Cartesian Components of Vectors

9.2

Introduction

It is useful to be able to describe vectors with reference to specific coordinate systems, such as the cartesian coordinate system. So, in this section, we show how this is possible by defining unit vectors in the directions of the x and y axes. Any other vector in the xy plane can then be represented as a combination of these *basis vectors*. The idea is then extended to three dimensional vectors. This is useful because most engineering problems arise in 3D situations.

Prerequisites

Before starting this Section you should ...

① distinguish between a vector and a scalar

② represent a vector as a directed line segment

③ use the cartesian coordinate system

Learning Outcomes

After completing this Section you should be able to ...

✓ explain the meaning of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$

✓ express two and three dimensional vectors in cartesian form

✓ find a ‘position vector’

✓ find the modulus of a vector expressed in Cartesian form

1. Two-dimensional Coordinate frames

Figure 1 shows a two-dimensional coordinate frame. Any point P in the plane of the figure can be defined in terms of its x and y coordinates.

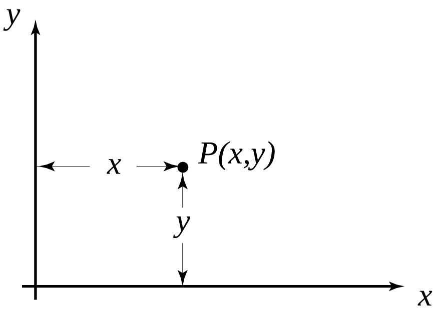


Figure 1.

A unit vector pointing in the positive direction of the x-axis is denoted by $\underline{i}$. (Note that it is common practice to write this particular unit vector without the hat ˆ). It follows that any vector in the direction of the x-axis will be a multiple of $\underline{i}$. Figure 2 shows vectors $\underline{i}$, $2\underline{i}$, $5\underline{i}$ and $-3\underline{i}$. In general a vector of length ℓ in the direction of the x-axis can be written $\ell\underline{i}$

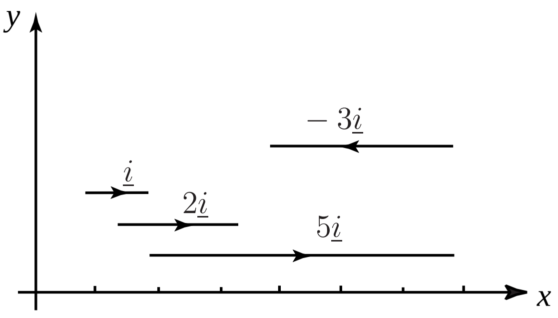


Figure 2. All these vectors are multiples of $\underline{i}$.

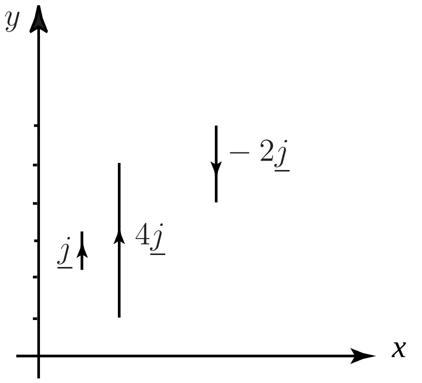


Figure 3. All these vectors are multiples of $\underline{j}$.

