Matrix Multiplication 7.2

Introduction

When we wish to multiply matrices together we have to ensure that the operation is possible and this is not always so. Also, unlike number arithmetic and algebra even when the product exists the order of multiplication may have an effect on the result. In this section we pick our way through the minefield of matrix multiplication.

Prerequisites

Before starting this Section you should ...

① understand the concept of a matrix and the terms associated with it.

Learning Outcomes

After completing this Section you should be able to ...

✓ know when the product AB exists

✓ recognise that $AB \neq BA$ in most cases

✓ carry out the multiplication AB

✓ understand what is meant by the identity matrix I

1. Multiplying row matrices and column matrices together

Let A be a 1×2 row matrix and B be a 2×1 column matrix:

$$A = \begin{bmatrix} a & b \end{bmatrix} \qquad B = \begin{bmatrix} c \\ d \end{bmatrix}$$

The product of these two matrices is written AB and is the 1×1 matrix defined by:

$$AB = \begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = [ac + bd]$$

so that corresponding elements are multiplied and the results are then added together. For example

$$\begin{bmatrix} 2 & -3 \end{bmatrix} \times \begin{bmatrix} 6 \\ 5 \end{bmatrix} = [12 - 15] = [-3]$$

This matrix product is easily generalised to other row and column matrices. For example if C is a 1×4 row matrix and D is a 4×1 column matrix:

$$C = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

then we would naturally define the product of C with D as

$$CD = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix} = [6 - 12 - 6 + 10] = [-2]$$

The only requirement that we make is the number of elements of the row matrix is the same as the number of elements of the column matrix.

2. Multiplying two 2×2 matrices

If A and B are two matrices then the product AB is obtained by multiplying the rows of A with the columns of B in the manner described above. This will only be possible if the number of elements in the rows of A are the same as the number of elements in the columns of B. In particular we define the product of two 2×2 matrices A and B to be another 2×2 matrix C whose elements are calculated according to the following pattern

$$\begin{array}{ccccc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \times & \begin{bmatrix} w & x \\ y & z \end{bmatrix} & = & \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} \\ A & & B & = & C \end{array}$$

The rule for calculating the elements of C is described in the following keypoint:

Key Point

Matrix Product

$$AB = C$$

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B

We illustrate this construction for the abstract matrices A and B given above:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \\ \\ \begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 3 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -6 & 10 \end{bmatrix}$$

Find the product AB where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$.

Write down row 1 of A, column 2 of B and form the product as described above.

Your solution

$[1, 2]$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$; their product is $1 \times (-1) + 2 \times 1 = 1$.

Now repeat the process for row 2 of A, column 1 of B.

Your solution

$[3, 4]$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Their product is $3 \times 1 + 4 \times (-2) = -5$

Find the two other elements of $C = AB$ and hence write down the matrix C.

Your solution

Row 1 column 1 is $1 \times 1 + 2 \times (-2) = -3$. Row 2 column 2 is $3 \times (-1) + 4 \times 1 = 1$

$C = \begin{bmatrix} -3 & 1 \\ -5 & 1 \end{bmatrix}$.

3. Some surprising results

We have already calculated the product AB where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Now complete the following guided exercise in which you are asked to determine the product BA, i.e. with the matrices in reverse order.

Form the products of

row 1 of B and column 1 of A row 1 of B and column 2 of A

row 2 of B and column 1 of A row 2 of B and column 2 of A

Now write down the matrix BA

Your solution

row 1, column 1 is $1 \times 1 + (-1) \times 3 = -2$ row 1, column 2 is $1 \times 2 + (-1) \times 4 = -2$

row 2, column 1 is $-2 \times 1 + 1 \times 3 = 1$ row 2, column 2 is $-2 \times 2 + 1 \times 4 = 0$

BA is $\begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$

It is clear that AB and BA are **not** in general the same. In fact it is the **exception** that $AB = BA$. (This is to be contrasted with multiplication of numbers in which ab always equals ba).

