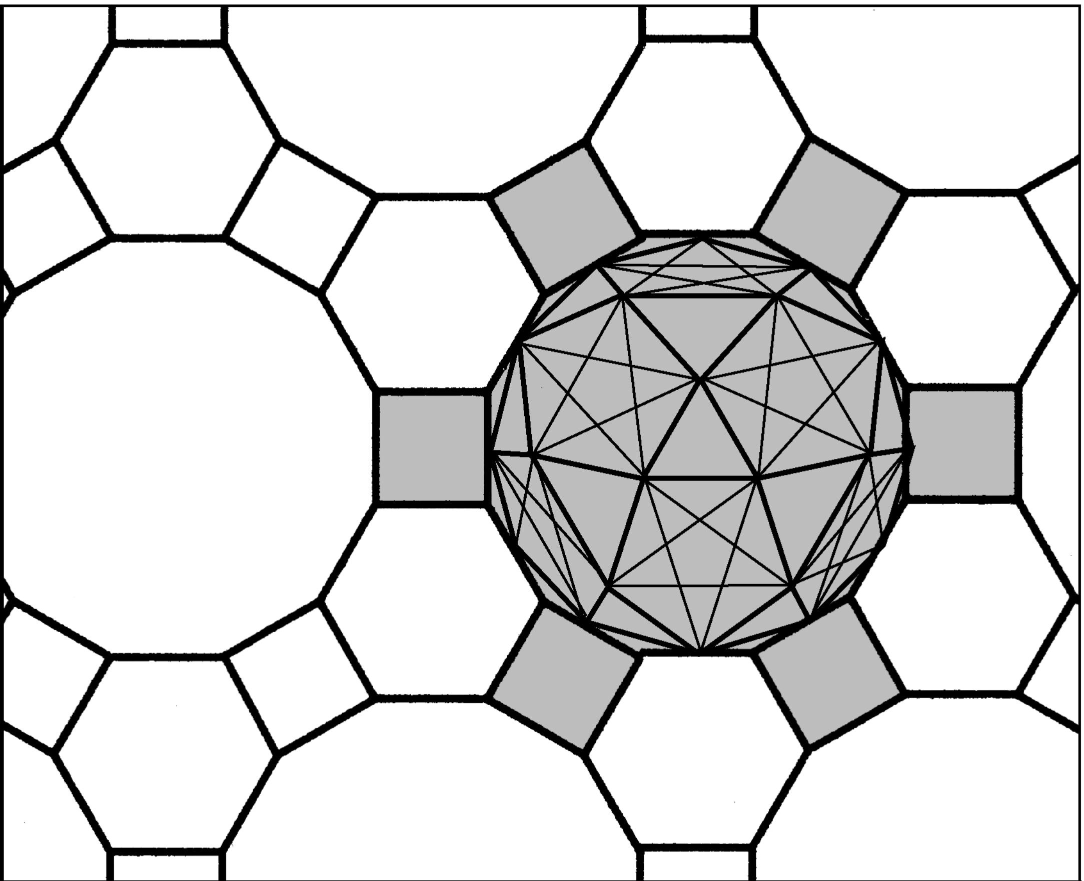
Workbook 5



Functions and Modelling

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

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Functions and Modelling

Learning outcomes

After studying the Workbook and completing associated Tasks and Exercises you should be able to: list and explain the stages of the modelling cycle; use linear, quadratic and power law functions in modelling where appropriate.

The Modelling Cycle and Functions

5.1

Introduction

In this Section we look at the process of modelling with mathematics which is vitally important in engineering. Knowledge of mathematics is not much use to an engineer unless it can be applied to engineering problems. After discussing the mathematical modelling process we discuss the use of linear models.

Prerequisites

Before starting this Section you should ...

- be competent at algebraic manipulation
- be familiar with linear functions

Learning Outcomes

On completion you should be able to ...

- understand the basics of the modelling process
- use linear functions to model motion under constant acceleration
- analyse motion under gravity

1. Functions and modelling

Engineers use mathematics to a considerable extent. Mathematical techniques offer ways of handling **mathematical models** of an engineering problem and coming up with a solution. Of course it is possible to model a problem in ways that are not mathematical e.g. by physical or scale modelling, but this Workbook is concerned exclusively with mathematical modelling, so we will drop the word 'mathematical' and refer just to modelling. This Section is intended to introduce some modelling ideas as well as to show applications of the functions and techniques introduced in HELM 2, HELM 3 and HELM 6. By **modelling** we mean the process by which we set up a mathematical model of a situation or of an assumed situation, use the model to make some predictions and then interpret the results in the original context. The mathematical techniques themselves contribute only to part of the modelling procedure. The modelling procedure can be regarded as a cycle. If we do not like the outcome for some reason we can try again. Five steps of a modelling cycle can be identified as follows:

Step 1 **Specify** the **purpose** of the model.

Step 2 **Create** the mathematical **model** after making and stating relevant assumptions.

Step 3 **Do** the resulting **mathematics**.

Step 4 **Interpret** the **results**.

Step 5 **Evaluate** the outcome, usually by comparing with reality and/or purpose and, if necessary, try again!.

Much of this first Section is concerned with steps 2 and 3 of the cycle: **creating a mathematical model** and **doing the maths**. Engineering case studies found in many Workbooks will aim to demonstrate the complete cycle. An important part of step 2 may include **choosing an appropriate function** based on the assumptions made also as part of this step. This choice will influence the kind of mathematical activity that is involved in step 3.

So far in your engineering mathematical studies you might have had little opportunity to think about what is 'appropriate', since the type of function to be studied and used has been chosen for you. Sometimes, however, you may be faced with making appropriate choices of function for yourself so it is important to have some understanding of what might be appropriate in any given circumstance. A well chosen function will be appropriate in two different ways. Firstly the function should be consistent with the purpose of the model, with known **data** or **theory** or **facts**, and with known or assumed **behaviours**. For example, the purpose might be to predict the future behaviour of a quantity which is expected to increase with time. In this case time can be identified as the independent variable since the quantity depends on time. The function chosen for mathematical activity should be one in which the value of the dependent variable increases with time. Secondly, bearing in mind that the modelling process is a **cycle** and so it is possible, and usual, to go round it **more than once**, the first choice of function should be as simple as allowed by the modelling context. The main reason for doing this is to avoid complication unless it is really necessary. Philosophically, an initial choice of a simple function is consistent with the fundamental belief that most phenomena may be modelled adequately by simple laws and theories. It is common engineering practice always to use the simplest model possible in a given situation. So, for the first trip around the cycle, the appropriate function should be the simplest that is consistent with known facts, behaviours, theory or data. If the quantity of interest is known not to be constant, this might be a linear function. If the first choice turns out to be inadequate at the stage of the cycle where the result is interpreted or the outcome is evaluated (step 5) then it is reasonable to try something more complicated; a quadratic function might be the second choice if the first choice was linear.

It is important to realise that sophistication is not necessarily a virtue in itself. The merits of complication depend upon the purpose for which the model is being formulated. A model of the weather that enables a decision on whether or not to take an umbrella to work on any particular day will be rather less sophisticated than that required to give an accurate prediction of the amount of rainfall in the vicinity of the workplace on that day.
In the next subsection we will look at various types of functions that have been introduced so far but in a different way, concentrating more on their graphical behaviour and their parameters. As mentioned earlier, appropriateness is determined by the extent to which the *behaviour* of the chosen function reflects the behaviour to be modelled as the independent variable varies. The behaviour of a function is determined by whether it is linear, non-linear, or periodic and its range of validity. An important task of this Workbook is to get you to think more and more in modelling terms about the forms and associated behaviours of functions. We shall also take the opportunity of deriving some generalities from specific examples.

2. Constant functions

There are two physical interpretations of constancy that are of interest here. A very common form is **constancy with time**. Motion under gravity may be modelled as motion with constant acceleration. By definition, Fixed Rate Mortgages (increasingly popular in the late 1990s) offer a constant rate of interest over a specified period. In these examples, the constancy will be limited to a certain time interval. Motion under gravity will only involve the constant acceleration due to the Earth's gravitational pull as long as the motion is close to the Earth's surface. In any case the acceleration will only be from the time the object is released to the time it stops. Unfortunately, increases in base interest rates eventually feed into mortgage rates. So mortgage lenders are only able to offer fixed rates for a certain time. A mathematical statement of these limits is a statement of the *range of validity* of the constant function model.

Another type of constancy is **constancy in space**. Long stretches of Roman roads were built in a fixed direction. For at least part of their lengths, roads have constant width. In modelling the formation and movement of seismic waves in the Earth's crust it is convenient to assume that the layers from which the Earth's crust is formed have constant thickness with respect to the Earth's surface. In these cases the assumption of constancy will only be valid within certain limits in space.

Example 1

The rate of flow of water from a tap is denoted as r (litres per minute). The time for which it is turned on is denoted by t (minutes). Suppose that a tap is turned on and that the rate of water running out of a tap is assumed to be constant at 3 litres per minute and that it is turned off after 10 minutes.

(a) Write down a mathematical statement of the model for the flow from the tap, including its range of validity.

(b) Sketch a graph of the variation of r with t.

(c) Find an equation for the number of litres of water that have run out of the tap after t minutes.

(d) Calculate the volume of water that has run out of the tap three minutes before it is turned off.

Solution

(a) $r = 3 \qquad (0 \leq t \leq 10)$

(b)

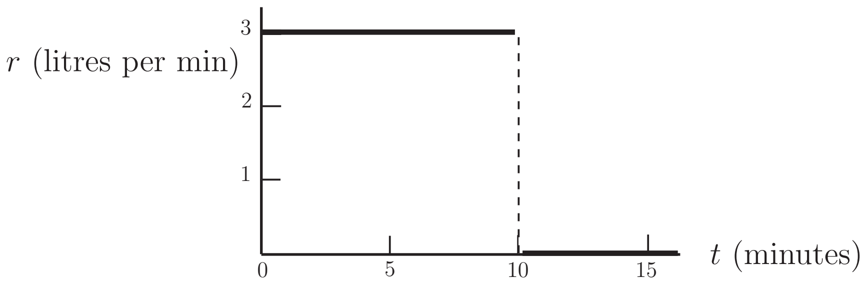


Figure 1: Flow from tap

(c) $V = rt \qquad (0 \leq t \leq 10)$

(d) The tap will have run for seven minutes; 3 litres per minute x 7 minutes = 21 litres

Note that a more sophisticated model would allow for the variation in flow rate as the tap is turned on and turned off.

3. Linear functions

HELM 2 has introduced linear functions of the form $y = ax + b$. Such functions give rise to straight-line graphs. The coefficient *a* is the slope. If *a* is positive the graph of *y* against *x* slopes upwards. If *a* is negative the graph slopes downwards. The coefficient *b* gives the intercept on the y-axis. The terms *a* and *b* may be called the **parameters** of the line. Note that this is a different use of the term 'parameter' than in the parametrisation of functions discussed in HELM 2.