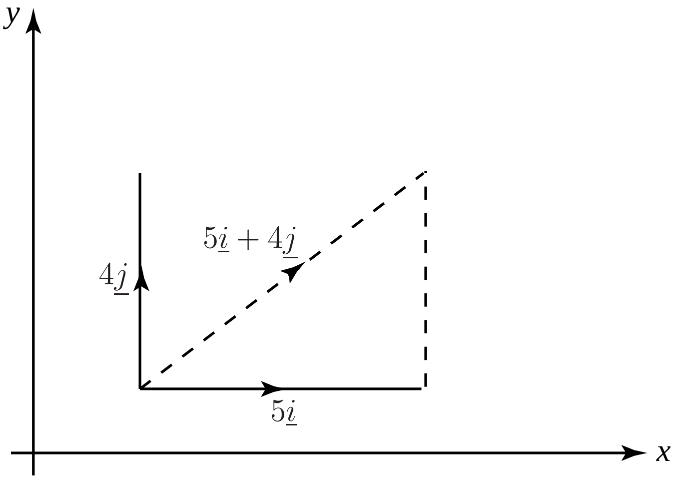
Similarly, a unit vector pointing in the positive y-axis is denoted by $\underline{j}$. Then, any vector in the direction of the y-axis will be a multiple of $\underline{j}$. Figure 3 shows $\underline{j}$, $4\underline{j}$ and $-2\underline{j}$. In general a vector of length ℓ in the direction of the y-axis can be written $\ell\underline{j}$.

Key Point

$\underset{\sim}{i}$ represents a unit vector in the direction of the positive x-axis
$\underset{\sim}{j}$ represents a unit vector in the direction of the positive y-axis

Example Draw the vectors $5\underset{\sim}{i}$ and $4\underset{\sim}{j}$. Use your diagram and the triangle law of addition to add these two vectors together. First draw the vectors $5\underset{\sim}{i}$ and $4\underset{\sim}{j}$. Then, by translating the vectors so that they lie head to tail, find the vector sum $5\underset{\sim}{i} + 4\underset{\sim}{j}$.

Solution



We now generalise the situation in the previous Example. Consider Figure 4.
It shows a vector $\underset{\sim}{r} = \overrightarrow{AB}$. We can regard $\underset{\sim}{r}$ as being the resultant of the two vectors $\overrightarrow{AC} = a\underset{\sim}{i}$, and $\overrightarrow{CB} = b\underset{\sim}{j}$. From the triangle law of vector addition

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB}\\ &= a\underset{\sim}{i} + b\underset{\sim}{j}\end{aligned}$$

We conclude that any vector in the xy plane can be expressed in the form $\underset{\sim}{r} = a\underset{\sim}{i} + b\underset{\sim}{j}$. The numbers a and b are called the **components** of $\underset{\sim}{r}$ in the x and y directions. Sometimes, for emphasis, we will use a_x and a_y instead of a and b to denote the components in the x- and y-directions respectively. In that case we would write $\underset{\sim}{r} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$.

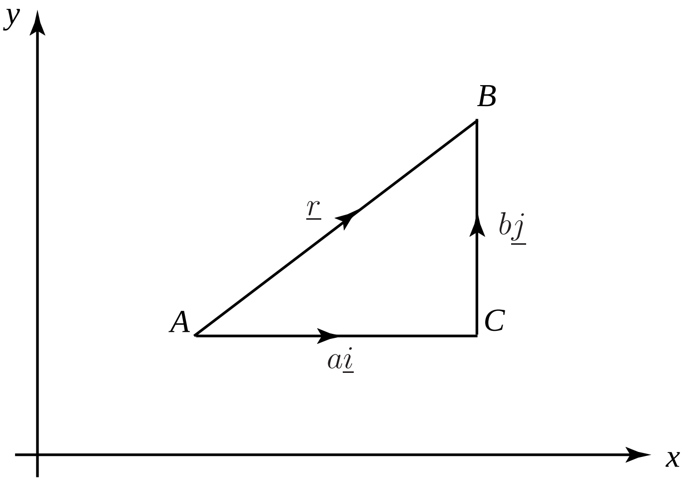


Figure 4. $\underline{AB} = \underline{AC} + \underline{CB}$ by the triangle law.

Column vector notation

An alternative, useful, and often briefer notation is to write the vector $\underline{r} = a\underline{i} + b\underline{j}$ in **column vector** notation as

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

a) Draw an xy plane and show the vectors $\underline{p} = 2\underline{i} + 3\underline{j}$, and $\underline{q} = 5\underline{i} + \underline{j}$.
b) Express $\underline{p}$ and $\underline{q}$ using column vector notation.
c) By translating one of the vectors apply the triangle law to show the sum $\underline{p} + \underline{q}$.
d) Express the resultant $\underline{p} + \underline{q}$ in terms of $\underline{i}$ and $\underline{j}$.

