The Scalar Product

9.3

Introduction

There are two kinds of multiplication involving vectors. The first is known as the **scalar product** or **dot product**. This is so-called because when the scalar product of two vectors is calculated the result is a scalar. The second product is known as the **vector product**. When this is calculated the result is a vector. The definitions of these products may seem rather strange at first, but they are widely used in applications. In this section we consider only the scalar product.

Prerequisites

Before starting this Section you should ...

① that a vector can be represented as a directed line segment

② how to express a vector in cartesian form

③ how to find the modulus of a vector

Learning Outcomes

After completing this Section you should be able to ...

✓ calculate, from its definition, the scalar product of two given vectors

✓ calculate the scalar product of two vectors given in cartesian form

✓ use the scalar product to find the angle between two vectors

✓ use the scalar product to test whether two vectors are perpendicular

1. Definition of the Scalar Product

Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 1.

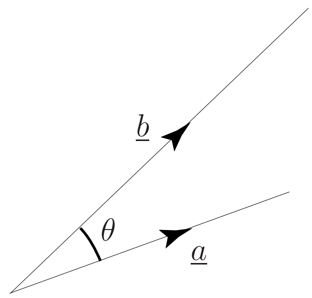


Figure 1. Two vectors subtend an angle θ.

Note that the tails of the two vectors coincide and that the angle between the vectors has been labelled θ. Their scalar product, denoted by $\underline{a}\cdot\underline{b}$, is defined as the product $|\underline{a}|\,|\underline{b}|\,\cos\theta$. It is very important to use the dot in the formula. The dot is the specific symbol for the scalar product, and is the reason why the scalar product is also known as the **dot product**. You should not use a $\times$ sign in this context because this sign is reserved for the vector product which is quite different.
The angle θ is always chosen to lie between 0 and π, and the tails of the two vectors should coincide. Figure 2 shows two incorrect ways of measuring θ.

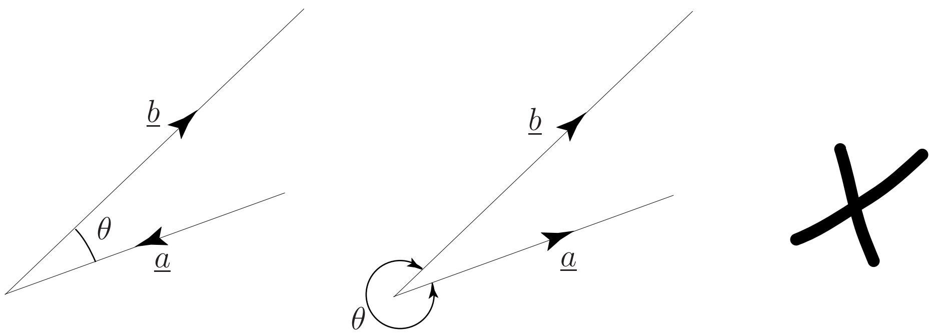


Figure 2. θ should not be measured in these ways.

Key Point

scalar product : $\quad \underline{a}\cdot\underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$

We can remember this formula as:

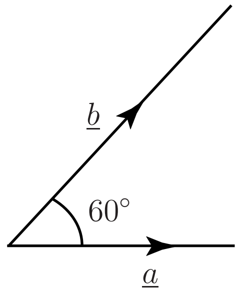
> "the modulus of the first vector, multiplied by the modulus of the second vector, multiplied by the cosine of the angle between them."

Clearly $\underline{b} \cdot \underline{a} = |\underline{b}|\,|\underline{a}|\,\cos\theta$ and so

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}.$$

Thus we can evaluate a scalar product in any order: the operation is said to be **commutative**.

Example Vectors $\underline{a}$ and $\underline{b}$ are shown in the figure below. Vector $\underline{a}$ has modulus 6 and vector $\underline{b}$ has modulus 7 and the angle between them is 60°. Calculate $\underline{a}.\underline{b}$.



