The Vector Product 9.4

Introduction

In this section we describe how to find the **vector product** of two vectors. Like the scalar product its definition may seem strange when first met but the definition is chosen because of its many applications. When vectors are multiplied using the vector product the result is always a vector.

Prerequisites

Before starting this Section you should ...

① that a vector can be represented as a directed line segment

② how to express a vector in cartesian form

③ how to evaluate 3×3 determinants

Learning Outcomes

After completing this Section you should be able to ...

✓ understand the right-handed screw rule

✓ calculate the vector product of two given vectors

✓ use determinants to calculate the vector product of two vectors given in cartesian form

1. The Right-handed Screw Rule

To understand how the vector product is formed it is helpful to consider first the right-handed screw rule. Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 1.

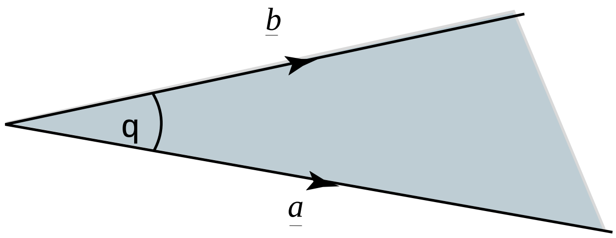


Figure 1.

The two vectors lie in a plane; this plane is shaded in Figure 1. Figure 2 shows the same two vectors and the plane in which they lie together with a unit vector, denoted $\underline{\hat{e}}$, which is perpendicular to this plane. Imagine turning a right-handed screw, aligned along $\underline{\hat{e}}$, in the sense from $\underline{a}$ towards $\underline{b}$ as shown. A right-handed screw is one which when turned clockwise enters the material into which it is being screwed (most screws are of this kind). You will see from Figure 2 that the screw will advance in the direction of $\underline{\hat{e}}$.

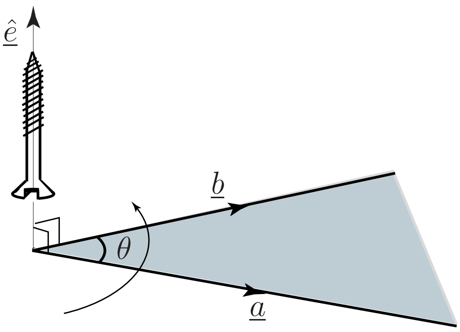


Figure 2.

On the other hand, if the right-handed screw is turned from $\underline{b}$ towards $\underline{a}$ the screw will retract in the direction of $\underline{\hat{f}}$ as shown in Figure 3.

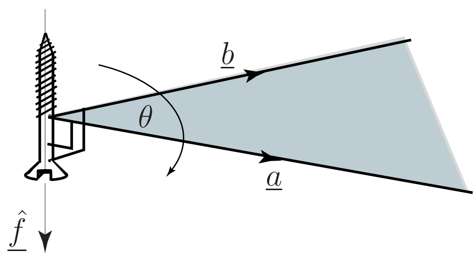


Figure 3.

We are now in a position to describe the vector product.

2. Definition of the Vector Product

We define the vector product of $\underline{a}$ and $\underline{b}$, written $\underline{a} \times \underline{b}$ as

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$$

By inspection of this formula note that this is a vector of magnitude $|\underline{a}|\,|\underline{b}|\,\sin\theta$ in the direction of the vector $\hat{\underline{e}}$, where $\hat{\underline{e}}$ is a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ in a sense defined by the right-handed screw rule. The quantity $\underline{a} \times \underline{b}$ is read as "$\underline{a}$ cross $\underline{b}$"and is sometimes referred to as the cross product. See Figure 4.

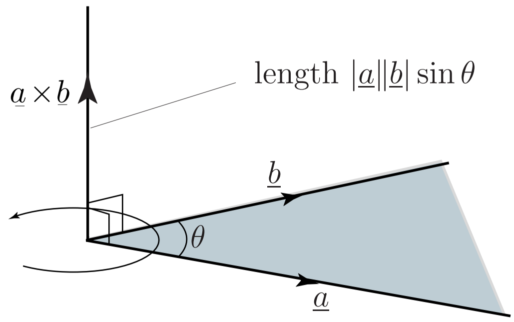


Figure 4. $\underline{a} \times \underline{b}$ is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$.

