

# The Inverse of a Matrix

## 7.4



### Introduction

In number arithmetic every number  $a \neq 0$  has a reciprocal  $b$  written as  $a^{-1}$  or  $\frac{1}{a}$  such that  $ba = ab = 1$ . Similarly a square matrix  $A$  may have an inverse  $B = A^{-1}$  where  $AB = BA = I$ . We develop a rule for finding the inverse of a  $2 \times 2$  matrix (where it exists) and we look at two methods of finding the inverse of a  $3 \times 3$  matrix (where it exists). Non-square matrices do not possess inverses.



### Prerequisites

Before starting this Section you should ...

- ① be familiar with the algebra of matrices
- ② be able to calculate a determinant
- ③ know what a cofactor is



### Learning Outcomes

After completing this Section you should be able to ...

- ✓ know the condition for the existence of an inverse matrix
- ✓ use the formula for finding the inverse of a  $2 \times 2$  matrix
- ✓ find the inverse of a  $3 \times 3$  matrix using row operations and using the determinant method

# 1. The inverse of a square matrix

We know that any non-zero number  $k$  has an inverse; for example 2 has an inverse  $\frac{1}{2}$  or  $2^{-1}$ . The inverse of the number  $k$  is usually written  $\frac{1}{k}$  or, more formally, by  $k^{-1}$ . This numerical inverse has the property that

$$k \times k^{-1} = k^{-1} \times k = 1$$

We now show that an inverse of a matrix can, in certain circumstances, also be defined.

Given an  $n \times n$  square matrix  $A$ , then an  $n \times n$  square matrix  $B$  is said to be the **inverse matrix** of  $A$  if

$$AB = BA = I$$

where  $I$  is, as usual, the identity matrix of the appropriate size.

**Example** Show that the inverse matrix of  $A = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$  is  $B = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$

## Solution

All we need do is to check that  $AB = BA = I$ .

$$AB = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reader should check that  $BA = I$  also.

We make a number of important remarks.

- Non-square matrices do not have an inverse.
- The inverse of  $A$  is usually written  $A^{-1}$ .
- Not all square matrices have an inverse.



Consider  $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ , and let  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be a candidate for the inverse of  $A$ . Find  $AB$  and  $BA$ .

## Your solution

$$AB =$$

$$BA =$$

$$\begin{bmatrix} 0 & 2c + a \\ 0 & 2d + a \end{bmatrix} = BA, \quad \begin{bmatrix} 2a & a \\ b & b \end{bmatrix} = AB$$



Equate the elements of  $AB$  to those of  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and solve the resulting equations.

**Your solution**

$$a = 1, b = 0, c = 0, d = 1. \text{ Hence } a = 1, b = 0, c = 0, d = 1.$$

Hence, we have a contradiction (as would also have been obtained by equating the elements of  $BA$  to those of  $I$ ). The matrix  $A$  therefore has no inverse and is said to be a **singular matrix**. A matrix which has an inverse is said to be **non-singular**.

- If a matrix has an inverse then that inverse is unique.  
Suppose  $B$  and  $C$  are both inverses of  $A$ . Then, by definition,

$$AB = BA = I \quad \text{and} \quad AC = CA = I$$

Consider the two ways of forming the product  $CAB$

1.  $CAB = C(AB) = CI = C$
2.  $CAB = (CA)B = IB = B.$

Hence  $B = C$  and the inverse is unique.

- There is no such operation as division in matrix algebra. In arithmetic we have

$$3^{-1} \times 6 = \frac{1}{3} \times 6 = 2 = 6 \times \frac{1}{3} = 6 \times 3^{-1}$$

but in matrix algebra we must write either

$$A^{-1}B \quad \text{or} \quad BA^{-1},$$

depending on the order required.

- Assuming that the square matrix  $A$  has an inverse  $A^{-1}$  then the solution of the system of equations  $AX = B$  is found by pre-multiplying both sides by  $A^{-1}$ .

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B, \\ \text{changing brackets } (A^{-1}A)X &= A^{-1}B \\ \text{i.e. } IX &= A^{-1}B, && \text{using } A^{-1}A = I \\ \text{finally } X &= A^{-1}B && \text{which is the solution we seek.} \end{aligned}$$

## 2. The inverse of a $2 \times 2$ matrix

In this section we show how (if it exists) the inverse of a  $2 \times 2$  matrix can be obtained.



Form the matrix products  $AB$  and  $BA$  where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

### Your solution

$$AB =$$

$$BA =$$

$$I(ad-bc) = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = BA \quad I(ad-bc) = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = AB$$

You will see that had we chosen  $C = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  instead of  $B$  then both products  $AC$  and  $CA$  will be equal to  $I$ . Hence this matrix  $C$  is the inverse of  $A$ . Further, if  $ad-bc = 0$  then  $A$  has no inverse. (Note that for the matrix  $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ , which occurred in the last guided exercise,  $ad-bc = 1 \times 0 - 0 \times 2 = 0$  confirming, as we found, that  $A$  has no inverse.)