Solution

The angle between the two vectors is 60°. Hence

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta = (6)(7)\cos 60^\circ = 21$$

The scalar product of $\underline{a}$ and $\underline{b}$ is equal to 21. Note that when finding a scalar product the result is always a scalar.

Example Find $\underline{i} \cdot \underline{i}$ where $\underline{i}$ is the unit vector in the direction of the positive x axis.

Solution

Because $\underline{i}$ is a unit vector its modulus is 1. Also, the angle between $\underline{i}$ and itself is zero. Therefore

$$\underline{i}.\underline{i} = (1)(1)\cos 0^\circ = 1$$

So the scalar product of $\underline{i}$ with itself equals 1. It is easy to verify that $\underline{j}.\underline{j} = 1$ and $\underline{k}.\underline{k} = 1$.

Example Find $\underline{i} \cdot \underline{j}$ where $\underline{i}$ and $\underline{j}$ are unit vectors in the directions of the x and y axes.

Solution

Because $\underline{i}$ and $\underline{j}$ are unit vectors they each have a modulus of 1. The angle between the two vectors is 90°. Therefore

$$\underline{i} \cdot \underline{j} = (1)(1)\cos 90^\circ = 0$$

That is $\underline{i} \cdot \underline{j} = 0$.

More generally, the following results are easily verified:

Key Point

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$$
$$\underline{i} \cdot \underline{j} = \underline{i} \cdot \underline{k} = \underline{j} \cdot \underline{k} = 0$$

Even more generally, whenever any two vectors are perpendicular to each other their scalar product is zero because the angle between the vectors is 90° and $\cos 90^\circ = 0$.

Key Point

The scalar product of perpendicular vectors is zero.

2. A Formula for Finding the Scalar Product

We can use the previous results to obtain a formula for finding a scalar product when the vectors are given in cartesian form. We consider vectors in the xy plane. Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$. Then

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_1\underline{i} + a_2\underline{j}) \cdot (b_1\underline{i} + b_2\underline{j}) \\ &= a_1\underline{i} \cdot (b_1\underline{i} + b_2\underline{j}) + a_2\underline{j} \cdot (b_1\underline{i} + b_2\underline{j}) \\ &= a_1b_1\underline{i} \cdot \underline{i} + a_1b_2\underline{i} \cdot \underline{j} + a_2b_1\underline{j} \cdot \underline{i} + a_2b_2\underline{j} \cdot \underline{j} \end{aligned}$$

Now, using the previous boxed results we can simplify this to give the following formula:

Key Point

if $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$ then

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2$$

Thus to find the scalar product of two vectors their $\underline{i}$ components are multiplied together, their $\underline{j}$ components are multiplied together and the results are added.

Example If $\underline{a} = 7\underline{i} + 8\underline{j}$ and $\underline{b} = 5\underline{i} - 2\underline{j}$, find the scalar product $\underline{a} \cdot \underline{b}$.

Solution

We use the previous boxed formula and multiply corresponding components together, adding the results.

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (7\underline{i} + 8\underline{j}) \cdot (5\underline{i} - 2\underline{j}) \\ &= (7)(5) + (8)(-2) \\ &= 35 - 16 \\ &= 19 \end{aligned}$$

The formula readily generalises to vectors in three dimensions as follows:

Key Point

if $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example If $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$, find $\underline{a} \cdot \underline{b}$.

Solution

Corresponding components are multiplied together and the results are added.

$$\underline{a} \cdot \underline{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar: there are no $\underline{i}$'s, $\underline{j}$'s, or $\underline{k}$'s in the answer.

If $\underline{p} = 4\underline{i} - 3\underline{j} + 7\underline{k}$ and $\underline{q} = 6\underline{i} - \underline{j} + 2\underline{k}$, find $\underline{p} \cdot \underline{q}$.