Formally we have

Key Point

vector product: $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$
modulus of vector product: $|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|\,\sin\theta$

Note that $|\underline{a}|\,|\underline{b}|\,\sin\theta$ gives the vector product its modulus whereas $\hat{\underline{e}}$ gives its direction.
Now study Figure 5 which is used to illustrate the calculation of $\underline{b} \times \underline{a}$. In particular note the direction of $\underline{b} \times \underline{a}$ arising through the application of the right-handed screw rule.
We see that $\underline{a} \times \underline{b}$ is not equal to $\underline{b} \times \underline{a}$ because their directions are oppositely directed. In fact $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.

Example If $\underline{a}$ and $\underline{b}$ are parallel, show that $\underline{a} \times \underline{b} = \underline{0}$.

Solution

If $\underline{a}$ and $\underline{b}$ are parallel then the angle θ between them is zero. Consequently $\sin\theta = 0$ from which it follows that $\underline{a} \times \underline{b} = \underline{0}$. Note that the result, $\underline{0}$, is the zero vector.

Note in particular the following important results:

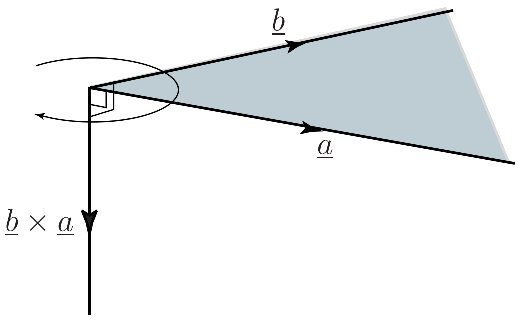


Figure 5. Calculation of $\underline{b} \times \underline{a}$.

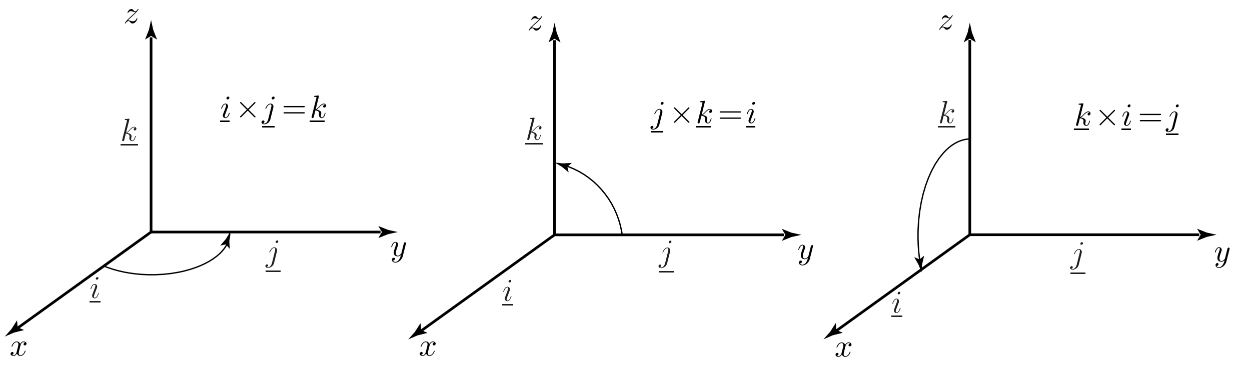
Key Point

$$\underline{i} \times \underline{i} = \underline{0} \qquad \underline{j} \times \underline{j} = \underline{0} \qquad \underline{k} \times \underline{k} = \underline{0}$$

Example Show that $\underline{i} \times \underline{j} = \underline{k}$ and find expressions for $\underline{j} \times \underline{k}$ and $\underline{k} \times \underline{i}$.

