Solving simultaneous equations using the inverse matrix

8.2

Introduction

The power of matrix algebra is seen in the representation of a system of simultaneous linear equations as a matrix equation. Matrix algebra allows us to write the solution of the system using the inverse matrix of the coefficients. In practice the method is suitable only for small systems. Its main use is the theoretical insight into such problems which it provides.

Prerequisites

Before starting this Section you should ...

- be familiar with the basic rules of matrix algebra
- be able to evaluate 2×2 and 3×3 determinants
- be able to find the inverse of 2×2 and 3×3 matrices

Learning Outcomes

After completing this Section you should be able to ...

- ✓ use the inverse matrix of coefficients to solve a system of two linear simultaneous equations
- ✓ use the inverse matrix of coefficients to solve a system of three linear simultaneous equations
- ✓ Recognise and identify cases where there is no unique solution

1. Using the inverse matrix on a system of two equations

If we have one linear equation

$$ax = b$$

in which the unknown is x and a and b are constants then there are just three possibilities

- $a \neq 0$ then $x = \frac{b}{a} \equiv a^{-1}b$. The equation $ax = b$ has a *unique solution* for x.
- $a = 0$, $b = 0$ then the equation $ax = b$ becomes $0 = 0$ and any value of x will do. There are *infinitely many solutions* to the equation $ax = b$.
- $a = 0$ and $b \neq 0$ then $ax = b$ becomes $0 = b$ which is a contradiction. In this case the equation $ax = b$ has *no solution* for x.

What happens if we have more than one equation and more than one unknown? In this section we copy the algebraic solution $x = a^{-1}b$ used for a single equation to solve a system of linear equations. As we shall see, this will be a very natural way of solving the system if it is first written in matrix form.
Consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 - 2x_2 &= -1. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \text{which is of the form} \qquad AX = B.$$

If A^{-1} exists then the solution is

$$X = A^{-1}B. \qquad \text{(compare the solution } x = a^{-1}b \text{ above)}$$

Given the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ find its determinant. What does this tell you about A^{-1}?

Your solution

$|A| = 2 \times (-2) - 1 \times 3 = -7$

since $|A| \neq 0$ then A^{-1} exists.

Now find A^{-1}

Your solution

$$A^{-1} = \frac{1}{(-7)} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

To solve the system $AX = B$ we use $X = A^{-1}B$

Solve the system $AX = B$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ and B is

(i) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Your solution

(i) $X = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $x_1 = 1$, $x_2 = 1$.

(ii) $X = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence $x_1 = 2$, $x_2 = -1$.

(iii) $X = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence $x_1 = 0$, $x_2 = 0$, as might have been expected.

2. Non-unique solutions

The key to obtaining a unique solution of the system $AX = B$ is to find A^{-1}. What happens when A^{-1} does not exist?
Consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \qquad \text{(i)} \\ 4x_1 + 6x_2 &= 10. \qquad \text{(ii)} \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Identify the matrix A and hence find A^{-1}.

Your solution

$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ and $|A| = 2 \times 6 - 4 \times 3 = 0$. Hence A^{-1} does not exist.

Looking at the original system we see that equation (ii) is simply equation (i) with all coefficients doubled. In effect we have only one equation for the two unknowns x_1 and x_2. The equations are **consistent** but have infinitely many solutions.
If we let x_2 assume a **particular** value, t say, then x_1 must take the value $x_1 = \frac{1}{2}(5-3t)$ i.e. the solution to the given equations is:

$$x_2 = t, \quad x_1 = \frac{1}{2}(5-3t)$$

For each value of t there are values for x_1 and x_2 which satisfy the original system of equations. For example, if $t = 1$, then $x_2 = 1$, $x_1 = 1$, if $t = -3$ then $x_2 = -3$, $x_1 = 7$ and so on.
Now consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \qquad \text{(i)} \\ 4x_1 + 6x_2 &= 9 \qquad \text{(ii)} \end{aligned}$$

Since the left-hand sides are the same as those in the previous system then A is the same and again A^{-1} does not exist. There is no unique solution to the equations (i) and (ii).
However, if we double equation (i) we obtain

$$4x_1 + 6x_2 = 10,$$

which conflicts with equation (ii). There are thus no solutions to (i) and (ii) and the equations are said to be **inconsistent**.