Your solution

Corresponding components are multiplied together and the results are added.

41

If $\underline{r} = 3\underline{i} + 2\underline{j} + 9\underline{k}$ find $\underline{r} \cdot \underline{r}$. Show that this is the same as $|\underline{r}|^2$.

Your solution

94. $|\underline{r}| = \sqrt{9 + 4 + 81} = \sqrt{94}$ hence $|\underline{r}|^2 = \underline{r} \cdot \underline{r}$.

Key Point

For any vector $\underline{r}$ we have $|\underline{r}|^2 = \underline{r} \cdot \underline{r}$

3. Resolving One Vector Along Another

The scalar product can be used to find the component of a vector in the direction of another vector. Consider Figure 3 which shows two arbitrary vectors $\underline{a}$ and $\underline{n}$. Let $\hat{\underline{n}}$ be a *unit vector* in the direction of $\underline{n}$.

Study the figure carefully and note that a perpendicular has been drawn from P to meet $\underline{n}$ at Q. The distance OQ is called the **projection** of $\underline{a}$ onto $\underline{n}$. Simple trigonometry tells us that the length of the projection is $|\underline{a}| \cos\theta$. Now by taking the scalar product of $\underline{a}$ with the unit vector $\hat{\underline{n}}$ we find

$$\underline{a} \cdot \hat{\underline{n}} = |\underline{a}|\,|\hat{\underline{n}}|\,\cos\theta = |\underline{a}|\cos\theta \qquad \text{since } |\hat{\underline{n}}| = 1$$

We conclude that

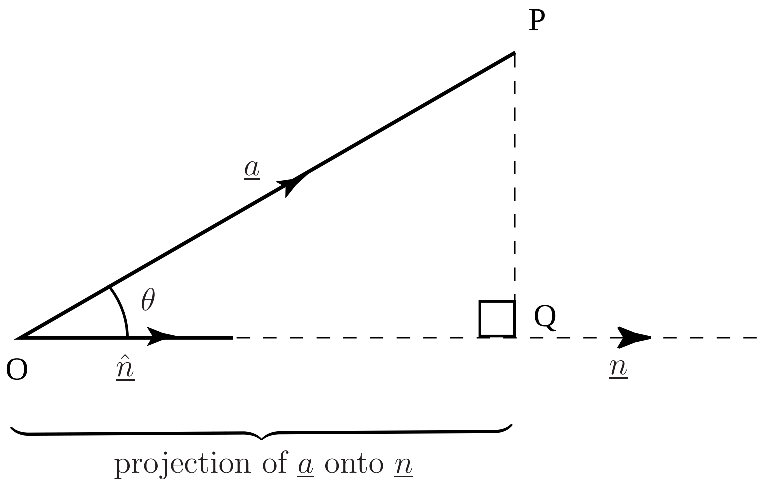
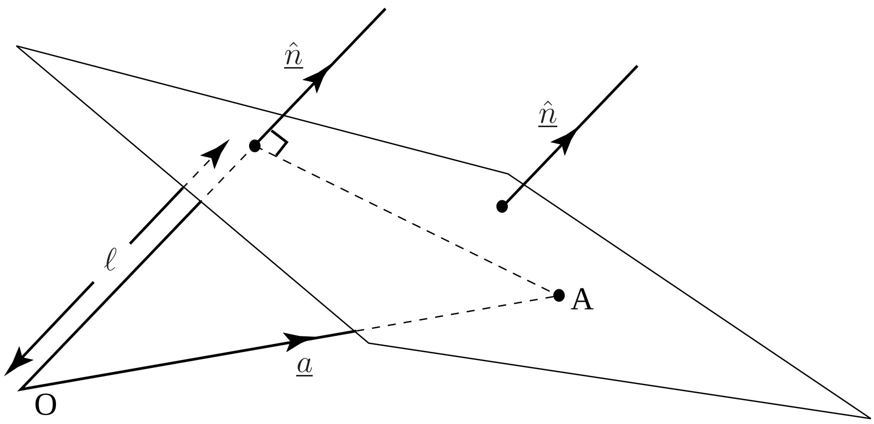


Figure 3.