Your solution

a) Draw the xy plane and the required vectors. They can be drawn from any point in the plane.

Your solution

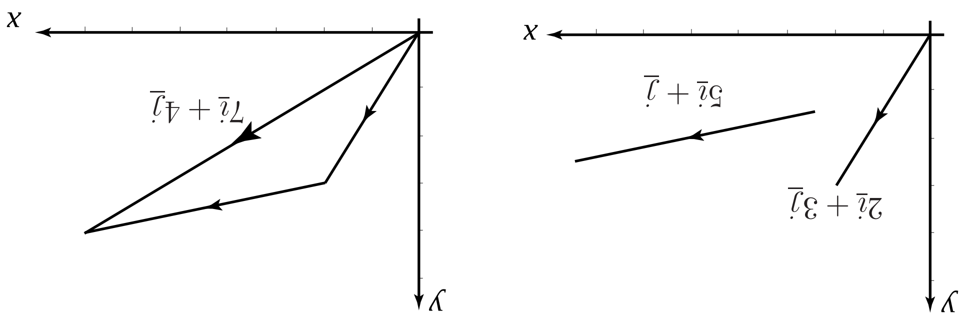
b) The column vector form of $\underline{p}$ is $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$. Write down the column vector form of $\underline{q}$.

b) $\underline{p} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ $\underline{q} = \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right)$

Your solution

c) Translate one of the vectors in part a) so that they lie head to tail, completing the third side of the triangle to give the resultant $\underline{p}+\underline{q}$.

c) Note that the vectors have not been drawn to scale.



Your solution

d) By studying your diagram note that the resultant has two components $7\underline{i}$, horizontally, and $4\underline{j}$ vertically. Hence write down an expression for $\underline{p}+\underline{q}$.

d) $7\underline{i}+4\underline{j}$

It is very important to note from the last example that when given vectors in cartesian form they can be added by simply adding their respective $\underline{i}$ and $\underline{j}$ components.
Thus, if $\underline{a}=a_x\underline{i}+a_y\underline{j}$ and $\underline{b}=b_x\underline{i}+b_y\underline{j}$ then

$$\underline{a}+\underline{b}=(a_x+b_x)\underline{i}+(a_y+b_y)\underline{j}$$

A similar and obvious rule applies when subtracting viz: $\underline{a}-\underline{b}=(a_x-b_x)\underline{i}+(a_y-b_y)\underline{j}$.

If $\underline{a}=9\underline{i}+7\underline{j}$ and $\underline{b}=8\underline{i}+3\underline{j}$ find a) $\underline{a}+\underline{b}$ b) $\underline{a}-\underline{b}$

Your solution

a) Simply add the respective components: $17\underline{i}+10\underline{j}$, b) Simply subtract the respective components: $\underline{i}+4\underline{j}$

Now consider the special case when $\underline{r}$ represents the vector from the origin to the point $P(a,b)$. This vector is known as the **position vector** of P and is shown in Figure 5.

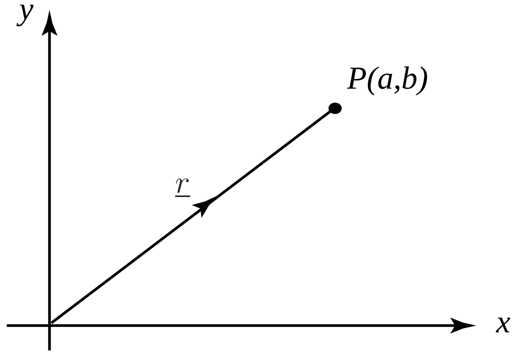


Figure 5.

Key Point

the position vector of P with coordinates (a,b) is $\underline{r}=\overrightarrow{OP}=a\underline{i}+b\underline{j}$

Unlike most vectors, position vectors cannot be freely translated. Because they indicate the position of a point they are fixed vectors in the sense that the tail of a position vector is always located at the origin as shown.

