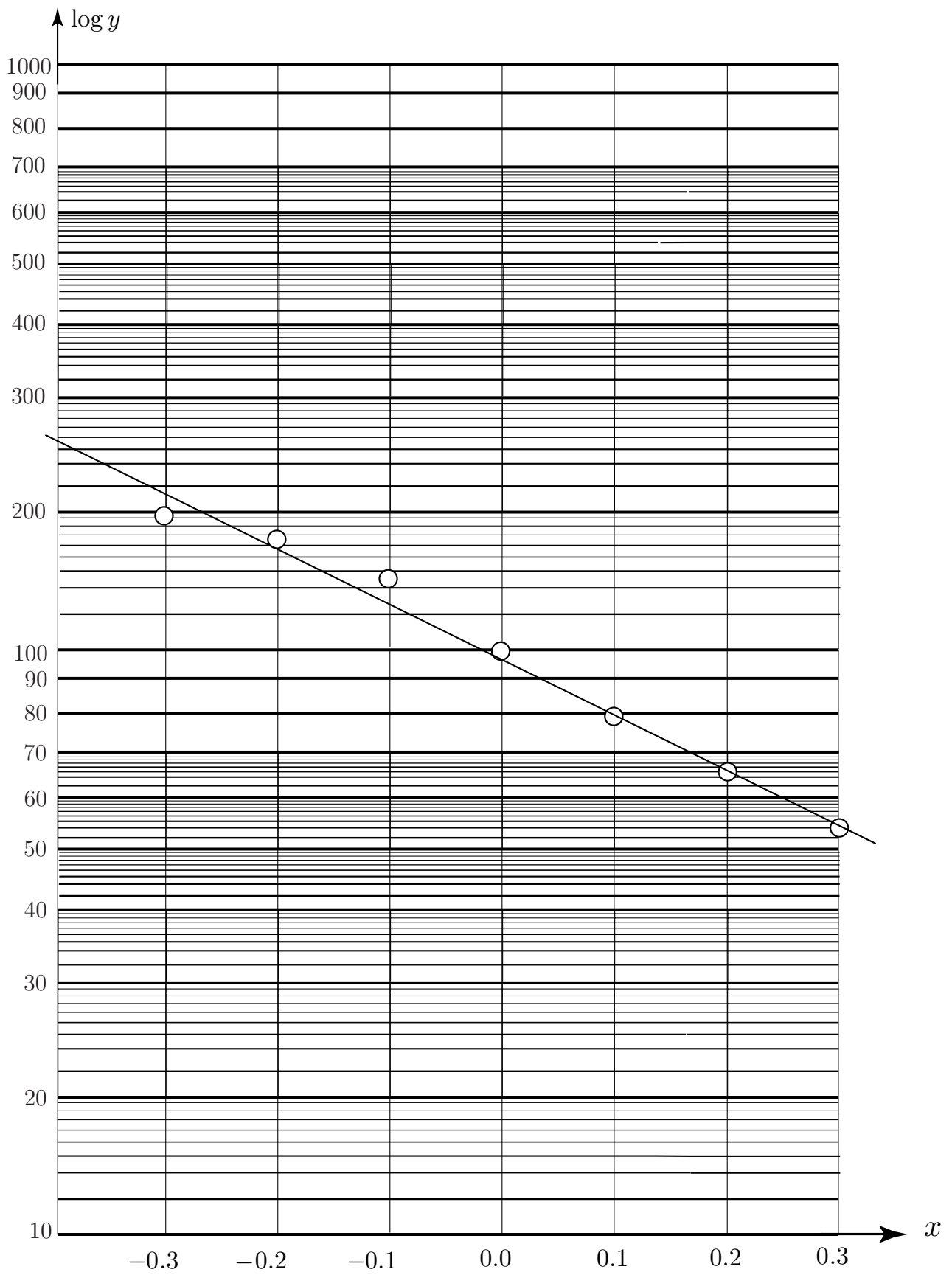
$k \approx 94$ (intercept on $x = 0$ line). The gradient is

$$\frac{\log 235 - \log 52}{-0.4 - 0.3} = -\frac{\log(235/52)}{0.7} = -\frac{0.655}{0.7} = -0.935$$

But the gradient is $-2 \log a$.

$$\text{Thus } -2 \log a = -0.935 \quad \text{which implies } a = 10^{0.468} = 2.93$$





Exercises

1. Estimate the values of k and a if $y = ka^x$ represents each of the following sets of data values:

(a)

x	0.5	1	2	3	4
y	5.93	8.8	19.36	42.59	93.70

(b)

x	1	2	3	4	5
y	5.4	9.7	17.5	31.5	56.7

2. Estimate the values of k and a if the relation $y = k(a)^{-x}$ is a good representation for the data values

x	2	2.5	3	3.5	4
y	7.9	3.6	1.6	0.7	0.3

2. (a) $k \approx 200$ $a \approx 5$
 1. (a) $k \approx 4$ $a \approx 2.2$ (b) $k \approx 3$ $a \approx 1.8$