In the special case in which $AB = BA$ we say that the matrices A and B **commute**.

Carry out the multiplication AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Your solution

$$AB = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We call B the 2×2 **zero matrix** written $\underline{0}$ so that $A \times \underline{0} = \underline{0} \times A = 0$ for any matrix A.

Now in the multiplication of numbers, the equation

$$ab = 0$$

implies that either a, or b, or both is zero. The following guided exercise shows that this is not necessarily true for matrices.

Carry out the multiplication AB where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Your solution

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here we have a zero product yet neither A nor B is the zero matrix. Thus the statement $AB = 0$ does **not** allow us to conclude that either $A = \underline{0}$ or $B = \underline{0}$.

Find the product AB where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Your solution

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix** or **unit matrix** of order 2×2, and is usually denoted by the symbol I. (Strictly we should write I_2, to indicate the size.)
I plays the same role in matrix multiplication as the number 1 does in number multiplication. Hence

as $\quad a.1 = 1.a = a$ for any number $a \quad$ so $\quad AI = IA = A$ for any matrix A.

4. Multiplying two 3×3 matrices

The definition of the product $C = AB$ where A and B are two 3×3 matrices is as follows

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar+bu+cx & as+bv+cy & at+bw+cz \\ dr+eu+fx & ds+ev+fy & dt+ew+fz \\ gr+hu+ix & gs+hv+iy & gt+hw+iz \end{bmatrix}$$

This looks a rather daunting amount of algebra but in fact the construction of the matrix on the right-hand side is straightforward if we follow the simple rule from the keypoint that the element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.
For example, to obtain the element in row 2, column 3 of C we take row 2 of A: $[d, e, f]$ and multiply it with column 3 of B in the usual way to produce $[dt + ew + fz]$.
By repeating this process we can quickly obtain every element of C.

Find the element in row 2 column 1 of the product

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

Your solution

Row 2 of A is $(3, 4, 0)$ column 1 of B is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

The combination required is $3 \times 2 + 4 \times 1 + (0) \times (0) = 10$

Now complete the multiplication to find all the elements of the matrix AB

Your solution

In full detail, the elements of AB are:

$$\begin{bmatrix} 1 \times 2 + 2 \times 1 + (-1) \times 0 & 1 \times (-1) + 2 \times (-2) + (-1) \times 3 & 1 \times 3 + 2 \times 1 + (-1) \times (-2) \\ 3 \times 2 + 4 \times 1 + 0 \times 0 & 3 \times (-1) + 4 \times (-2) + 0 \times 3 & 3 \times 3 + 4 \times 1 + 0 \times (-2) \\ 1 \times 2 + 5 \times 1 + (-2) \times 0 & 1 \times (-1) + 5 \times (-2) + (-2) \times 3 & 1 \times 3 + 5 \times 1 + (-2) \times -2 \end{bmatrix}$$

i.e. $AB = \begin{bmatrix} 4 & -8 & 7 \\ 10 & -11 & 13 \\ 7 & -17 & 12 \end{bmatrix}$

The 3×3 unit matrix is:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and as in the 2×2 case this has the property that

$$AI = IA = A$$

The 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Multiplying non-square matrices together

So far, we have just looked at multiplying 2×2 matrices and 3×3 matrices. However, products between non-square matrices may be possible.

Key Point

General Matrix Products

The general rule is that an $n \times p$ matrix A can be multiplied by a $p \times m$ matrix B to form an $n \times m$ matrix $AB = C$.

In words:

'for the matrix product AB to be defined the number of columns of A must equal the number of rows of B'

The elements of C are found in the usual way:

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B

Example Find the product AB if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix}$

Solution

Since A is a 2×3 and B is a 3×2 matrix the product AB can be found and results in a 2×2 matrix.