Linear models for falling rocks

In modelling it is wise to use a notation which fits in with the application. When modelling velocity under constant acceleration, we shall replace the dependent variable *y* by v (for velocity), and the independent variable x by t (for time). The acceleration will be denoted by the symbol a. Consider the motion of a rock dislodged from the top of a cliff (35 m high) by a villain during the filming of a thriller. The film producer might be interested in how long the rock would take to fall to the ground below the cliff and how fast it would be travelling at ground impact. The rock may be assumed to have a constant downward acceleration of 9.8 m s^{-2} which the acceleration due to gravity. The velocity (v m s^{-1}) of a rock, falling from the top of a cliff 35 m high, can be modelled by the equation

$$v = 9.8t \qquad (0 \leq t \leq 2.7)$$

where t is the time in seconds after the rock starts to fall. This follows from the fact that acceleration is the rate of change of velocity with time. If the acceleration is constant and the object starts from rest, then the velocity is given simply by the product of acceleration and time. The upper limit for t is the time at which the rock hits the ground measured with a stop-watch (about 2.7 s in this case). Figure 2 shows v as a function of t. Velocity is a linearly increasing function of time and its graph is a straight line passing through $t = 0, v = 0$. Note that various assumptions are needed to obtain the quoted result of a linear variation in speed with time: it is assumed that there is no air resistance, no spinning and no wind.

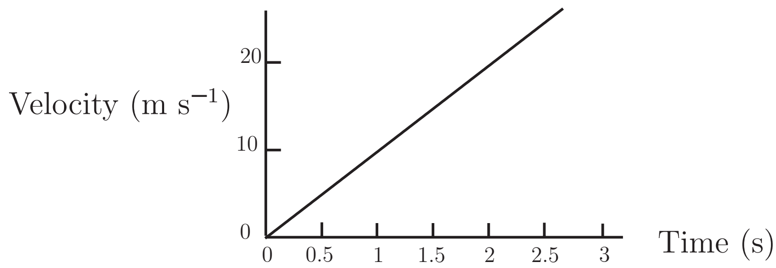


Figure 2: Graph of $v = 9.8t$ for the falling rock

In what way should the equation for v be altered if the villain were able to **throw** the rock downwards at 5 m s^{-1}? Provided we are measuring position or displacement downwards, a downwards velocity is positive. Now we have that $v = 5$ when $t = 0$. So a new model for v is

$$v = 9.8t + 5 \qquad (0 \leq t \leq T_1)$$

Since they are both downwards, the initial velocity simply adds to the velocity at any time resulting from falling under gravity. Note that T_1 is being used now for the upper limit on t (instead of 2.7) because 2.7 is (approximately) the time taken to fall 35 m **from rest** rather than with an initial downwards velocity. (Using the symbol T_1 saves us trying to work out its value for the moment!) Note that a general form of the model for motion under constant acceleration of magnitude a m s^{-2} given an initial speed b m s^{-1} is $v = at + b$. In the model just considered $a = 9.8$ and $b = 5$.

Task

For the above example modelling a falling rock:

(a) Determine whether T_1 is more or less than 2.7.

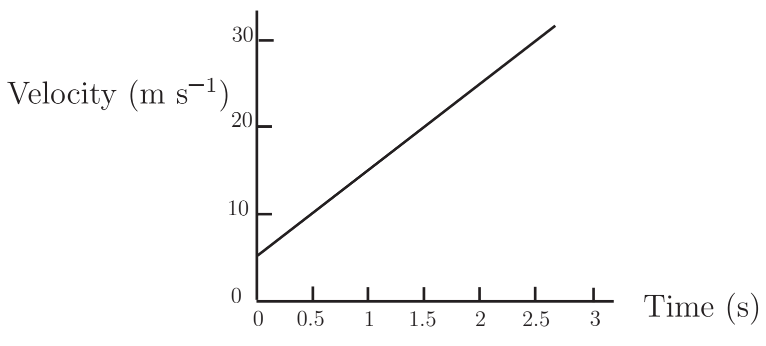
(b) Sketch a graph of v for $0 \leq t \leq T_1$.

Your solution

Answer

(a) T_1 will be less than 2.7 since the rock will be moving faster throughout its descent.

(b) The graph is still a straight line but displaced upwards compared with Figure 2.



Graph of $v = 9.8t + 5$ for the falling rock

Consider now how the function for v will change if the villain is even mightier than we previously thought and throws the rock **upwards** with an initial speed of 5 m s^{-1} instead of simply dislodging it or throwing it downwards. In this circumstance, the initial velocity is directed upwards, and since position is being measured downwards, the initial velocity is negative. We can use the equation $v = 9.8t + b$ again. This time $v = -5$ when $t = 0$, leading to $b = -5$ and

$$v = 9.8t - 5 \qquad (0 \leq t \leq T_2)$$

The new time at which the rock hits the ground is denoted by T_2. The rock will rise before falling to the gound this time so T_2 will be larger than T_1.
From the modelling point of view, there is one other significant time before the rock hits the ground. Figure 3 shows the new graph of v against t. Notice that there is a time at which v (which starts at -5) is zero. What does this mean?

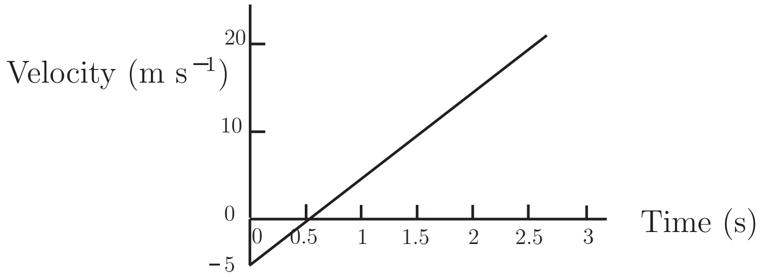


Figure 3: Graph of $v = 9.8t - 5$ for the falling rock

As time goes by, the fact that gravity is causing the rock to accelerate downwards means that the rock's upward motion will slow. Its velocity will decrease in magnitude until it reaches zero. At this particular instant the rock will be at its highest point and its velocity will change from upwards (negative) to downwards (positive) passing instantaneously through zero in the process.
We can calculate this time by substituting zero for v and working out the corresponding t.

$$0 = 9.8t - 5, \quad \text{so} \qquad t = \frac{5}{9.8} = 0.51.$$

This means that the rock is stationary about a half second after being thrown upwards. Subsequently the rock will fall until it hits the ground. But there is yet one more time that may be significant in the modelling context chosen here. During its journey to the ground 35 m beneath the cliff-top, the rock will pass the top of the cliff again. Note that we are modelling the motion of a particular point, say the lowest point, on the rock. A real rock, with appreciable size, will only pass the top of the cliff, without landing on it or hitting it, if it is thrown a little forward as well as up. Anyway, in principle we could use the function that we started with, representing the velocity of an object falling from rest under gravity, to work out how long the rock will take to pass the top of the cliff having reached the highest point in its path. A simpler method is to argue that, as long as the rock is thrown from the cliff top level (this requires the villain to be lying down!), the rock should take exactly the same time (approximately 0.5 s) to return to the level of the cliff top as it took to rise above the cliff top to the highest point in its path. So we simply double 0.5 s to deduce that the rock passes the cliff top again about 1 s after being thrown.

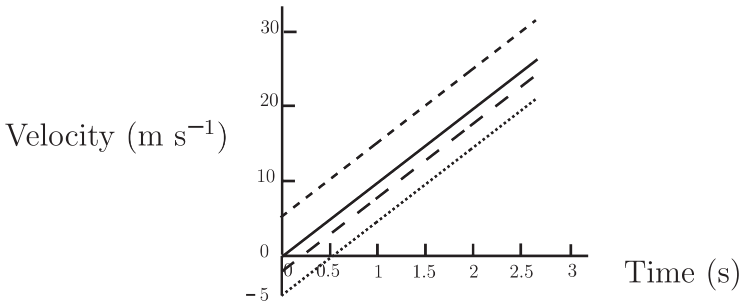
Task

This Task concerns the falling rock model just discussed.

(a) Add lines to a sketch version of Figure 3 to represent velocity as a function of time if the rock is (i) dislodged (ii) thrown with velocity 3 m s^{-1} downwards (iii) thrown with velocity -2 m s^{-1}:

Your solution

Answer



(b) What do you deduce about the effect of the initial velocity on the graph of velocity against time?

Your solution

Answer
The effect of changing the initial velocity (in size or in sign) is simply to displace the straight line upwards or downwards without changing its slope.

(c) Imagine that the filming was on the Moon with roughly one-sixth the gravitational pull of Earth. Find a linear function that would describe the velocity of a dislodged rock:

Your solution

Answer
$$v = \frac{9.8}{6}t \approx 1.6t$$

(d) What do you deduce about the effect of changing the acceleration due to gravity on the graph of velocity against time?

Your solution

Answer
The graph of velocity agains time is still linear but the change in the acceleration due to gravity changes the slope.

So, in the context of modelling motion under gravity, the **initial velocity** determines the vertical displacement of the line, its **intercept** on the v-axis, and the **acceleration** determines the **slope**. Again, given the modelling context, both of these influence the range of validity of the model since they alter the time taken for the rock to reach the ground and this fixes the upper limit on time.
Like velocity, acceleration has direction as well as magnitude. As long as position is being measured downwards, and only gravity is considered to act, falling objects do not provide any examples of negative accelerations - but rocket motion does. Where downwards accelerations are represented as positive, an upwards acceleration will be negative. So a model of the motion of a rocket accelerating away from the Earth could include a constant negative acceleration. Horizontal acceleration, say of a road vehicle, in the same direction as position as being measured, is represented as positive. Deceleration, for example when this vehicle is being braked, implies that velocity is decreasing with time, and is represented as negative. In mathematical modelling, it is usual to refer to *acceleration*, whether it represents positive acceleration or deceleration.

Suppose that we are describing the motion of a rocket taking off vertically during its initial booster stage of 10 s. We might model the acceleration as a constant -20 m s^{-2}. The negative sign arises because downwards is being taken as the positive direction but the acceleration is upwards. Since the rocket is starting from rest, an appropriate function is

$$v = -20t \qquad (0 \leq t \leq 10)$$

This should describe the variation of its velocity with time until the end of the initial booster stage of its flight. Figure 4 shows the corresponding graph of velocity against time. Note the way in which the graph slopes downwards to the right. This function describes an increasingly *negative* velocity as time passes, consistent with an increasing *upwards* velocity. The corresponding graph for a positive acceleration of the same magnitude would slope upwards towards the right.