Solution

Note that $\underline{i}$ and $\underline{j}$ are perpendicular so that the angle between them is 90°. So the modulus of $\underline{i} \times \underline{j}$ is $(1)(1)\sin 90° = 1$. The unit vector perpendicular to $\underline{i}$ and $\underline{j}$ in the sense defined by the right-hand screw rule is $\underline{k}$ as shown in the figure below (left diagram). Therefore $\underline{i} \times \underline{j} = \underline{k}$ as required.



The vector $\underline{k}$ is perpendicular to both $\underline{i}$ and $\underline{j}$.

Similarly you should verify (see middle and right-hand diagram of the above figure) that $\underline{j} \times \underline{k} = \underline{i}$ and $\underline{k} \times \underline{i} = \underline{j}$.

Key Point

$$\underline{i}\times\underline{j}=\underline{k},\qquad \underline{j}\times\underline{k}=\underline{i},\qquad \underline{k}\times\underline{i}=\underline{j}$$

$$\underline{j}\times\underline{i}=-\underline{k},\qquad \underline{k}\times\underline{j}=-\underline{i},\qquad \underline{i}\times\underline{k}=-\underline{j}$$

To help remember these results you might like to think of the vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ written in alphabetical order like this:

$$\underline{i}\ \underline{j}\ \underline{k}\ \underline{i}\ \underline{j}\ \underline{k}$$

Moving left to right yields a positive result: e.g. $\underline{k}\times\underline{i}=\underline{j}$. Moving right to left yields a negative result as in $\underline{j}\times\underline{i}=-\underline{k}$

3. A Formula for Finding the Vector Product

We can use the boxed results of the previous section to develop a formula for finding the vector product of two vectors given in cartesian form: Suppose $\underline{a}=a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$ and $\underline{b}=b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$ then

$$\begin{aligned}\underline{a}\times\underline{b} &= (a_1\underline{i}+a_2\underline{j}+a_3\underline{k})\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})\\ &= a_1\underline{i}\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})\\ &\quad +a_2\underline{j}\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})\\ &\qquad +a_3\underline{k}\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})\\ &= a_1b_1(\underline{i}\times\underline{i})+a_1b_2(\underline{i}\times\underline{j})+a_1b_3(\underline{i}\times\underline{k})\\ &\quad +a_2b_1(\underline{j}\times\underline{i})+a_2b_2(\underline{j}\times\underline{j})+a_2b_3(\underline{j}\times\underline{k})\\ &\qquad +a_3b_1(\underline{k}\times\underline{i})+a_3b_2(\underline{k}\times\underline{j})+a_3b_3(\underline{k}\times\underline{k})\end{aligned}$$

Using the *Key Point* results, on page 5, (above) this expression simplifies to

$$\underline{a}\times\underline{b}=(a_2b_3-a_3b_2)\underline{i}-(a_1b_3-a_3b_1)\underline{j}+(a_1b_2-a_2b_1)\underline{k}$$

Key Point

If $\underline{a}=a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$ and $\underline{b}=b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$ then

$$\underline{a}\times\underline{b}=(a_2b_3-a_3b_2)\underline{i}-(a_1b_3-a_3b_1)\underline{j}+(a_1b_2-a_2b_1)\underline{k}$$

Example Evaluate the vector product $\underline{a} \times \underline{b}$ if $\underline{a} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ and $\underline{b} = 7\underline{i} + 4\underline{j} - 8\underline{k}$.

Solution

Identifying $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$ we find

$$\begin{aligned}\underline{a} \times \underline{b} &= ((-2)(-8) - (5)(4))\underline{i} - ((3)(-8) - (5)(7))\underline{j} + ((3)(4) - (-2)(7))\underline{k} \\ &= -4\underline{i} + 59\underline{j} + 26\underline{k}\end{aligned}$$

Use the **Key Point** formula directly above to find the vector product of $\underline{p} = 3\underline{i} + 5\underline{j}$ and $\underline{q} = 2\underline{i} - \underline{j}$.