### Key Point

The Inverse of a  $2 \times 2$  Matrix

If  $ad-bc \neq 0$  then the  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has a (unique) inverse given by

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that  $ad-bc = |A|$ , the determinant of the matrix  $A$ .

In words: to find the inverse of a  $2 \times 2$  matrix  $A$  we effectively interchange the diagonal elements  $a$  and  $d$ , change the sign of the other two elements and then divide by the determinant of  $A$ .



Which of the following matrices has an inverse?

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Your solution**

Therefore,  $A$ ,  $B$  and  $D$  each have an inverse.  $C$  does not because it has a zero determinant.  
 $|A| = 1 \times 3 - 0 \times 2 = 3$ ;  $|B| = 1 + 1 = 2$ ;  $|C| = 2 - 2 = 0$ ;  $|D| = 1 - 0 = 1$ .



Find the inverse of the matrices  $A$ ,  $B$  and  $D$  above.

Use the keypoint result

**Your solution**

$$A^{-1} =$$

$$B^{-1} =$$

$$C^{-1} =$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D$$

It can be shown that the matrix  $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  represents an anti-clockwise rotation through an angle  $\theta$  of points in an  $xy$ -plane about the origin. The inverse matrix  $B$  represents a rotation clockwise through an angle  $\theta$ . It is given therefore by

$$B = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$



Form the products  $AB$  and  $BA$  for these ‘rotation matrices’.

**Your solution**

Effectively: a rotation through an angle  $\theta$  followed by a rotation through angle  $-\theta$  is equivalent to zero rotation ( $\theta = 0$ ).

similarly,  $BA = I$

$$AB = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \cos \theta \sin \theta - \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### 3. The inverse of a $3 \times 3$ matrix - Gauss elimination method

It is true, in general, that if the determinant of a matrix is zero then that matrix has no inverse. If the determinant is non-zero then the matrix has a (unique) inverse. In this section and the next section we look at two ways of finding that inverse using a  $3 \times 3$  matrix; larger matrices can be inverted by the same methods - the process is more tedious and takes longer. The  $2 \times 2$  case could be handled similarly but as we have seen we have a simple formula to use.

The method we now describe for finding the inverse of a matrix has many similarities to a technique (introduced in section 8.3 and known as the Gaussian elimination method) used to obtain solutions of simultaneous equations. This method involves operating on the rows of a matrix in order to reduce it to a unit matrix.

The Gauss row operations we shall use are

- (i) interchanging two rows
- (ii) multiplying a row by a constant factor
- (iii) adding a multiple of one row to another.

Note that in (ii) and (iii) the multiple could be negative or fractional, or both.

The Gauss elimination method is outlined in the following keypoint:



## Key Point

### Matrix inverse – Gauss elimination method

We use the result, quoted without proof, that:

*if a sequence of row operations applied to a square matrix  $A$  reduce it to the identity matrix  $I$  of the same size then the **same** sequence of operations applied to  $I$  reduce it to  $A^{-1}$ .*

Three points to note:

- If we cannot reduce  $A$  to  $I$  then  $A^{-1}$  does not exist. This will become evident by the appearance of a row of zeros.
- There is no unique route from  $A$  to  $I$  and it is experience which selects the optimum route.
- It is more efficient to do the two reductions,  $A$  to  $I$  and  $I$  to  $A^{-1}$ , simultaneously.

Suppose we wish to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}.$$

We first place  $A$  and  $I$  adjacent to each other.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now proceed by changing the columns of  $A$  *left to right* to reduce  $A$  to the form  $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$

where  $*$  can be any number. This form is called an **upper triangular** form.

First we subtract row 1 from row 2 and twice row 1 from row 3. ‘Row’ refers to both matrices.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R2 - R1 \\ R3 - 2R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Now subtract row 2 from row 3

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad R3 - R2 \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Phase two of the task consists of continuing the row operations to reduce the elements above the leading diagonal to zero.

We proceed *right to left*. We subtract 3 times row 3 from row 1 (the elements in row 2 column 3 is already zero.)

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad R1 - 3R3 \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Finally we subtract 3 times row 2 from row 1.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad R1 - 3R2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

Then we claim  $A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ .

(This can be verified by showing that  $AA^{-1}$  and  $A^{-1}A$  both equal  $I$ .)



Consider  $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ ,  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Use the Gauss elimination approach to obtain  $A^{-1}$ .

First interchange rows 1 and 2, then carry out the operation  $\text{row } 3 + \frac{1}{2} \text{ row } 1$

**Your solution**

$$\begin{aligned} & \begin{bmatrix} 1 & \frac{2}{1} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & \frac{2}{2} & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \xRightarrow{R3 + \frac{1}{2}R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \xRightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$





Now carry out the operation  $\text{row } 3 - \frac{7}{2} \text{row } 2$  followed by  $\text{row } 1 - \frac{1}{3} \text{row } 3$  and  $\text{row } 2 + \frac{1}{3} \text{row } 3$ .