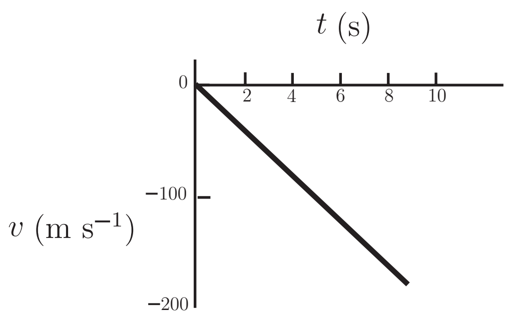


Figure 4: Variation of velocity of rocket during the initial booster stage.

Task

Imagine that a satellite is falling towards Earth at 5 m s^{-1} when a booster rocket is fired for 5 s accelerating it away from the Earth at 10 m s^{-2}.

(a) Write down a corresponding linear function that would describe its velocity during the booster stage.

Your solution

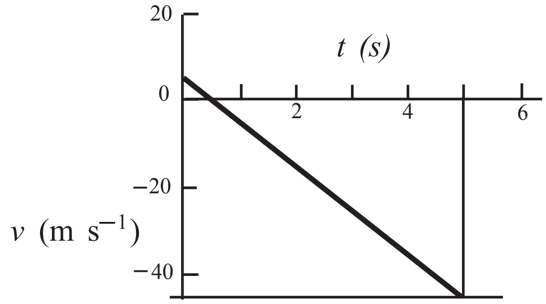
Answer

If position is measured downwards, acceleration away from the Earth may be written as -10 m s^{-2}. The initial velocity towards the Earth may be denoted by $(+)5$ m s^{-1} so $v = -10t+5 \quad (0 \leq t \leq 5)$. If position is measured upwards $v = 10t - 5 \quad (0 \leq t \leq 5)$.

(b) Sketch the corresponding graph of velocity against time if position is measured **downwards** towards Earth:

Your solution

Answer

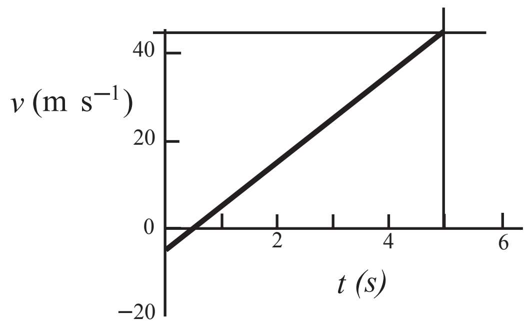


Satellite velocity (position measured downwards towards Earth)

(c) Sketch the corresponding graph of velocity against time if position is measured **upwards** away from the Earth:

Your solution

Answer



Satellite velocity (position measured upwards away from Earth)

(d) At what time would the velocity of the satellite be zero?

Your solution

Answer

When v is 0, $0 = -10t + 5$, so $t = 0.5$. The satellite has zero velocity towards the Earth after 0.5 s.

(e) What is the value of the velocity at the end of the booster stage?

Your solution

Answer

When $t = 5$, either $v = -10 \times 1 + 5 = -5$, so the velocity is 5 m s^{-1} away from the Earth, or, using the second equation in (a), $v = 10 - 5 = 5$, leading to the same conclusion.

Other contexts for linear models

Linear functions may arise in other contexts. In each of these situations, the slope and intercept values will have some modelling significance. Indeed the behaviour and hence the suitability of a linear function, of the form $y = ax + b$, when modelling any given situation will be determined by the values of *a* and *b*.

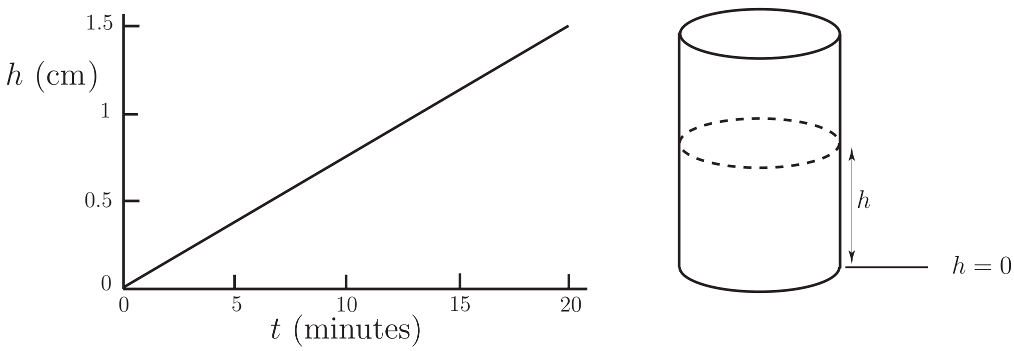
Task

During 20 minutes of rain, a cylindrical rain barrel that is initially empty is filled to a depth of 1.5 cm.

(a) Choose variables to represent the level of water in the barrel and time. Sketch a graph representing the level of water in the barrel if the intensity of rainfall remains constant over the 20 minute period.

Your solution

Answer
In this answer *h* cm is used for the level of water measured from the bottom of the barrel and *t* minutes for time.



Height (depth) of rainwater in a barrel.

(b) Write down a linear function that represents the level of water in the vessel together with its range of validity.

Your solution

Answer
The intensity of rainfall is stated to be constant, so the rate at which the barrel fills may be taken as constant. The gradient of an appropriate linear function relating level of water (h cm) measured from the bottom of the vessel and time (t, minutes) measured would be $\frac{1.5}{20} = 0.075$ and an appropriate linear function would be $h = 0.075t + c$. Since the barrel is empty to start with, $h = 0$ when $t = 0$, implying that $c = 0$. So the appropriate linear function and its range of validity are expressed by $h = 0.075t, \quad (0 \leq t \leq 20)$.

(c) State any assumptions that you have made:

Your solution

Answer
It is assumed that the barrel has a uniformly cylindrical cross section, that no water is removed during the rainfall and there are no holes or leaks up to 1.5 cm depth.

(d) Write down the amended form of your answer to (b), if the vessel contains 2 cm of water initially.

Your solution

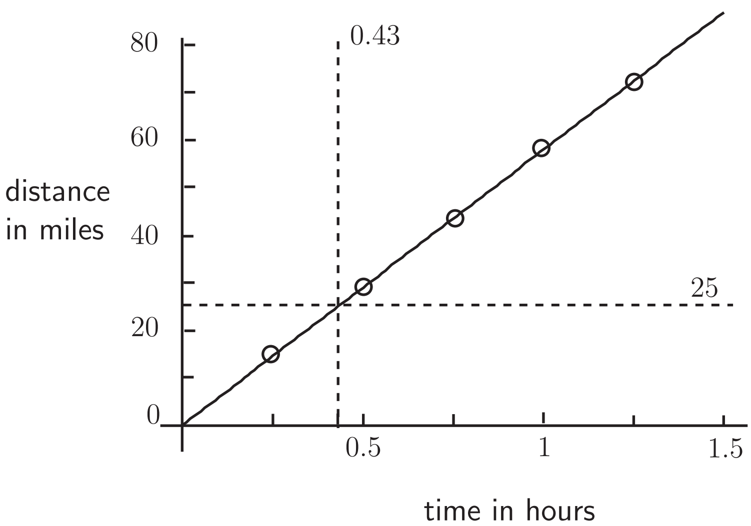
Answer
$h = 0.075t + 2 \qquad (0 \leq t \leq 20)$

Suppose that you travel often from Nottingham to Milton Keynes which is a distance of 87 miles almost all of which is along the M1 motorway. Usually it takes 1.5 hours. Suppose also that, on one occasion, you have agreed to pick someone up at the Leicester junction (21) of the M1. This is 25 miles from the start of your journey in Nottingham. If you start your journey at 8 a.m., what time should you advise for the pick-up?

Your solution

Answer
(Graphical method)

Assume a constant speed for the whole journey. This means that if 87 miles is covered in 1.5 hours, then half the distance (43.5 miles) is covered in 0.75 hours and so on.



A distance of 25 miles will be covered in 0.43 hours

The average speed is $\dfrac{87}{1.5} = 58$ mph.

This is also the *gradient* of the graph

Answer
(Symbolic method)

Let *d* miles be the distance travelled in time *t* hours. Then $\dfrac{d}{t} = 58t$. This is valid only for the duration of the journey $(0 \leq t \leq 1.5)$. The equation can be used to find the time at which *d* = 25. Now $25 = 58t$ and so $t = \dfrac{25}{58} = 0.43103448 = 0.43$ (to two decimal places). Either way, given that 0.43 h is about 26 minutes, a possible suggestion is that the passenger should be advised 8.26 a.m. for the pick-up. But the assumption of constant speed has its limitations. It would be safer to say "be there by 8.20 a.m. but be prepared to wait perhaps until 8.30 a.m."

Task

A local authority has flood control plans in which the emergency and rescue services are alerted when the river level rises to critical values. A linear model is used to estimate the variation of height with time. After a period of continuous heavy rain the level one day was 1.5 m at 8 a.m. and 1.8 m at 2 p.m.

(a) Use a linear model to write down an equation for estimating the level of the river at different times of the day:

Your solution

Answer

If the level of water is represented by L m and time by t hours after 8 a.m. then a linear model for the level as a function of time may be written

$$L = at + b$$

where a and b are constants to be found from the other information in the problem. Specifically, it is stated that $L = 1.5$ when $t = 0$ and $L = 1.8$ when $t = 6$. The first statement implies that

$$1.5 = 0 + b \qquad \text{or} \qquad b = 1.5$$

The second statement implies that

$$1.8 = 6a + b$$

or, after substituting for b,

$1.8 = 6a + 1.5$ or $0.3 = 6a$ or $a = 0.05$

So the equation for estimating the level of the river at different times is

$$L = 0.05t + 1.5$$

(b) Suggest a suitable range of values of time for which the model could be used:

Your solution

Answer

The model is valid between 8 a.m. and 2 p.m. and, subsequently, only as long as the river level rises steadily.

(c) What time does the model predict that the level of the river will reach 2 m?

Your solution

Answer

The model will predict a level of 2 m at time t given by

$$2 = 0.05t + 1.5 \qquad \text{or} \qquad t = \frac{0.5}{0.05} = 10$$

i.e. 10 hours after 8 a.m. which is 6 p.m.

Task

During one winter, the roads in a rural area were completely free from snow when it started snowing at midnight. It snowed heavily all night and day. By 10 a.m. it was 19 cm deep.

To save money the local authorities wait until the snow is 30 cm deep before ploughing the snow away from the roads. Forecast when ploughing should start, stating any assumptions you have made.

Your solution

Answer

If the depth of snow is represented by D cm and time by t hours after midnight then a linear model for the depth as a function of time may be written:

$$D = at + b$$

where a and b are constants to be found from the other information in the problem or from assumptions. As there was no snow at midnight

$$0 = 0 + b \qquad \text{or} \qquad b = 0$$

It is stated that $D = 19$ when $t = 10$, i.e.

$$19 = 10a \qquad \text{or} \qquad a = 1.9$$

So the equation for estimating the depth of snow at different times is

$$D = 1.9t$$

The model will predict a level of 30 cm at a time t given by

$$30 = 1.9t \qquad \text{or} \qquad t = \frac{30}{1.9} = 15.789$$

i.e. 15.789 hours after midnight which is a little after 3.47 p.m.