Your solution

Note that in this example there are no $\underline{k}$ components so a_3 and b_3 are both zero. Apply the formula:

$$\underline{p} \times \underline{q} =$$

$-13\underline{k}$

4. Using Determinants to Evaluate a Vector Product

Evaluation of a vector product using the previous formula is very cumbersome. A more convenient and easily remembered method is to use determinants. Recall that, for a 3×3 determinant,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The vector product of two vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ can be found by evaluating the determinant:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

in which $\underline{i}$, $\underline{j}$ and $\underline{k}$ are (temporarily) treated as if they were scalars.

Key Point

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2b_3 - a_3b_2) - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

Example Find the vector product of $\underline{a} = 3\underline{i} - 4\underline{j} + 2\underline{k}$ and $\underline{b} = 9\underline{i} - 6\underline{j} + 2\underline{k}$.

Solution

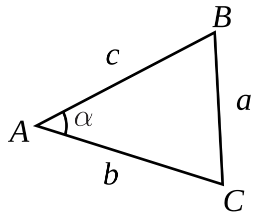
We have

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$\underline{a} \times \underline{b} = \underline{i}(-8 - (-12)) - \underline{j}(6 - 18) + \underline{k}(-18 - (-36)) = 4\underline{i} + 12\underline{j} + 18\underline{k}$$

Example *The area of a triangle* The area A_T of the triangle shown in the figure below is given by the formula $A_T = \frac{1}{2}bc\sin\alpha$. Show that an equivalent formula is $A_T = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.



Solution

From the definition of the vector product

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\alpha$$

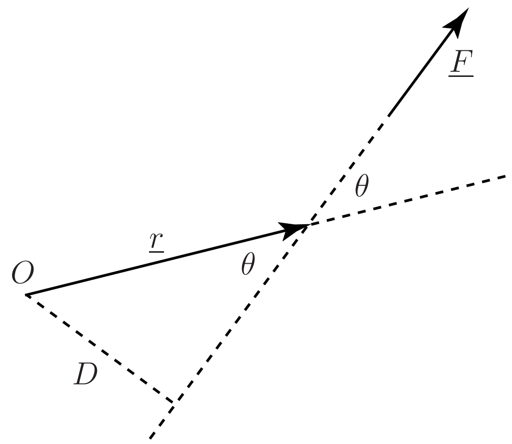
since α is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Furthermore $|\overrightarrow{AB}| = c$ and $|\overrightarrow{AC}| = b$. The required result follows immediately.

Moments

The moment (or torque) of the force $\underline{F}$ about a point O is defined as

$$\underline{M}_o = \underline{r} \times \underline{F}$$

where $\underline{r}$ is a position vector from O to *any* point on the line of action of $\underline{F}$.

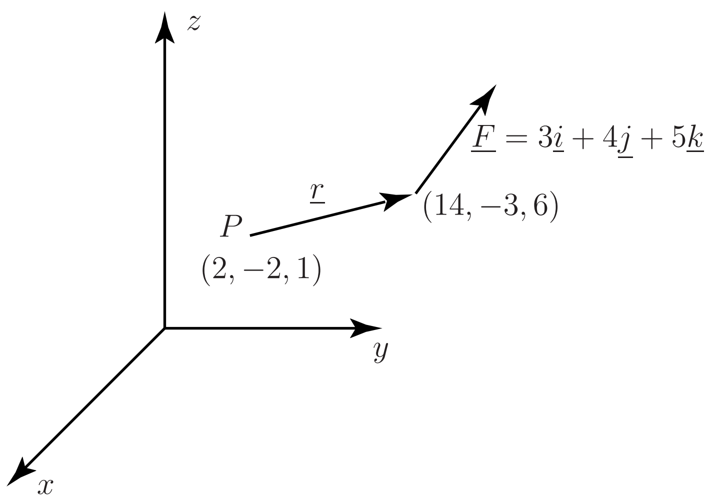


It may seem strange that any point on the line of action may be taken but it is easy to show that exactly the same vector is obtained.
By the properties of the cross product the direction of $\underline{M}_o$ is perpendicular to the plane containing $\underline{r}$ and $\underline{F}$ (i.e. out of the paper). The magnitude of the moment is

$$|\underline{M}_0| = |\underline{r}||\underline{F}| \sin\theta$$

From the diagram $|\underline{r}| \sin\theta = D$. Hence $|\underline{M}_o| = D|\underline{F}|$. This would be the same no matter which point on the line of action of $\underline{F}$ was chosen.