**Your solution**

$$\begin{array}{ccc}
 \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \xRightarrow{R3 - \frac{7}{2}R2} & \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
 \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \xRightarrow{R1 - \frac{1}{3}R3, R2 + \frac{1}{3}R3} & \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
 \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \xRightarrow{R3 + \frac{2}{3}R1, R2 + \frac{1}{3}R3} & \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
 \end{array}$$



Next, subtract 3 times row 2 from row 1 then, divide row 1 by 2 and row 3 by  $(-3)$ . Finally identify  $A^{-1}$ .

**Your solution**

$$\begin{aligned}
 & \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R1 - 3R2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R1 \div 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R3 \div (-3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 & \begin{bmatrix} 10 & 2 & 6 \\ 2 & 1 & 6 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R1 - 3R2} \begin{bmatrix} 4 & -1 & -6 \\ 2 & 1 & 6 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R1 \div 2} \begin{bmatrix} 2 & -\frac{1}{2} & -3 \\ 2 & 1 & 6 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R1 - R2} \begin{bmatrix} 0 & \frac{1}{2} & -9 \\ 2 & 1 & 6 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R2 \div 2} \begin{bmatrix} 0 & \frac{1}{2} & -9 \\ 1 & \frac{1}{2} & 3 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{1}{2} & -9 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R3 - 3R1} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{1}{2} & -9 \\ 0 & -\frac{1}{2} & -6 \end{bmatrix} \xrightarrow{R3 + R2} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{1}{2} & -9 \\ 0 & 0 & -15 \end{bmatrix} \xrightarrow{R3 \div (-15)} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{1}{2} & -9 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \div \frac{1}{2}} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 + 18R3} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 - \frac{1}{2}R2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 - 3R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Hence  $A^{-1} = \begin{bmatrix} \frac{5}{9} & \frac{1}{9} & -\frac{2}{3} \\ -\frac{1}{9} & \frac{1}{9} & \frac{1}{3} \\ \frac{7}{9} & -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$

### 4. The inverse of a 3x3 matrix - determinant method

This method which employs determinants, is of importance from a theoretical perspective. The numerical computations involved are too heavy for matrices of higher order than 3 x 3 and in such cases the Gauss elimination approach is preferred.

To obtain A<sup>-1</sup> using the determinant approach the steps in the following keypoint are followed:



#### Key Point

##### Matrix inverse – the determinant method

Given a square matrix A:

- find |A|. If |A| = 0 then, as we know, A<sup>-1</sup> does not exist. If |A| ≠ 0 we can proceed to find the inverse matrix.
- replace each element of A by its cofactor (see section 7.3).
- transpose the result to form the **adjoint matrix**, adj(A)
- then A<sup>-1</sup> =  $\frac{1}{|A|}$  adj(A).



Find the inverse of  $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

First find  $|A|$ .

**Your solution**

$$6 = 2 \times 1 + (1-1) \times 1 + 5 \times 0 = |A|$$

Now replace each element of  $A$  by its minor.

**Your solution**

$$\begin{bmatrix} 5 & -1 & -4 \\ 1 & 1 & -2 \\ 7 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$



Now attach the signs from the array

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

(so that where a + sign is met no action is taken and where a - sign is met the sign is changed) to obtain the matrix of cofactors. Then transpose the result to obtain the adjoint matrix.

**Your solution**

$$\begin{bmatrix} 5 & -1 & 7 \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{bmatrix} \text{ and, transposing, } \text{adj}(A) = \begin{bmatrix} 5 & -1 & 7 \\ -1 & 1 & -1 \\ 7 & -1 & -2 \end{bmatrix}.$$



You can now obtain  $A^{-1}$ .

### Your solution

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{6}{1} \begin{bmatrix} 5 & -1 & 7 \\ -1 & 1 & -1 \\ 7 & -1 & -2 \end{bmatrix} \text{ as before.}$$

### Exercises

1. Find the inverses of the following matrices

(a)  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$     (b)  $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$     (c)  $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

2. Use the determinant method and also the Gauss elimination method to find the inverse of the following matrices

(a)  $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{bmatrix}$     (b)  $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

1. (a)  $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$     (b)  $\begin{bmatrix} 0 & \frac{1}{4} \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$     (c)  $\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

2. (a)  $A^{-1} = -\frac{1}{2} \begin{bmatrix} 0 & -2 & 1 \\ -2 & 4 & 2 \\ 0 & -1 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & -2 & 1 \\ -2 & 4 & 2 \\ 0 & -2 & 0 \end{bmatrix}$

(b)  $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^T$

Answers