Assuming that the snow build up is steady e.g. no drifting or change in precipitation, this suggests that ploughing should start about 3.45 p.m.

Exercises

1. A cross-channel ferry usually takes 2 hours to make the 40 km crossing from England and France.

   (a) What is the boat's average speed?

   (b) Derive a linear model connecting distance from England and time since leaving port. State any limitations of the model.

   (c) According to this model, when will the boat be 15 km and 35 km from England?

2. During one winter, the roads in the country district were completely free from snow when it started snowing at 2:30 a.m. and it snowed steadily all day. At 7:30 a.m. it was 14 cm deep. To save money, the local practice was to wait until the snow was 20 cm deep before ploughing the roads. Forecast when ploughing would start, stating any assumptions.

3. In a drought, the population of a particular species of water beetle in a pond is observed to have halved when the volume of water in the pond has fallen by half. Make a simple assumption about the relationship between the beetle population and the volume of water in the pond and express this in symbols as an equation. What would your model predict for the population when the water volume is only one third of what it was originally.

4. A firm produces a specialised instrument and, although it has the facilities to produce 100 instruments per week, it rarely produces more than 50. It is finding it difficult to assess the cost of producing the instruments and to set realistic prices. The firm's accountant estimates that the firm pays out £5000 per week on fixed costs (overheads, salaries etc.) and that the additional cost of producing each instrument is £50.

   (a) Derive and use a linear model for the variation in total costs with the quantity of instruments produced. State any limitation of this model.

   (b) What is the model's prediction for the cost of producing 80 instruments per week?

Answers

1. (a) A distance of 40 km is covered in 2 hours. So the average speed is 40/2 = 20 km h^{-1}.

   (b) A linear model assumes that the boat is a point moving at constant speed and will only be valid for 2 hours (or 40 km) while the boat is travelling from England to France. It does not allow for variations in speed. If the distance from the English port at any time t hours is denoted by d km, then $d = 20t$.

   (c) When the boat is 15 km from England $15 = 20t$, so $t = 15/20 = 0.75$, so the boat is 0.75 hour (45 minutes) from port. When the boat is 35 km from England, $35 = 20t$, so $t = 35/20 = 1.75$, so the boat is 1.75 hour (1 hour 45 minutes) from port.

2. Assume that there is no snow at 2:30 a.m. and that the rate of accumulation of snow is constant. Then, if the snow is 14 cm deep at 7:30 a.m., the rate of accumulation is 2.8 cm per hour. A linear model for the depth (d cm) of snow t hours after 2:30 a.m. is $d = 2.8t$. d will be 20 when $20 = 2.8t$, i.e. $t = 20/2.8 = 7.143$. This corresponds to about 9:39 a.m. So ploughing should start at about 9:40 a.m.

3. Denote population by P and volume of pond by V. Then P is proportional to V so $P = kV$ where k is a constant of proportionality. When V becomes $V/3$, then P becomes $P/3$.

4. (a) Denote the number of instruments made per week by N and the corresponding cost by $£C$. Asume that C increases at a constant rate with N (i.e. C is proportional to N). Then a linear model for total costs ($£T$) is $T = 5000+50N$. This will be valid only for $0 \leq N \leq 100$.

   If $N = 80$, then $T = 5000 + 50 \times 80 = 9000$.

   (b) The predicted total cost is £9000.

4. Methods for calculating gradient

Occasionally you may be faced with two different pairs of values or *coordinates* with which to determine the parameters of a linear function. Put another way, two pairs of values are needed to determine the two (unknown) parameters. Perhaps, unconsciously, you might have used this result already when carrying out the rain barrel task. The gradient, written as $\dfrac{1.5}{20}$ in the answer, may be expressed also as $\dfrac{1.5-0}{20-0}$ since the line connects the (time, level of water) coordinates (20, 1.5) with (0, 0). In general the gradient is given by

$$\frac{\text{the change in the dependent variable}}{\text{the corresponding change in the independent variable}}$$

Once the gradient of the line has been calculated, it can be used with one of the known points to determine the intercept. If one of the points is (0, 0) the intercept is zero.

Suppose that a new type of automatic car is being road tested. The measuring team wants to know the maximum acceleration between 0 and 30 m s^{-1}. It plans to calculate this by assuming that it is constant and measuring the time taken from rest to achieve a speed of 30 m s^{-1} at maximum acceleration. In their first test the speedometer reading is 30 m s^{-1} after 12 s from start of timing and motion. We can think of these values in terms of (time, velocity) coordinates. At the start of timing the coordinates are $(0,0)$. When the speedometer reads 30 m s^{-1} the coordinates are $(12,30)$. If the acceleration is constant then its magnitude will be given by the gradient of the line joining these two points. Using the 'change in variable idea', the gradient is $\dfrac{30-0}{12-0}=2.5$, and so the magnitude of the acceleration is 2.5 m s^{-2}. The 'change in variable' route to calculating the gradient is an abridged version of a more general method. The two pairs of coordinates may be used with the general equation of a line to work out the parameters of the particular line that passes through these two points. The assumption of constant acceleration leads to a linear relationship between the velocity (v m s^{-1}) and time (t s) of the form $v = at + b$ where a and b are the parameters corresponding to gradient and intercept respectively. The road test gives $v =$ 0 when $t =$ 12. These may be substituted into the general form to give

$0 = 0 + b$ and $30 = 12a + b$.

You may recognise that these are simultaneous equations. The first gives $b = 0$ which may be substituted into the second to give $a =$ 2.5, corresponding to an acceleration of 2.5 m s^{-2} as before.

Suppose that the test team carry out a second test. In this test they note when speeds of 15 m s^{-1} and 27 m s^{-1} are reached and assume constant acceleration between these times and speeds. The speedometer reads 15 m s^{-1}, after 4 seconds from the start of motion and 27 m s^{-1} after 9 s from the start of motion. We apply the general method to the data from this test. The (time, velocity) coordinates corresponding to the readings are $(4,15)$ and $(9,27)$. The equations resulting from substitutions in the general form are

$$15 = 4a + b$$

$$27 = 9a + b$$

We use the elimination method of solving these simultaneous equations (HELM 3). The first of these equations may be subtracted from the second to eliminate b.

$$27 - 15 = 9a + b - 4a - b$$

or

$$a = 2.4.$$

The resulting value of a may be substituted into either of the equations expressing the data to calculate b. In the first, $15 = 4 \times 2.5 + b$, so $b = 5$. The resulting model is

$$v = 2.4t + 5 \qquad (4 \leq t \leq 9).$$

This model predicts an acceleration of 2.4 m s^{-2}, which is fairly close to the previous result of 2.5 m s^{-2} but if we try to use this model at $t =$ 0, what do we predict? The model predicts that $v = 5$ when $t = 0$. This is not consistent with $t = 0$ being the time at which the vehicle starts to move! So, even if the acceleration is constant between 15 and 27 m s^{-1}, it does not have the same values between 0 m s^{-1} and 15 m s^{-1} as either between 15 m s^{-1} and 27 m s^{-1} and 30 m s^{-1}. A more general principle is illustrated by this example. It may be dangerous to use a model based on certain data at points other than those given by these data! The business of using a model outside the range of data for which is is known to be valid is called **extrapolation**. Use of the model between the data points on which it is based is called **interpolation**. So the general principle may also be stated as that **it is very risky to extrapolate** and **it can be risky to interpolate**. Nevertheless extrapolation or interpolation may be part of the purpose for a mathematical model in the first place.

The method of finding gradient and intercept just exemplified may be generalised. Suppose that we are specifying a linear function $y = ax + b$ where the dependent variable is y and the independent variable is x. We represent two known points by (p, q) and (r, s). The gradient, a, for the straight line, may be calculated either from $\dfrac{p-r}{q-s}$ or by substituting $y = q$ when $x = r$ in $y = ax + b$ to obtain two simultaneous equations. Subtraction of these eliminates b and allows a to be calculated. The intercept of the line on the y-axis, b, may be found by substitution in $y = ax + b$, of either p, q and a or r, s and a.

Task

Use the general method to deduce the different accelerations (assuming that they are constant) between the start of motion and 15 m s^{-1} and between velocities of 27 m s^{-1} and 30 m s^{-1}.

Your solution

Answer

For the (time, velocity) coordinates $(0, 0)$ and $(4, 15)$,

$$0 = 0a + b$$

$$15 = 4a + b$$

From the first of these b = 0 and hence, in the second, $a = \dfrac{15}{4} = 3.75$. So the acceleration up to 15 m s^{-1} is 3.756 m s^{-2}. For the (time, velocity) coordinates $(9, 27)$ and $(12, 30)$,

$$27 = 9a + b$$

$$30 = 12a + b$$

Subtracting the first from the second gives

$$3 = 3a \text{ so } a = 1,$$

so the acceleration between 27 m s^{-1} and 30 m s^{-1} is 1 m s^{-2}

Linear functions may be useful in economics. A lot of attention is paid to the way in which **demand** for a product varies with its price. A measure of demand is the number of items sold, if available, in a given period. For example, the purpose might be to determine the best price for a product given certain details about costs and with certain assumptions about the way the number of items sold per month varies with price. The price affects the profit and hence, in turn, the number manufactured in response to the demand. The number of items manufactured in a given period is known as the **supply**. Information about the variation of demand or supply with price may be obtained from market surveys. Constant functions are not appropriate in this context since both demand and supply vary with price. In the absence of other information the simplest way to model the variation of either demand or supply with price is to use a linear function.

Task

When the price of a luxury consumer item is $£1000$, a market survey reveals that the demand is 100,000 items per year. However another survey has shown that at a price of $£600$, the demand for the item is 200,000 items per year. Assuming that both surveys are valid, find a linear function that relates demand Q to price P. What demand would be predicted by the linear function at a price of $£750$? Comment on the validity of both predictions.

Your solution

Answer

The linear function will be of the form

$$Q = aP + b \qquad (600 \leq P \leq 1000)$$

The limits on P represent the given range of data on price. Substituting the first pair of values of Q and P:

$$100000 = 100a + b$$

Substituting the second pair of values:

$$200000 = 600a + b$$

Subtracting the first expression from the second:

$$100000 = -400a \qquad \text{so} \qquad a = -250$$

Note that the negative gradient is consistent with the fact that demand falls as price increases. Check that the 'Change in variable' definition for finding a works.