Example State the position vectors of the points with coordinates a) $P(2,4)$, b) $Q(-1,5)$,
c) $R(-1,-7)$, d) $S(8,-4)$.

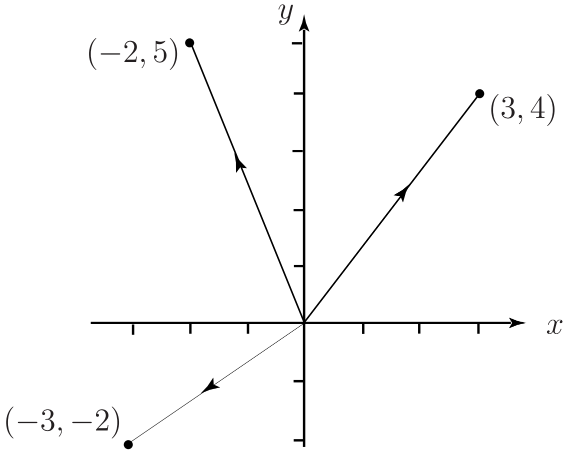
Solution

a) The position vector of P is $2\underline{i}+4\underline{j}$. b) The position vector of Q is $-\underline{i}+5\underline{j}$. c) The position vector of R is $-\underline{i}-7\underline{j}$. d) The position vector of S is $8\underline{i}-4\underline{j}$.

Example Sketch the position vectors $\underline{r}_1 = 3\underline{i} + 4\underline{j}$, $\underline{r}_2 = -2\underline{i} + 5\underline{j}$, $\underline{r}_3 = -3\underline{i} - 2\underline{j}$.

Solution

The vectors are shown below. Note that all position vectors start at the origin.



The **modulus** of any vector $\underline{r}$ is equal to its length. As we have noted earlier the modulus of $\underline{r}$ is usually denoted by $|\underline{r}|$. When $\underline{r} = a\underline{i} + b\underline{j}$ the modulus can be obtained using Pythagoras' theorem. If $\underline{r}$ is the position vector of point P then the modulus is clearly the distance of P from the origin.

Key Point

if $\underline{r} = a\underline{i} + b\underline{j}$ then $|\underline{r}| = \sqrt{a^2 + b^2}$

Example Find the modulus of each of the vectors shown in the previous example.

Solution

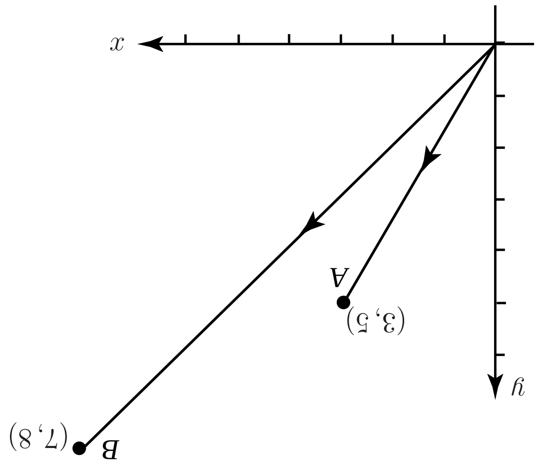
a) The modulus of $\underline{r}_1 = |3\underline{i} + 4\underline{j}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. b) The modulus of $\underline{r}_2 = |-2\underline{i} + 5\underline{j}|$ $= \sqrt{(-2)^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$. c) Similarly $|\underline{r}_3| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

Point A has coordinates $(3, 5)$. Point B has coordinates $(7, 8)$.
a) Depict these points on a diagram.
b) State the position vectors of A and B.
c) Find an expression for $\overrightarrow{AB}$.
d) Find $|\overrightarrow{AB}|$.