What can you conclude about the solutions of the systems

$$\text{(i)} \quad \begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 - 6x_2 &= 3 \end{aligned} \qquad\qquad \text{(ii)} \quad \begin{aligned} 3x_1 + 2x_2 &= 7 \\ -6x_1 - 4x_2 &= 5 \end{aligned}$$

First write the systems in matrix form and find $|A|$.

Your solution

(i) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad |A| = -6 + 6 = 0;$

(ii) $\begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad |A| = -12 + 12 = 0.$

Now compare the equations in each system.

Your solution

(i) The second equation is 3 times the first equation. There are infinitely many solutions of the form
$x_2 = t$, $x_1 = 1 + 2t$ where t is arbitrary.
(ii) If we multiply the first equation by (-2) we obtain $-6x_1 - 4x_2 = -14$ which is in conflict with the second equation. The equations are inconsistent and have no solution.

3. Three equations in three unknowns

It is much more tedious to use the inverse matrix to solve a system of three equations although in principle, the method is the same as for two equations.
Consider the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12. \end{aligned}$$

We met this system in section 8.1 where we found that

$$|A| = 10.$$

Hence A^{-1} exists.

Find A^{-1} by the method of determinants. First form the matrix where each element of A is replaced by its minor.

Your solution

$$\begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{bmatrix}.$$

Now use the 3×3 array of signs to obtain the matrix of cofactors.

Your solution

The array of signs is $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ so that we obtain $\begin{bmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{bmatrix}$.

Now transpose this matrix and divide by $|A|$ to obtain A^{-1}.

Your solution

Transposing gives $\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$. Finally, $A^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$.

Now use $X = A^{-1}B$ to solve the system of linear equations.

Your solution

$X = \frac{1}{10}\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ Then $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

Comparing this approach to the use of Cramer's rule for three equations in section 2 of section 8.1 we can say that the two methods are both rather tedious.

Equations with no unique solution

If $|A| = 0$, A^{-1} does not exist and therefore, early on, it is easy to see that the system of equations has no unique solution. But it is not obvious whether this is because the equations are inconsistent or whether they have infinitely many solutions.

Consider the systems

$$\text{(i)}\quad \begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 7 \end{aligned} \qquad\qquad \text{(ii)}\quad \begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ x_1 - 11x_2 + 9x_3 &= 13 \end{aligned}$$

In system (i) add the first equation to the second. What does this tell you about the system?

Your solution

The sum is $3x_1 + 2x_2 - x_3 = 7$, which is identical to the third equation. Thus, essentially, there are only two equations $x_1 - x_2 + x_3 = 4$ and $3x_1 + 2x_2 - x_3 = 7$. Now adding these two gives $4x_1 + x_2 = 11$ or $x_2 = 11 - 4x_1$ and then

$$x_3 = 4 - x_1 + x_2 = 4 - x_1 + 11 - 4x_1 = 15 - 5x_1$$

Hence if we give x_1 a value, t say, then $x_2 = 11 - 4t$ and $x_3 = 15 - 5t$. Thus there is an infinite number of solutions (one for each value of t).

In system (ii) take the combination 5 times the first equation minus 2 times the second equation. What does this tell you about the system?

Your solution

The combination is $x_1 - 11x_2 + 9x_3 = 14$, which conflicts with the third equation. There are thus no solutions.

In practice, systems containing three or more linear equations are best solved by the method which we shall introduce in section 8.3.

Exercises

1. Solve the following using the inverse matrix approach:

(a)
$$\begin{array}{rcrcl} 2x & - & 3y & = & 1 \\ 4x & + & 4y & = & 2 \end{array}$$

(b)
$$\begin{array}{rcrcl} 2x & - & 5y & = & 2 \\ -4x & + & 10y & = & 1 \end{array}$$

(c)
$$\begin{array}{rcrcl} 6x & - & y & = & 0 \\ 2x & - & 4y & = & 1 \end{array}$$

2. Solve the following equations using matrix methods:

(a)
$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$$

(b)
$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

Answers 1. (a) $x = \frac{1}{2}$, $y = 0$ (b) A^{-1} does not exist. (c) $x = -\frac{1}{22}$, $y = -\frac{3}{11}$

2. (a) $x_1 = \frac{8}{3}$, $x_2 = -4$, $x_3 = \frac{4}{3}$ (b) $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$