Change in dependent variable $\quad (Q) = 200000 - 100000 = 100000.$

Corresponding change in independent variable $\quad (P) = 600 - 1000 = -400$. The ratio of these changes is $\dfrac{100000}{-250} = -250$

This value of a may be used with the first pair of values,

$$100000 = -250000 + b$$

so

$$b = 350000$$

and the linear function relating demand and price is

$$Q = 350000 - 250P.$$

[A precautionary check is to make sure that this result is consistent with the other pairs of values. When $P = 600,\ Q = 350000 - 250 \times 600 = 350000 - 150000 = 200000$, as required.] When P = 750:

$$Q = 350000 - 250 \times 750 = 350000 - 187500 = 162500.$$

So a linear relationship between demand and the price for this luxury suggests a demand of 162500 items per year when the price per item is £750. At a price of £500, P = 500, and the model predicts that

$$Q = -250 \times 500 + 350000 = 225000.$$

So the linear model suggests a demand of 225,000 items per year when the price per item is £500. Such a price however is outside the range of given data. Consequently the corresponding demand prediction represents an extrapolation and this might not be reliable. On the other hand, the price of £750 lies within the given range of data and the corresponding demand prediction is an interpolation. If the given data points are close to each other then interpolation between these points is more reliable than extrapolation to points further away.

Quadratic Functions and Modelling

5.2

Introduction

This Section describes forms of equations for quadratic functions (also called parabolas), ways in which quadratic functions can be used to model motion involving projectiles, and certain kinds of problem involving a single maximum or minimum.

Prerequisites

Before starting this Section you should ...

- be competent at algebraic manipulation
- be familiar with quadratic functions

Learning Outcomes

On completion you should be able to ...

- use quadratic functions to model motion under constant acceleration
- express the equation of a parabola in a general form

1. Quadratic functions

Quadratic functions and parabolas

Graphs of y against x resulting from quadratic functions (HELM 2.8, Table 1) are called **parabolas**. These take the general form: $y = ax^2 + bx + c$. The coefficients a, b and c influence the shape, form and position of the graph of the associated parabola. They are the **parameters** of the parabola. In particular the magnitude of a determines how wide the parabola opens (large a implies a narrow parabola, small a implies a wide parabola) and the sign of a determines whether the parabola has a lowest point (minimum) or highest point (maximum). Negative a implies a parabola with a highest point. The most useful form of equation for determing the graphical appearance of a parabola is $y - C = A(x - B)^2)$. To see the relation between this form and the general form simply expand:

$$y = Ax^2 - 2ABx + AB^2 + C$$

so, comparing with $y = ax^2 + bx + c$ we have:

$$a \equiv A, \quad b \equiv -2AB \quad c \equiv AB^2 + C$$

We deduce that the relation between the two sets of constants A, B, C and a, b, c is:

$$A = a \qquad B = -\frac{b}{2a} \quad \text{and} \quad C = c - \frac{b^2}{4a}$$

This new form for the parabola enables the coordinates of the highest or lowest point, known as the **vertex** to be written down immediately. The coordinates of the vertex are given by (B, C). Changing the value of B shifts the vertex, and hence the whole parabola, up or down. Changing the value of C shifts the vertex, and hence the whole parabola, to left or right.

Task

Assume the variation of an object's location with time is represented by a quadratic function:-

$$s = \frac{t^2}{9} \qquad (0 \leq t \leq 30)$$

Compare this function with the general form $y - C = A(x - B)^2$.

(a) What variables correspond to y and x in this case?

(b) What are the values of C, A and B?

Your solution

Answer

(a) s corresponds to y, and t corresponds to x (b) $C = 0, A = \dfrac{1}{9}$ and $B = 0$

2. Modelling with parabolas

The function

$$s = \frac{t^2}{9} \qquad (0 \leq t \leq 30)$$

is part of a parabola starting at the origin ($s = 0$ and $t = 0$) and rising to $s = 100$ at the end of its range of validity. s represents the distance of the object from the origin - N.B. Do not confuse this s with the symbol for seconds. 'Negative' time corresponds to time before the motion of the object is being considered. What would this parabolic function have predicted if it were valid up to 30 s before the 'zero' time? The answer to this can be deduced from the left-hand part of the graph of the function shown as a dashed curve, for in Figure 4, i.e. the part corresponding to $-30 \leq t \leq 0$.

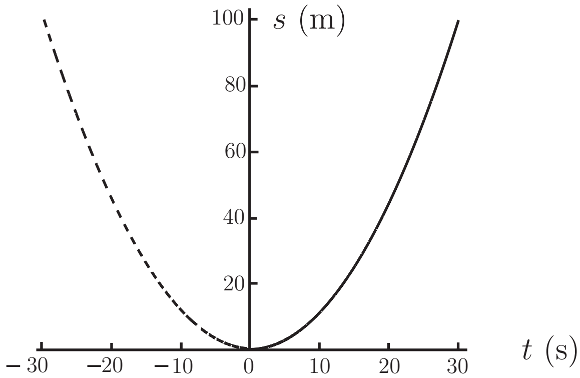


Figure 4: Graph of $s = \frac{t^2}{9}$ for $-30 \leq t \leq 30$

The parabolic form predicts that at $t = -30$, the object was 100 m away and for $(-30 \leq t \leq 0)$ it was moving towards the point at which the original timing started. The rate of change of position, or instantaneous velocity, is given by the gradient of the position-time graph. Since the gradient of the parabola for s is steeper near $t = -30$ than near $t = 0$, the chosen function for s and new range of validity suggests that the object was moving quickly at the start of the motion, slows down on approaching the initial starting point, and then moves away again accelerating as it does so. Note that the velocity (i.e. the gradient) for $(-30 \leq t \leq 0)$ is negative while for $(0 \leq t \leq 30)$ it is positive. This is consistent with the change in direction at $t = 0$.

We will consider falling objects again and return to the context of the thriller film and the villain on a cliff-tip dislodging a rock. Suppose that, as film director, you are considering a variation of the plot whereby, instead of the ground, the rock hits the roof of a vehicle carrying the hero and heroine. This means that you might be interested in the position as well as the velocity of the rock at any time. We can start from the linear function relating velocity and time for the dislodged rock,

$$v = 9.8t \qquad (0 \leq t \leq T)$$

where T represents the time at which the rock hits the roof of the vehicle. The precise value of T will depend upon the height of the vehicle. If s is measured *from the cliff-top* and timing starts with release of the rock, so that $s = 0$ when $t = 0$, the resulting function is

$$s = 4.9t^2 \qquad (0 \leq t \leq T)$$

(Note that $s = 4.9t^2$ is a particular case of a standard model for falling objects: $s = \frac{1}{2}gt^2$.)

Task

This Task refers to the model discussed above.

(a) What kind of function is $s = 4.9t^2$?

Your solution

Answer

Quadratic, or parabolic

(b) If the vehicle roof is 2 m above the ground and the cliff-top is 35 m above the ground, calculate a value for T, the time when a rock falling from the cliff-top hits the car roof:

Your solution

Answer

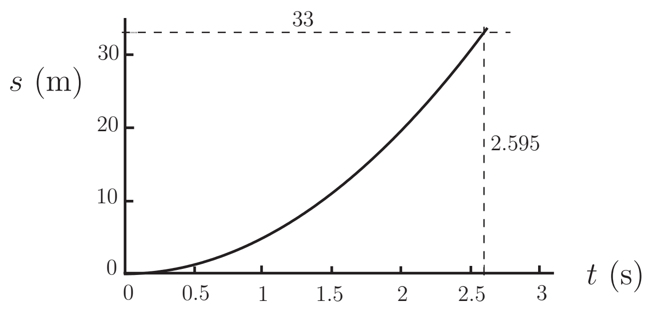
$t = T$ when $s = 35 - 2 = 33$

$$\Rightarrow \quad 33 = 4.9T^2 \qquad \text{so} \quad T = \sqrt{\frac{33}{4.9}} = 2.5951 \approx 2.6 \quad \text{(only positive } T \text{ makes sense)}$$

(c) Given this value for T sketch the function:

Your solution

Answer



In this modelling context, negative time would correspond to time before the villain dislodges the rock. It seems likely that the rock was stationary before this instant. The parabolic function would not be appropriate for $t \leq 0$ since it would predict that the rock was moving. An appropriate function would have two parts to its domain:

For $t \leq 0, s$ would be constant $(= 0)$ and for $0 \leq t \leq T,\ s = 4.9t^2$.

The corresponding graph would also have two parts:

A flat line along the $s = 0$ axis for $t \leq 0$ and part of a parabola for $0 < t \leq T$.

A different form of quadratic function for position is appropriate if position is measured **upwards** as height (h) above the ground below the cliff-top. This is given as

$$h = 35 - 4.9t^2 \qquad (0 \leq t \leq 2.6)$$

Note that once $t = 2.6$ then $h = 0$ and the rock cannot fall any further. When position is measured upwards, velocities and accelerations, which are downwards for falling objects, will be negative.

Task

This Task refers to the model discussed above.

By comparing $h = 35 - 4.9t^2$ with $y = ax^2 + bx + c$, deduce values for a, b and c and determine whether the parabola corresponding to this function has a highest or lowest point:

Your solution

Answer

Here h corresponds to y and t to x in the general form. The coefficient corresponding to a is $-4.9 \times b = 0$ and $c = 35$. The value of a is negative so the parabola opens downwards.

(b) Write down an appropriate function for the variation of h with t if height is measured upwards from the top of a 2 m high vehicle:

Your solution

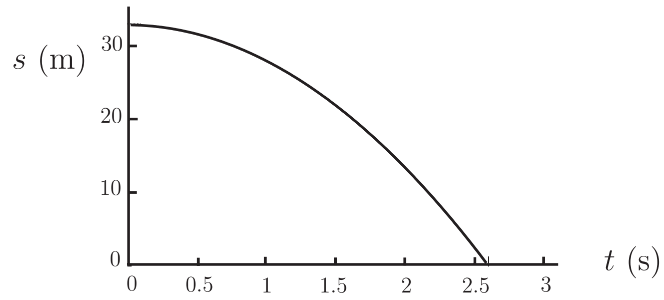
Answer

$h = 33 - 4.9t^2 \qquad (0 \leq t \leq 2.5951) = 2.60$ to 2 d.p.

(c) Sketch this function:

Your solution

Answer



Consider the situation in which position is measured **downwards** from the cliff-top again but the villain is lying down on the cliff-top and throws the rock **upwards** with speed 5 m s^{-1}. The distance it would travel in time t seconds if gravity were not acting would be $-5t$ metres (distance is speed multiplied by time but in the negative s direction in this case). To obtain the resulting distance in the presence of gravity we add this to the distance function $s = 4.9t^2$ that applies when the rock is simply dropped. The appropriate quadratic function for s is now

$$s = 4.9t^2 - 5t \qquad (0 \leq t \leq T)$$

The nature of this quadratic function means that for any given value of s there are **two** possible values of t. If we write the function in a slightly different way, taking out a common factor of t,

$$s = t(4.9t - 5) \qquad (0 \leq t \leq T)$$

it is possible to see that $s = 0$ at two different times. These are when $t = 0$ and when $4.9t - 5 = 0$. The first possibility is consistent with the initial position of the rock. The second possibility gives $t = \dfrac{5}{4.9}$ which is a little more than 1. The rock will be at the cliff-top level at two different times. It is there at the instant when it is thrown. It rises until its speed is zero and then descends, passing cliff-top level again on its way to impact with the ground below or with the vehicle roof. Since the initial motion of the rock is upwards and position is defined as positive downwards, the initial part of the rock's path corresponds to negative s. The parabola associated with the appropriate function crosses the $s = 0$ axis twice and has a vertex at which s is negative. A sketch of s against t for this case is shown in Figure 5.