Your solution

a) Draw a diagram which shows points A and B.

a)



Your solution

b) State the position vectors of A and B

$\overrightarrow{OA} = \underline{a} = 3\underline{i} + 5\underline{j}$, $\overrightarrow{OB} = \underline{b} = 7\underline{i} + 8\underline{j}$

Your solution

c) Referring to your figure and using the triangle law you can write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ so that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Hence write down an expression for $\overrightarrow{AB}$ in terms of the unit vectors $\underline{i}$ and $\underline{j}$.

$\overrightarrow{AB} = (7\underline{i} + 8\underline{j}) - (3\underline{i} + 5\underline{j}) = 4\underline{i} + 3\underline{j}$

Your solution

d) The length of $\overrightarrow{AB} = |4\underline{i} + 3\underline{j}| =$

$|\overrightarrow{AB}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Exercises

1. Explain the distinction between a position vector, and a more general, or free vector.
2. What is meant by the symbols $\underline{i}$ and $\underline{j}$?
3. State the position vectors of the points with coordinates a) $P(4,7)$, b) $Q(-3,5)$, c) $R(0,3)$, d) $S(-1,0)$
4. State the coordinates of the point P if its position vector is a) $3\underline{i} - 7\underline{j}$, b) $-4\underline{i}$, c) $-0.5\underline{i} + 13\underline{j}$, d) $a\underline{i} + b\underline{j}$
5. Find the modulus of each of the following vectors.
   a) $\underline{r} = 7\underline{i} + 3\underline{j}$, b) $\underline{r} = 17\underline{i}$, c) $\underline{r} = 2\underline{i} - 3\underline{j}$, d) $\underline{r} = -3\underline{j}$, e) $\underline{r} = a\underline{i} + b\underline{j}$, f) $\underline{r} = a\underline{i} - b\underline{j}$
6. Point P has coordinates (7,8). Point Q has coordinates $(-2,4)$.
   a) Draw a sketch showing P and Q
   b) State the position vectors of P and Q.
   c) Find an expression for $\overrightarrow{PQ}$
   d) Find $|\overrightarrow{PQ}|$

Answers 1. Free vectors may be translated provided their direction and length remain unchanged. Position vectors must always start at the origin. 2. $\underline{i}$ is a unit vector in the direction of the positive x-axis. $\underline{j}$ is a unit vector in the direction of the positive y-axis. 3. a) $4\underline{i} + 7\underline{j}$, b) $-3\underline{i} + 5\underline{j}$, c) $3\underline{j}$, d) $-\underline{i}$. 4. a) $(3,-7)$, b) $(-4,0)$, c) $(-0.5,13)$, d) (a,b) 5. a) $\sqrt{58}$, b) 17, c) $\sqrt{13}$, d) 3, e) $\sqrt{a^2+b^2}$, f) $\sqrt{a^2+b^2}$. 6. $\underline{p} = 7\underline{i} + 8\underline{j}$. $\underline{q} = -2\underline{i} + 4\underline{j}$, $\overrightarrow{PQ} = -9\underline{i} - 4\underline{j}$. $|\overrightarrow{PQ}| = \sqrt{97}$.

2. Three-dimensional Coordinate Frames

The real world is three-dimensional and in order to solve many engineering problems it is necessary to develop expertise in the mathematics of three-dimensional space. An important application of vectors is their use to locate points in three dimensions. When two distinct points are known we can draw a unique straight line between them. Three distinct points which do not lie on the same line form a unique plane. Vectors can be used to describe points, lines, and planes in three dimensions. These mathematical foundations underpin much of the technology associated with computer graphics and the control of robots. In this section we shall introduce the vector methods which underlie these applications.