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 22 & 13 \\ 38 & 25 \end{bmatrix}$$

Obtain the product AB if $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

Your solution

We expect AB to be a 2×3 matrix.

$$AB = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} [\,1 \;\; -2\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,1 \;\; -2\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} & [\,1 \;\; -2\,]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [\,2 \;\; -3\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,2 \;\; -3\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} & [\,2 \;\; -3\,]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 1 \\ -14 & 5 & 2 \end{bmatrix}$$

6. The Rules of Matrix Multiplication

It is worth noting that the process of multiplication can be continued to form products of more than two matrices.
Although two matrices may not commute (i.e. in general $AB \neq BA$) the **associative law** always holds i.e. for matrices which **can** be multiplied,

$$A(BC) = (AB)C.$$

The general principle is **keep the order left to right**, but within that any two adjacent matrices can be multiplied.
It is important to note that it is not always possible to multiply together any two given matrices.
For example if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ then $AB = \begin{bmatrix} a+2d & b+2e & c+2f \\ 3a+4d & 3b+4e & 3c+4f \end{bmatrix}$.
However $BA = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is **not defined** since each row of B has three elements whereas each column of A has two elements and we cannot multiply these elements in the manner described.

Given $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

State which of the products $AB,\ BA,\ AC,\ CA,\ BC,\ CB,\ (AB)C,\ A(CB)$ is defined and state the size of the product when defined.

Your solution

$\underset{2\times 3}{A}\ \underset{2\times 2}{B}$ not possible

$\underset{2\times 2}{B}\ \underset{2\times 3}{A}$ possible; result 2×3

$\underset{2\times 3}{A}\ \underset{3\times 2}{C}$ possible; result 2×2

$\underset{3\times 2}{C}\ \underset{2\times 3}{A}$ possible; result 3×3

BC not possible

CB possible; result 3×2

$(AB)C$ not possible, since AB not defined.

$\underset{2\times 3}{A}\ \underset{3\times 2}{(C\ B)}$ possible; result 2×2

We now list together some properties of matrix multiplication and compare them with corresponding properties for multiplication of numbers.

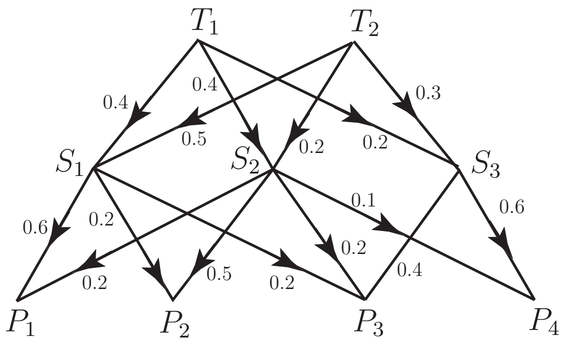
Key Point

Matrix algebra	Number algebra
$A(B+C) = AB + AC$	$a(b+c) = ab + ac$
$AB \neq BA$	$ab = ba$
$A(BC) = (AB)C$	$a(bc) = (ab)c$
$AI = IA = A$	$1.a = a.1 = a$
$A\underline{0} = \underline{0}A = \underline{0}$	$0.a = a.0 = 0$

Application of Matrices to Networks

A network is a collection of points (nodes) some of which are connected together by lines (paths). The information contained in a network can be conveniently stored in the form of a matrix.

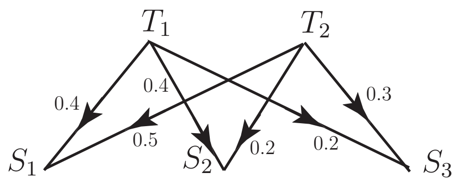
Example Petrol is delivered to terminals T_1 and T_2. They distribute the fuel to 3 storage depots ($S_1,\ S_2,\ S_3$). The network diagram below shows what fraction of the fuel goes from each terminal to the three storage depots. In turn the 3 depots supply fuel to 4 petrol stations as shown in the next diagram:



Show how this situation may be described using matrices.