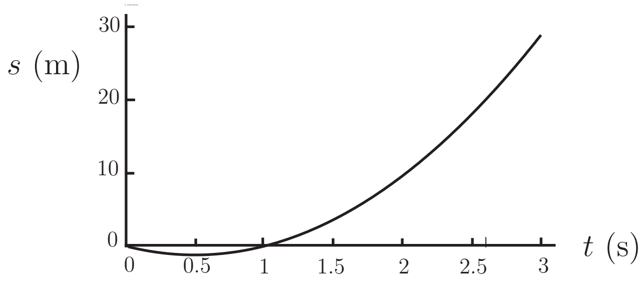


Figure 5: Graph of rock's position (measured downwards) when rock is thrown upwards

Task

For the above modelling of falling rocks, calculate how high the rock rises after being thrown upwards at 5 m s^{-1}. (Hint: use the previously determined value of the time when the rock reaches its highest point.)

Your solution

Answer

The value of t at which the rock's velocity is zero was worked out as $t = \dfrac{5}{9.8}$. This value can be used in the function for s to give

$$s = \frac{5}{9.8}(4.9 \times \frac{5}{9.8} - 5) = -\frac{2.5}{19.6} = -1.2755$$

So the rock rises to a little less than 1.28 m above the cliff-top.

Note that the form of the parabola makes it inevitable that, as long as it is plotted over a sufficiently wide range, and apart from its vertex, there will **always be two values** on the curve for each value of one of the variables. Which of these values makes sense in a mathematical model will depend on the modelling context. In each of the contexts mentioned so far in this Section each context has determined the part of the parabola that is of interest.

Note also that there is a connection between the vertex on a parabola and the point where the gradient of that parabola is zero. In fact these points are the same!

3. Parabolas and optimisation

Because the vertex may represent a highest or lowest point, a quadratic function may be the appropriate type of function to choose in a modelling problem where a maximum or a minimum is involved (optimisation problems for example). Consider the problem of working out the selling price for the product of a cottage industry that would maximise the profit, given certain details of costs and assumptions about market behaviour. A possible function relating profit $(£M)$ to selling price $(£P)$, is

$$M = -10P^2 + 320P - 2420 \qquad (12 \le P \le 20).$$

Note that this is a quadratic function. By comparing this function with the form $y = ax^2 + bx + c$ it is possible to decide whether the corresponding parabola that would result from graphing M against P, would open upwards or downwards. Here M corresponds to y and P to x. The coefficient corresponding to a in the general form is -10. This is negative, so the resulting parabola will open downwards. In other words it will have a **highest point** or **maximum** for some value of P. This is comforting in the context of an optimisation problem! We can go further in specifying the resulting parabola by reference to the other general form: $y - C = A(x - B)^2$. If we multiply out the bracket on the right hand side we get (as seen at the beginning of HELM 5.2)

$$y - C = Ax^2 - 2ABx + AB^2$$

or

$$y = Ax^2 - 2ABx + AB^2 + C.$$

Comparing this general form with the function relating profit and price for the cottage industry:

$$\begin{array}{rcccc} y = & Ax^2 & - 2ABx & + AB^2 + C \\ & \downarrow & \downarrow & \downarrow \\ M = & -10P^2 + & 320P & - 2420 \end{array}$$

Using the equivalances suggested by the arrows, we see that

$$A = -10,$$

$$2AB = -320$$

$$AB^2 + C = -2420.$$

These are three equations for three unknowns. Putting $A = -10$ in the second equation gives $B = 16$. Putting $A = -10$ and $B = 16$ in the third equation gives

$$-2560 + C = -2420,$$

and so

$$C = 140.$$

This means that the equation for M may also be written in the form

$$M - 140 = -10(P - 16)^2,$$

corresponding to the general form $y - C = A(x - B)^2$. In the general form, C corresponds to the value of y at the vertex of the parabola. Since y in the general form corresponds to M in the current modelling context, we deduce that $M = 140$ at the highest point on the parabola. B represents the value of x at the lowest or highest point of the general parabola. Here x corresponds to P, so we deduce thet $P = 16$ at the vertex of the parabola corresponding to the function relating profit and price. These deductions mean that a maximum profit of £140 is obtained when the selling price is £16.

4. Finding the equation of a parabola

Consider a parabola that has its vertex at $s = 50$ when $t = 0$ and rises to $s = 100$ when $t = 30$. In coordinate terms, we need the equation of a parabola that has its lowest point or vertex at (0, 50) and passes through (30, 100). The general form

$$y - C = A(x - B)^2$$

is useful here.
In this case y corresponds to s and x to t. So the equation relating s and t is

$$s - C = A(t - B)^2$$

According to the general form, the coordinates of the vertex are (B, C). We know that the coordinates of the vertex are (0, 50). So we can deduce that $B = 0$ and $C = 50$. It remains to find A. The fact that the parabola must pass through (30, 100) may be used for this purpose. These values together with those for B and C may be substituted in the general equation:

$$100 - 50 = A(30 - 0)^2$$

so $50 = 900A$ or $A = \dfrac{1}{18}$ and the function we want is

$$s = 50 + \frac{1}{18}t^2 \qquad (0 \leq t \leq 30)$$

Task

Find the equation of a parabola with vertex at $(0, 2)$ and passing through the point $(4, 4)$.

Your solution

Answer

Using the general form, with $B = 0$ and $C = 2$,

$$y - 2 = A(x - 0)^2 \text{ or } y - 2 = Ax^2$$

Then using the point $(4, 4)$

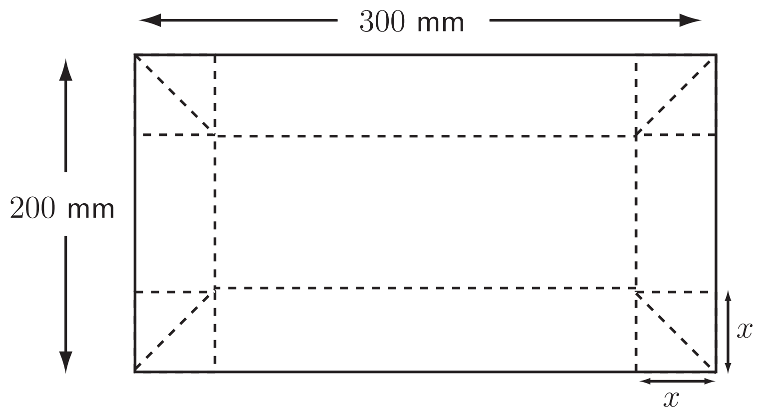
$$4 - 2 = 16A \qquad \text{so} \qquad A = \frac{2}{16} = \frac{1}{8}$$

and the required equation is

$$y = 2 + \frac{1}{8}x^2$$

Exercise

An open-topped carton is constructed from a 200 mm $\times$ 300 mm sheet of cardboard, using simple folds as shown in the diagram.



Cardboard folds to make an open-topped carton

(a) Show that the volume of the carton (in cm^3) is

$$V = \frac{x(300-2x)(200-2x)}{1000}$$

so $\quad V = \dfrac{x^3}{250} - x^2 + 60x \qquad \ldots(*)$

(b) Sketch Equation (1) as V vs x and hence estimate the maximum volume of carton that may be obtained by folding the cardboard sheet.

(c) A carton with a volume of 1000 cm^3 is to be made from the cardboard sheet.

(i) Show that one solution is to use a height $x = 50$ mm.

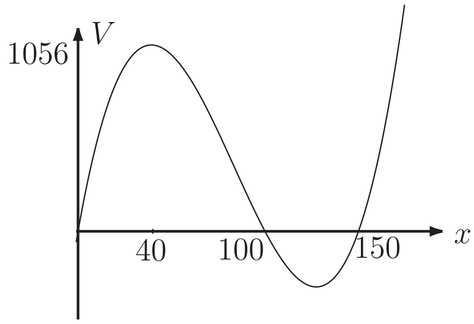
(ii) By factorisation of Equation (*) for $V = 1000$ cm^3, find a second solution for x which would give the same carton volume.

(iii) Why does the third root have no physical meaning?

Answer

(a) $$V = \frac{x(300-2x)(200-2x)}{1000} = \frac{x^3}{250} - x^2 + 60x \quad (\text{cm}^3)$$

(b)



$V_{\text{max}} \approx 1056$ cm^3 when $x \approx 39.2$

(c) (i) $x = 50$ mm $\Rightarrow$ $V = 1000$ cm^3 as required.

(ii) $$\frac{x^3 - 250x^2 + 15000x}{250} - 1000 = 0$$ factorises to

$$(x-50)(x^2 - 200x + 5000) = 0$$

so $x = 50$ or $x = 100 \pm 10\sqrt{50} \approx 29.3$ or 170.7. The second root is 29.3.

(iii) The third root 170.7 is impossible as $200 - 2x$ must be a positive distance.

Oscillating Functions and Modelling

5.3

Introduction

This Section describes ways in which trigonometric functions can be used to model situations involving periodic motion, which occur in a wide variety of scientific and engineering situations, and in nature.

Prerequisites

Before starting this Section you should . . .

- be competent at algebraic manipulation
- be familiar with trigonometric functions

Learning Outcomes

On completion you should be able to . . .

- use trigonometric functions to model periodic motion
- define terms associated with the description of periodic motion

1. Oscillating functions: amplitude, period and frequency

Particular types of periodic functions (HELM 2.2) that are especially important in engineering are the sine and cosine functions. These are possible choices when modelling behaviour that involves oscillation or motion in a circle. The usefulness of these functions is rather limited if we confine our attention only to sin(x) and cos(x). Use of functions such as 3sin$(2x)$, 5cos$(3x)$ and so on, and other functions made up of sums of functions of this type, enables the modelling of a great variety of situations where the quantity being modelled is known to change in a periodic way. Here we will examine the behaviour of sine and cosine functions and consider a modelling context where choice of a sine function is appropriate. Figure 6 shows how the terms **amplitude**, **period** and **frequency** are defined with respect to a general **sinusoid** (the name for any general sine or cosine function).