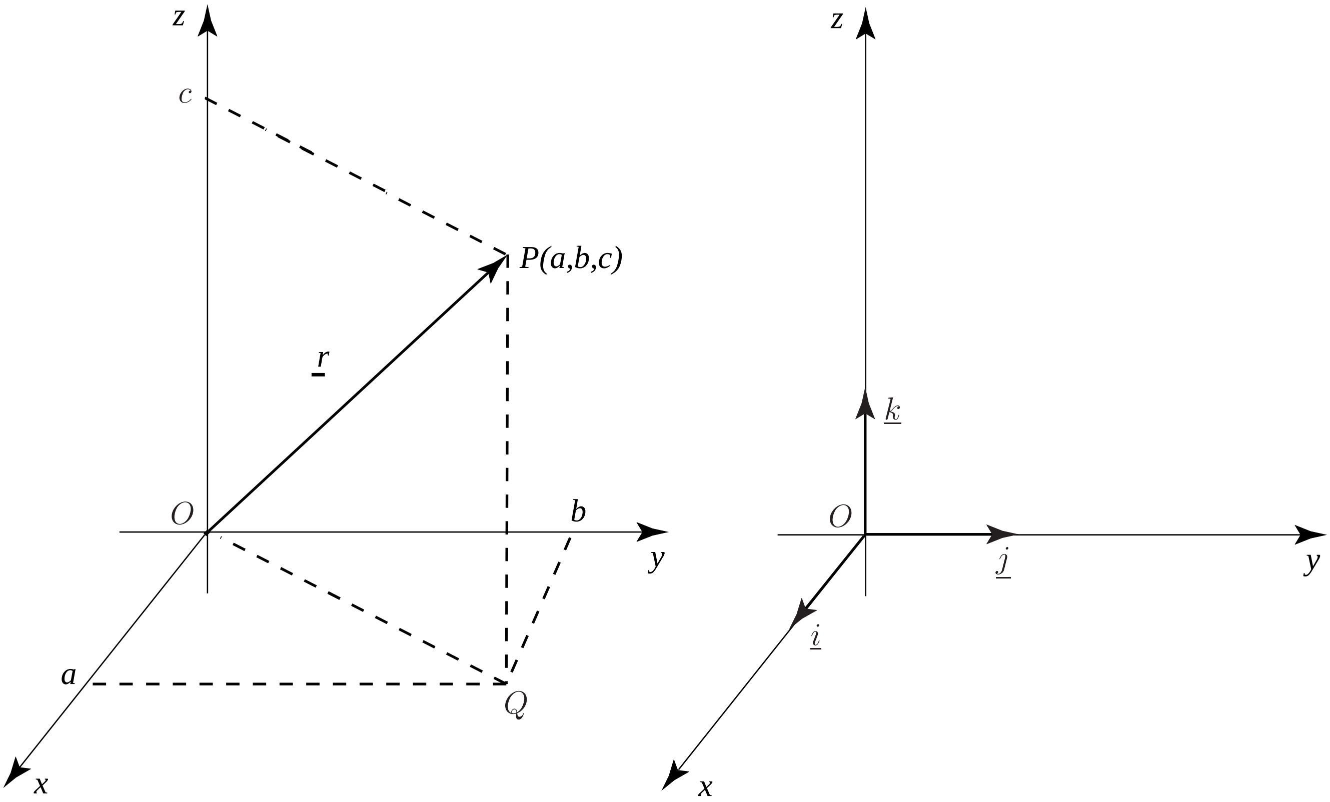


Figure 6.

Figure 6 shows a three-dimensional coordinate frame. Note that the third dimension requires the addition of a third axis, the z-axis. Whilst these three axes are drawn in the plane of the paper you should remember that we are now thinking of three-dimensional situations. Just as in two-dimensions the x and y axes are perpendicular, in three dimensions the x,y and z axes are all perpendicular to each other. We say they are **mutually perpendicular.** There is no reason why we could not have chosen the z-axis in the opposite direction. However it is conventional to choose the directions shown in Figure 6. Any point in the three dimensions can be defined in terms of its x, y and z coordinates. Consider the point P with coordinates (a, b, c) as shown. The vector from the origin to the point P is known as the **position vector** of P, $\overrightarrow{OP}$ or $\underline{r}$. To arrive at P from O we can think of moving a units in the x direction, b units in the y direction and c units in the z direction.