Example Find the moment of the force given by $\underline{F} = 3\underline{i} + 4\underline{j} + 5\underline{k}$ (N) acting at the point $(14, -3, 6)$ about the point $P(2, -2, 1)$.



Solution

The position vector $\underline{r}$ can be any vector from the point P to any point on the line of action of $\underline{F}$. We can take (in metres) $\underline{r} = (14-2)\underline{i} + (-3-(-2))\underline{j} + (6-1)\underline{k} = 12\underline{i} - \underline{j} + 5\underline{k}$. The moment is

$$\underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 12 & -1 & 5 \\ 3 & 4 & 5 \end{vmatrix} = -25\underline{i} - 45\underline{j} + 51\underline{k} \text{ (Nm)}$$

Exercises

1. Show that if $\underline{a}$ and $\underline{b}$ are parallel vectors then their vector product is the zero vector.
2. Find the vector product of $\underline{p} = -2\underline{i} - 3\underline{j}$ and $\underline{q} = 4\underline{i} + 7\underline{j}$.
3. If $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} + 2\underline{k}$ find $\underline{a} \times \underline{b}$. Show that $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$.
4. Points A, B and C have coordinates $(9, 1, -2)$, (3,1,3), and $(1, 0, -1)$ respectively. Find the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$.
5. Find a vector which is perpendicular to both of the vectors $\underline{a} = \underline{i} + 2\underline{j} + 7\underline{k}$ and $\underline{b} = \underline{i} + \underline{j} - 2\underline{k}$. Hence find a unit vector which is perpendicular to both $\underline{a}$ and $\underline{b}$.
6. Find a vector which is perpendicular to the plane containing $6\underline{i} + \underline{k}$ and $2\underline{i} + \underline{j}$.
7. For the vectors $\underline{a} = 4\underline{i} + 2\underline{j} + \underline{k}$, $\underline{b} = \underline{i} - 2\underline{j} + \underline{k}$, and $\underline{c} = 3\underline{i} - 3\underline{j} + 4\underline{k}$, evaluate both $\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{a} \times \underline{b}) \times \underline{c}$. Deduce that, in general, the vector product is not associative.
8. Find the area of the triangle with vertices at the points with coordinates $(1, 2, 3)$, $(4, -3, 2)$ and $(8, 1, 1)$.
9. For the vectors $\underline{r} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{s} = 2\underline{i} - 2\underline{j} - 5\underline{k}$, and $\underline{t} = \underline{i} - 3\underline{j} - \underline{k}$, evaluate
   a) $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r}$. b) $(\underline{r} \times \underline{s}) \times \underline{t}$. Deduce that $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r} = (\underline{r} \times \underline{s}) \times \underline{t}$.

Answers 2. $-2\underline{k}$ 3. $-5\underline{i} + 10\underline{j} - 5\underline{k}$ 4. $5\underline{i} - 34\underline{j} + 6\underline{k}$ 5. $-11\underline{i} + 9\underline{j} - \underline{k}$, $\frac{1}{\sqrt{203}}(-11\underline{i} + 9\underline{j} - \underline{k})$
6. $-\underline{i} + 2\underline{j} + 6\underline{k}$ for example. 7. $7\underline{i} - 17\underline{j} + 6\underline{k}$, $-42\underline{i} - 46\underline{j} - 3\underline{k}$ 8. $\frac{1}{2}\sqrt{1106}$ 9. $-29\underline{i} - 10\underline{j} + \underline{k}$