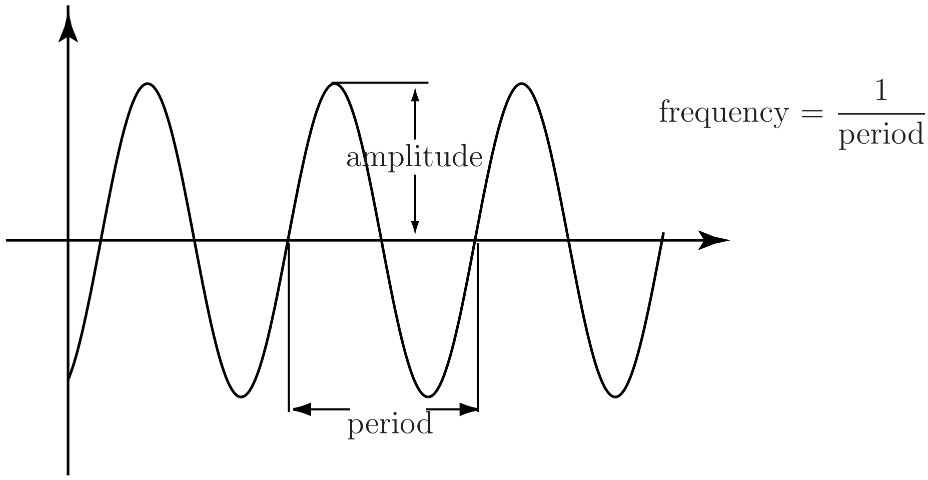


Figure 6: Defining amplitude and period for a sinusoid

The amplitude represents the difference between the maximum (or minimum) value of a sinusoidal function and its mean value (which is zero in Figure 6). The frequency represents the number of complete cycles of the function in each unit change in x. The period is such that $f(x+T) = f(x)$ for all x, e.g. for $\sin x$, $T = 2\pi$.

Example 2

Sketch the sinusoids:

(a) $y = \sin x$ (b) $y = 2\sin x$ (c) $y = \cos x$ (d) $y = \cos \dfrac{x}{2}$

Solution

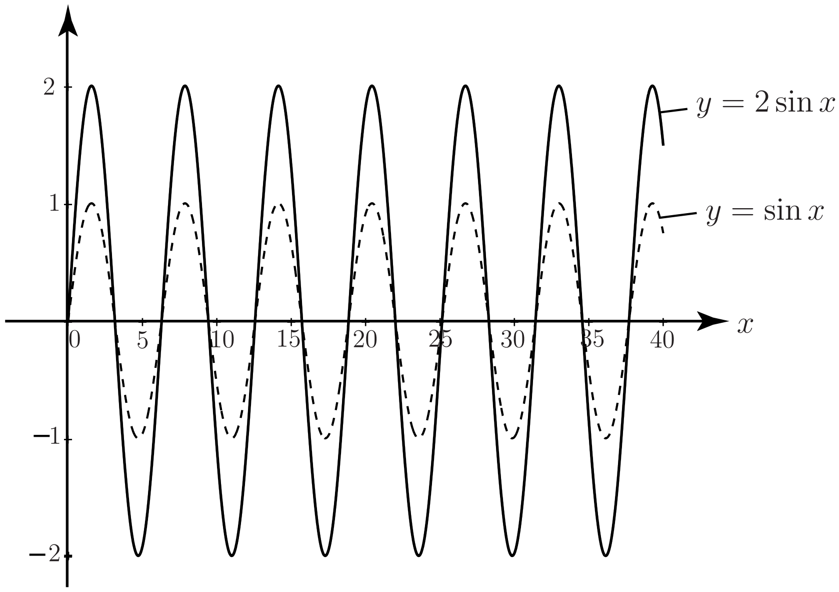


Figure 7

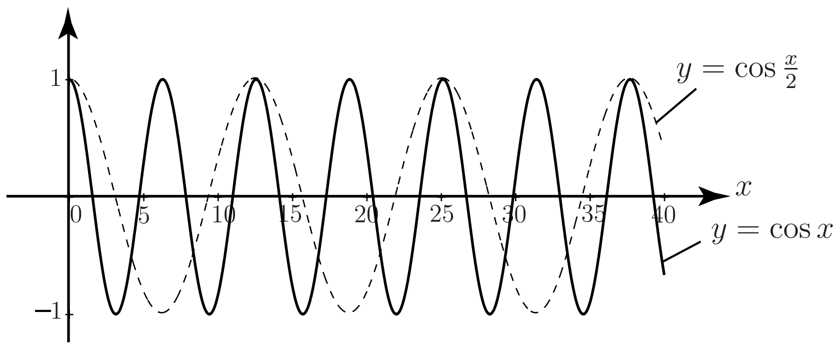


Figure 8

Task

Using the graphs in Figures 7 and 8 on page 37, state the amplitude, frequency and period of

(a) $\sin x$ (b) $2\sin x$ (c) $\cos x$ (d) $\cos\dfrac{x}{2}$

Give frequency and period in terms of π.

Your solution

Answer

(a) amplitude $= 1$, frequency $= 1/2\pi$, period $= 2\pi$.

(b) amplitude $= 1$, frequency $= 1/2\pi$, period $= 2\pi$.

(c) amplitude $= 2$, frequency $= 1/2\pi$, period $= 2\pi$.

(d) amplitude $= 1$, frequency $= 1/4\pi$, period $= 4\pi$.

See Figure 7 for the sine functions and Figure 8 for the cosine functions.

Note that (b) has twice the amplitude of (a) and (d) has half the frequency and twice the period of (c).

Note that the cosine functions $\cos nx$ have the same shape as the sine functions $\sin nx$ but, at $x = 0$, the cosine functions have a peak or maximum, whereas the sine functions have the value zero, which is the mean value for both of these functions. Indeed the graph of $y = \cos x$ is exactly like that for $y = \sin x$ with all the x values displaced by $\pi/2$.

More general forms of sine and cosine function are given by $y = a\sin(bx)$, and $y = a\cos(bx)$ where a and b are arbitary constants. These are functions with frequency $\dfrac{b}{2\pi}$, period $\dfrac{2\pi}{b}$ and amplitude a. The peak values of the sine functions occur at x values equal to $\dfrac{\pi}{2}$, $\dfrac{5\pi}{2}$, $\dfrac{9\pi}{2}$ etc. The minimum values occur at x values equal to $\dfrac{3\pi}{2}$, $\dfrac{7\pi}{2}$, $\dfrac{11\pi}{2}$ etc.

When the period is measured in seconds, frequency is measured in cycles per second or Hz which has units of 1/time.

Exercises

1. Figure 7 on page 37 shows on the same axes the graphs of $y = \sin x$ and $y = 2\sin x$.

   (a) State in words how the graph of $y = 2\sin x$ relates to the graph of $y = \sin x$

   (b) Sketch the graphs of (i) $y = \dfrac{1}{2}\sin x$, (ii) $y = \dfrac{1}{2}\sin x + \dfrac{1}{2}$

2. Figure 8 on page 37 shows on the same axes the graph $y = \cos x$ and $y = \cos\dfrac{x}{2}$

   (a) State in words how the graph of $y = \cos x$ relates to the graph of $y = \cos\dfrac{x}{2}$

   (b) Sketch graphs of (i) $y = \cos 2x$, (ii) $y = 2\cos x$

Answers

1. $y = \sin 2x$ has the same form as $y = \sin x$ but all the y values are doubled. The graph is 'stretched' vertically.

2. $y = \cos\dfrac{x}{2}$ has the same form as $y = \cos x$ but all the y values are halved. The graph is 'shrunk' vertically.

2. Oscillating functions: modelling tides

We consider how the function

$$h = 3.2\sin(2.7t + 8.5)$$

might be used to model the rise and fall of the tide in a harbour.
Figure 9 shows a graph of this function for $(0 \geq t \geq 5)$.

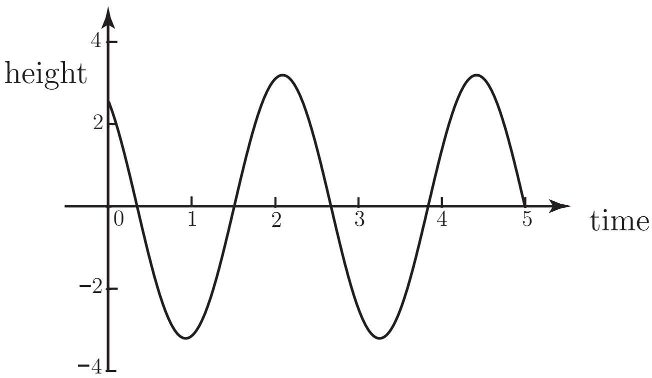


Figure 9

We consider some aspects of this graph and model. It seems reasonable to suppose that the tide creates an oscillation of the water level in the harbour of hm about some mean value represented on the graph by $h = 0$. There seems to be a low tide near $t = 1$ and another low tide just after $t = 3$. Since we expect intervals of 12 to 14 hours between low tides around the U.K., this suggests that time in this graph is specified in 6-hour intervals.

Write down the amplitude, period and frequency of $h = 3.2\sin(2.7t + 8.5)$

Your solution

Answer

The amplitude of the change in water level in the harbour is 3.2 m. The period of the function is given by $2\pi/2.7 = 2.3271$ between successive high tides or successive low tides. This corresponds to 2.3271×6 hours $= 13.96$ hours between high tides.The frequency of the function is $2.7/2\pi = 0.4297$.

The peak levels of the graph correspond to times when the sine function has the value 1. The lowest points correspond to times when the sine function is -1. At these times the arguments of the sine function (i.e. $2.7t + 8.5$) are an odd number of $\pi/2$ starting at $3\pi/2$ for the first low tide.

So far all of this may be deduced from the general form $y = a\sin(bx)$ and from the modelling context. However there is an additional term in the function being considered here. This is a constant 8.5 within the sine function. When $t = 0$ the presence of this constant means that the intercept on the height axis is 3.2sin$(8.5) =$ 2.56, implying that the water level is 2.56 m above the mean value at the start of timing. The constant 8.5 has displaced the sine curve sideways. This constant is known as the **phase** of the function. Phase is measured in radians as it is an angle.

As remarked earlier, at $t = 0$, this function has the value $3.2\sin(8.5)$. Since $\sin(8.5) = \sin(8.5 - 2\pi) \approx \sin(2.2168)$, we can replace the constant 8.5 by 2.2168 without altering the values on the graph. This means that the function

$$h = 3.2\sin(2.74t + 2.2168)$$

does just as well as the original function in representing the tidal variation in the harbour. We now rewrite this latest form of the function, representing the variation of water level in the harbour, so that time is measured in hours rather than in six-hourly invervals. The effect of changing the units of time to hours from 6 hours is to decrease the coefficient of t in the sine function by a factor of 6, so that the new function is

$h = 3.2\sin(0.45t + 2.168)$. See Figure 10.