Solution

If the amount of fuel, in litres, flowing from T_1 is denoted by t_1 and from T_2 by t_2 and the quantity being received by S_i by s_i for $i = 1, 2, 3$. This situation is described in the following diagram:

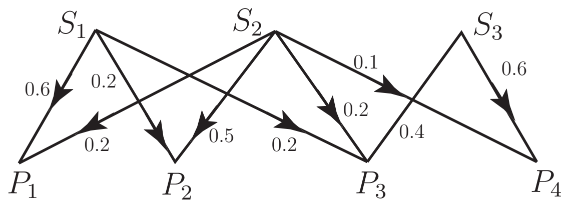


From this diagram we see that

$$\begin{aligned} s_1 &= 0.4t_1 + 0.5t_2 \\ s_2 &= 0.4t_1 + 0.2t_2 \\ s_3 &= 0.2t_1 + 0.3t_2 \end{aligned} \qquad \text{or, in matrix form:} \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Solution

In turn the 3 depots supply fuel to 4 petrol stations as shown in the next diagram:



If the petrol stations receive p_1, p_2, p_3 litres respectively then from the diagram we have:

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ p_2 &= 0.2s_1 + 0.5s_2 \\ p_3 &= 0.2s_1 + 0.2s_2 + 0.4s_3 \\ p_4 &= 0.1s_2 + 0.6s_3 \end{aligned} \quad \text{or, in matrix form:} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Combining the equations

$$\begin{aligned} s_1 &= 0.4t_1 + 0.5t_2 \\ s_2 &= 0.4t_1 + 0.2t_2 \\ s_1 &= 0.2t_1 + 0.3t_2 \end{aligned} \quad \text{and} \quad \begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ p_2 &= 0.2s_1 + 0.5s_2 \\ p_3 &= 0.2s_1 + 0.2s_2 + 0.4s_3 \\ p_4 &= 0.1s_1 + 0.6s_2 \end{aligned} \quad \text{we get:}$$

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 = 0.6(0.4t_1 + 0.5t_2) + 0.2(0.4t_1 + 0.2t_1) \\ &= 0.32t_1 + 0.34t_2 \end{aligned}$$

with similar results for p_2, p_3 and p_4. This is equivalent to combining the two networks. The results can be obtained more easily by multiplying the matrices:

$$\begin{aligned} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} &= \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.32 & 0.34 \\ 0.28 & 0.20 \\ 0.24 & 0.26 \\ 0.16 & 0.20 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0.32t_1 + 0.34t_2 \\ 0.28t_1 + 0.20t_2 \\ 0.24t_1 + 0.26t_2 \\ 0.16t_1 + 0.20t_2 \end{bmatrix} \end{aligned}$$

Exercises

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$ find

   (a) AB, (b) AC, (c) $(A+B)C$, (d) $AC+BC$ (e) $2A-3C$

2. If a rotation through an angle θ is represented by the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and a second rotation through an angle ϕ is represented by the matrix $B = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ show that a rotation through an angle $\theta+\phi$ is represented either by AB or by BA.

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find AB and BC.

4. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ verify $A(BC) = (AB)C$.

5. A square matrix A is said to be **symmetric** if $A = A^T$, where A^T is the **transpose** of A.
   If $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ then show that AA^T is symmetric.

6. If $A = \begin{bmatrix} 11 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ verify that $(AB)^T = \begin{bmatrix} 0 & 1 \\ 11 & 3 \\ 22 & 7 \end{bmatrix} = B^T A^T$

Answers

1. (a) $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ (b) $AC = \begin{bmatrix} 4 & -7 \\ 8 & -15 \end{bmatrix}$ (c) $(A+B)C = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$

   (d) $AC + BC = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 7 \\ 0 & 17 \end{bmatrix}$

2. $$AB = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$
   $$= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$
   which clearly represents a rotation through angle $\theta+\phi$.

3. $AB = \begin{bmatrix} 15 & 26 \\ -6 & -12 \\ 12 & 24 \end{bmatrix}$, $BC = \begin{bmatrix} 8 & 10 \\ 0 & 3 \\ 16 & 17 \end{bmatrix}$

4. $A(BC) = (AB)C = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$