Key Point

$\underline{a} \cdot \hat{\underline{n}}$ is the component of $\underline{a}$ in the direction of $\underline{n}$

Example The figure below shows a plane containing the point A with position vector $\underline{a}$. The vector $\hat{\underline{n}}$ is a unit vector perpendicular to the plane (such a vector is called a **normal** vector). Find an expression for the perpendicular distance of the plane from the origin.



Solution

From the diagram note that the perpendicular distance ℓ of the plane from the origin is the projection of $\underline{a}$ onto $\hat{\underline{n}}$ and is thus $\underline{a} \cdot \hat{\underline{n}}$.

4. Using the Scalar Product to Find the Angle Between Two Vectors

We have two distinct ways of calculating the scalar product of two vectors. From the first Key Point of this section $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$ whilst from the last Key Point of Section 2 we have $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$. Both methods of calculating the scalar product are entirely equivalent and will always give the same value for the scalar product. We can exploit this correspondence to find the angle between two vectors. The following example illustrates the procedure to be followed.

Example Find the angle between the vectors $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$.

Solution

The scalar product of these two vectors has already been found in the example on page 5 to be -9. The modulus of $\underline{a}$ is $\sqrt{5^2 + 3^2 + (-2)^2} = \sqrt{38}$. The modulus of $\underline{b}$ is $\sqrt{8^2 + (-9)^2 + 11^2} = \sqrt{266}$. Substituting these into the formula for the scalar product we find

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}|\,\cos\theta \\ -9 &= \sqrt{38}\,\sqrt{266}\cos\theta \end{aligned}$$

from which

$$\cos\theta = \frac{-9}{\sqrt{38}\sqrt{266}} = -0.0895$$

so that

$$\theta = \cos^{-1}(-0.0895) = 95.14^\circ$$

In general, the angle between two vectors can be found from the following formula:

Key Point

The angle θ between vectors $\underline{a}$, $\underline{b}$ is such that:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|}$$

Exercises

1. If $\underline{a} = 2\underline{i} - 5\underline{j}$ and $\underline{b} = 3\underline{i} + 2\underline{j}$ find $\underline{a} \cdot \underline{b}$ and verify that $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.
2. Find the angle between $\underline{p} = 3\underline{i} - \underline{j}$ and $\underline{q} = -4\underline{i} + 6\underline{j}$.
3. Use the definition of the scalar product to show that if two vectors are perpendicular, their scalar product is zero.
4. If $\underline{a}$ and $\underline{b}$ are perpendicular, simplify $(\underline{a} - 2\underline{b}) \cdot (3\underline{a} + 5\underline{b})$.
5. If $\underline{p} = \underline{i} + 8\underline{j} + 7\underline{k}$ and $\underline{q} = 3\underline{i} - 2\underline{j} + 5\underline{k}$, find $\underline{p} \cdot \underline{q}$.
6. Show that the vectors $\frac{1}{2}\underline{i} + \underline{j}$ and $2\underline{i} - \underline{j}$ are perpendicular.
7. The work done by a force $\underline{F}$ in moving a body through a displacement $\underline{r}$ is given by $\underline{F} \cdot \underline{r}$. Find the work done by the force $\underline{F} = 3\underline{i} + 7\underline{k}$ if it causes a body to move from the point with coordinates $(1, 1, 2)$ to the point $(7, 3, 5)$.
8. Find the angle between the vectors $\underline{i} - \underline{j} - \underline{k}$ and $2\underline{i} + \underline{j} + 2\underline{k}$.

Answers 1. -4. 2. $142.1°$, 4. $3a^2 - 10b^2$. 5. 22. 7. 39 units. 8. $101.1°$