A unit vector pointing in the positive direction of the z-axis is denoted by $\underline{k}$. See the right-hand diagram of Figure 6. Noting that $\overrightarrow{OQ} = a\underline{i} + b\underline{j}$ and that $\overrightarrow{QP} = c\underline{k}$ we can state

$$\begin{aligned} \underline{r} = \overrightarrow{OP} &= \overrightarrow{OQ} + \overrightarrow{QP} \\ &= a\underline{i} + b\underline{j} + c\underline{k} \end{aligned}$$

We conclude that the position vector of the point with coordinates (a, b, c) is $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$. (We might, for convenience, sometimes use a subscript notation. For example we might refer to the position vector $\underline{r}$ as $\underline{r} = r_x\underline{i} + r_y\underline{j} + r_z\underline{k}$ in which (r_x, r_y, r_z) have taken the place of (a, b, c).)

Key Point

If P has coordinates (a, b, c) then its position vector is

$$\underline{r} = \overrightarrow{OP} = a\underline{i} + b\underline{j} + c\underline{k}$$

State the position vector of the point with coordinates $(9, -8, 6)$.

Your solution

$$6\underline{i} - 8\underline{j} + 9\underline{k}$$

The modulus of the vector $\overrightarrow{OP}$ is equal to the distance OP which can be obtained by Pythagoras' theorem. This is easily shown to be

Key Point

If $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ then

$$|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$$

Find the modulus of the vector $\underline{r} = 4\underline{i} + 2\underline{j} + 3\underline{k}$.

Your solution

$|\underline{r}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{16 + 4 + 9} = \sqrt{29}$

Example Points A, B and C have coordinates $(-1, 1, 4)$, $(8, 0, 2)$ and $(5, -2, 11)$ respectively.

a) Find the position vectors of A, B and C.

b) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

c) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

Solution

a) Denoting the position vectors of A, B and C by $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively, we find

$$\underline{a} = -\underline{i} + \underline{j} + 4\underline{k}, \quad \underline{b} = 8\underline{i} + 2\underline{k}, \quad \underline{c} = 5\underline{i} - 2\underline{j} + 11\underline{k}$$

b) $\overrightarrow{AB} = \underline{b} - \underline{a} = 9\underline{i} - \underline{j} - 2\underline{k}$. $\overrightarrow{BC} = \underline{c} - \underline{b} = -3\underline{i} - 2\underline{j} + 9\underline{k}$. c) $|\overrightarrow{AB}| = \sqrt{9^2 + (-1)^2 + (-2)^2} = \sqrt{86}$. $|\overrightarrow{BC}| = \sqrt{(-3)^2 + (-2)^2 + 9^2} = \sqrt{94}$.

Exercises

1. State the position vector of the point with coordinates $(4, -4, 3)$.
2. Find the modulus of each of the following vectors.
   a) $7\underline{i} + 2\underline{j} + 3\underline{k}$, b) $7\underline{i} - 2\underline{j} + 3\underline{k}$, c) $2\underline{j} + 8\underline{k}$, d) $-\underline{i} - 2\underline{j} + 3\underline{k}$, e) $a\underline{i} + b\underline{j} + c\underline{k}$,
3. Points P, Q and R have coordinates (9,1,0), (8,-3,5), and (5,5,7) respectively.
   a) Find the position vectors of P, Q and R.
   b) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$
   c) Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QR}|$.

Answers 1. $4\underline{i} - 4\underline{j} + 3\underline{k}$ 2. a) $\sqrt{62}$, b) $\sqrt{62}$, c) $\sqrt{68}$, d) $\sqrt{14}$, e) $\sqrt{a^2 + b^2 + c^2}$. 3. a) $\underline{p} = 9\underline{i} + \underline{j}$, $\underline{q} = 8\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{r} = 5\underline{i} + 5\underline{j} + 7\underline{k}$. b) $\overrightarrow{PQ} = -\underline{i} - 4\underline{j} + 5\underline{k}$, $\overrightarrow{QR} = -3\underline{i} + 8\underline{j} + 2\underline{k}$ c) $|\overrightarrow{PQ}| = \sqrt{42}$. $|\overrightarrow{QR}| = \sqrt{77}$