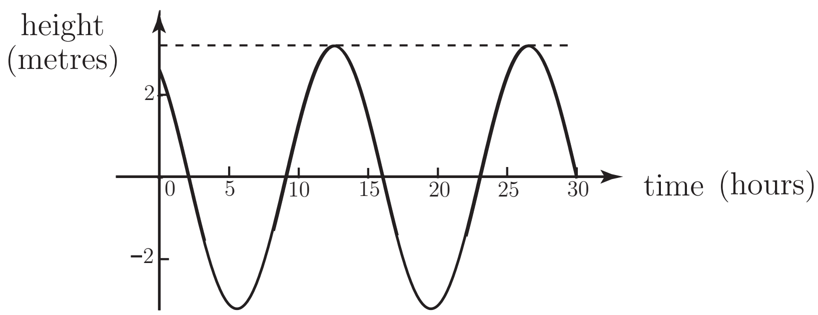


Figure 10

We can use the latest form of the function to calculate the time of the first low tide assuming that $t = 0$ corresponds to midnight.

At the first low tide, $h = -3.2$ and $\sin(0.45t + 2.2168) = -1$,

Using the fact that $\sin(\frac{3\pi}{2}) = -1$, we have

$$0.45t + 2.2168 = 3\pi/2, \text{ giving } \quad t = 5.5458 = 5.55 \text{ to 2 d.p.}$$

so the first low tide is at 5:30 a.m.

For the above tide modelling situation, assume that $t = 0$ corresponds to midnight. Calculate

(a) the time of the first high tide after midnight

(b) the times either side of midnight at which the water is at its mean level.

Your solution

Answer

(a) At the first high tide, $h = 3.2$ and $\sin(0.45t + 2.2168) = 1$, so $0.45t + 2.2168 = 5\pi/2$ giving $t = 12.5271$ so the first high tide is at half past midday

(b) When the water level is at the mean value,

$$\sin(0.45t + 2.2168) = 0.$$

At the mean level before midnight, using the fact that $\sin(0) = 0$ we have

$$0.45t = -2.2168 \quad \text{so} \quad t = -4.9262 = -4.93 \text{ to 2 d.p.}$$

So this mean level occurs nearly 5 hours before midnight, i.e. about 7 p.m. the previous day. The next mean level will occur one period, or 13.963 hours, later, at approximately 9 a.m.

There are various rules connected with sine and cosine functions that can be summarised at this point.

(1) Placing a multiplier before $\sin x$ or $\cos x$ (e.g. $2\sin x$) changes the amplitude without changing the period.

(2) Placing a multiplier before x in $\sin x$ or $\cos x$, (e.g. $\sin 3x$), changes the period or frequency without changing the amplitude.

(3) As with any function, the addition of a constant (e.g. $4 + \sin x$) raises or lowers the whole graph of the sine or cosine function. It alters the mean value without changing the amplitude.

(4) Changing the sign within a cosine function has no effect, (e.g. $\cos(-x) = \cos x$).

(5) Changing the sign within a sine function changes the sign of the function, (e.g. $\sin(-x) = -\sin x$).

(6) Placing a constant or altering the constant b in $\sin(ax + b)$ or $\cos(ax + b)$ changes the phase and shifts the sine or cosine function along the x-axis.

Task

(a) Write down the amplitude and period of $y = \sin(3x)$

(b) Write down the amplitude and frequency of $y = 3\sin(2x)$

(c) Write down the amplitude, period and frequency of $y = a\sin(bx)$

(d) Write down the amplitude, period, frequency and phase of

$$y = 4\sin(2x + 7).$$

(e) Write down an equivalent expression to that in (d) but with the phase less than 2π.

Your solution

Answer

(a) amplitude $= 1$ period $= 2\pi/3$

(b) amplitude $= 3$ frequency $= \dfrac{2}{2\pi} = \dfrac{1}{\pi}$

(c) amplitude $= a$ period $= 2\pi/b$ frequency $= b/2\pi$

(d) amplitude $= 4$ period $= \dfrac{2\pi}{2} = \pi$ frequency $= \dfrac{1}{\pi}$ phase $= 7$

(e) $y = 4\sin(2x + 7 - 2\pi) = 4\sin(2x + 0.7168)$

Task

Write down a function relating water level (L m) in a harbour to time (T hours), starting when the level is equal to the mean level of 5 m, that has an amplitude of 2 m and has a period of twelve hours.

Your solution

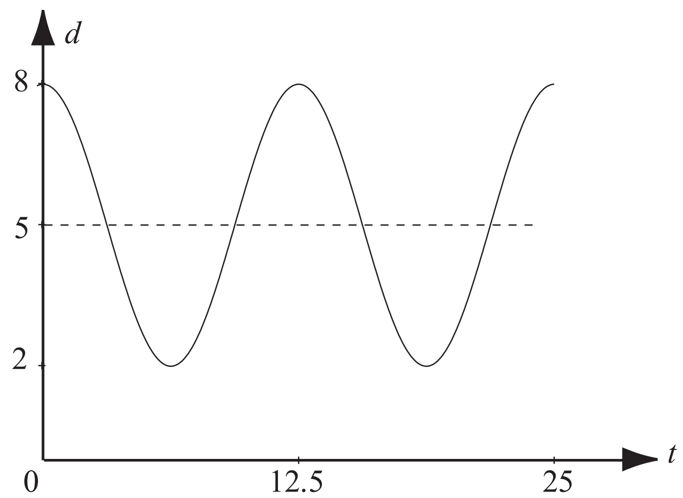
Answer

In the general form $y = a\sin(bx + c) + d$, the phase $c = 0$, the period $\dfrac{2\pi}{b} = 12$, so $b = \dfrac{\pi}{6}$ the amplitude $a = 2$, the mean value $d = 5$.

$L = 2\,\sin(\dfrac{\pi}{6}T) + 5 \qquad (T \geq 0)$

Task

The diagram shows a graph of a typical variation of the depth (d metres) of water in a particular harbour with time (t hours) as the depth changes with the tide.



(a) Find a suitable equation for the curve in the diagram:

Your solution

Answer

Equation is of the form

$$h = a + b\cos(\omega t) \qquad \text{(or} \quad h = a + b\sin(\omega t + \frac{\pi}{2}))$$

By inspection, $a = 5$ and $b = 3$.

The period $T = 12.5 = \dfrac{2\pi}{\omega}$ so $\omega = \dfrac{4\pi}{25}(= 0.502655)$

so the equation of the curve is $h = 5 + 3\cos(\dfrac{4\pi}{25}t)$

(b) A boat enters the harbour in late morning on a day when the high tide is at 2 p.m. The boat needs a water depth of 4 m to sail safely. What advice would you give to its pilot about when to leave the harbour if the boat is not to be forced to wait in the harbour through the evening low tide?

Your solution

Answer

Put $h = 4$ into the equation:

$$4 = 5 + 3\cos(\frac{4\pi}{25}t) \quad \text{implying} \quad -\frac{1}{3} = \cos(\frac{4\pi}{25}t)$$

Now, inverting the cosine:

$$\frac{4\pi}{25}t = \cos^{-1}(-\frac{1}{3}) = 1.91063 \;\; \text{giving} \;\; t = 3.80108 \text{ hours.}$$

So the advice to the pilot should be that he needs to be clear of the harbour by 5:45 pm at the very latest - and that he should allow a safety margin.

(c) State two modelling assumptions you have made:

Your solution

Answer

Assumptions likely are:

The tide on the day in question is *typical*.
No waves.
A sinusoidal function accurately models the effect of the tide on sea level.

Inverse Square Law Modelling

5.4

Introduction

This Section describes how functions involving a constant numerator and a squared variable denominator can be used in adding sound energies of different sources.

Prerequisites

Before starting this Section you should ...

- be competent at algebraic manipulation
- be familiar with polynomial functions
- be able to use Pythagoras' theorem
- be able to use the formula for solving quadratics

Learning Outcomes

On completion you should be able to ...

- model inverse square problems
- use a graphical method to solve a quadratic equation

1. Introduction

Many aspects of physics and engineering involve inverse square law dependence. For example gravitational forces and electrostatic forces vary with the inverse square of distance from the mass or charge. The following short case study illustrates this and concerns the dependence of sound intensity on distance from a source.

Engineering Example 1

Sound intensity

Introduction

For a single source of sound power W (watts) the dependence of sound intensity magnitude I (W m^{-2}) on distance r (m) from a source is expressed as

$$I = \frac{W}{4\pi r^2}$$

The way in which sounds from different sources are added depends on whether or not there is a phase relationship between them. There will be a phase relationship between two loudspeakers connected to the same amplifier. A stereo system will sound best if the loudspeakers are in phase. The loudspeaker sources are said to be **coherent** sources. Between such sources there can be reinforcement or cancellation depending on position. Usually there is no phase relationship between two separate items of industrial equipment. Such sources are called **incoherent**. For two such incoherent sources A and B the combined sound intensity magnitude (I_C W m^{-2}) at a specific point is given by the sum of the magnitudes of the intensities due to each source at that point. So

$$I_C = I_A + I_B = \frac{W_A}{4\pi r_A^2} + \frac{W_B}{4\pi r_B^2}$$

where W_A and W_B are the respective sound powers of the sources; r_A and r_B are the respective distances from the point of interest. Note that sound intensity is directional. So if A and B are on opposite sides of the receiver's position their intensity contributions will have opposite directions.

Problem in words

With reference to the situation shown in Figure 11, given incoherent point sources A and B, with sound powers 1.9 W and 4.1 W respectively, 6 m apart, find the sound intensity magnitude at points C and D at distances p and q from the line joining A and B and find the locations of C, D and E that correspond to sound intensity magnitudes of 0.02, 0.06 and 0.015 W m^{-2} respectively.

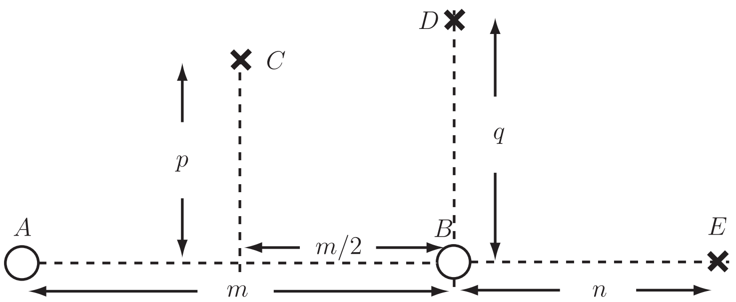


Figure 11

Mathematical statement of problem

(a) Write down an expression for the sound intensity magnitudes at point C due to the independent sources A and B with powers W_A and W_B, taking advantage of the symmetry of their locations about the line through C at right-angles to the line joining A and B.

(b) Find the expression for p in terms of I_C, W_A, W_B and m.

(c) If $W_A = 1.9$ W, $W_B = 4.1$ W and $m = 6$m calculate the distance p at which the sound intensity is 0.02 W m^{-2}?

(d) Find an expression for the intensity magnitude at point D.

(e) Find the value for q such that the intensity magnitude at D is 0.06 W m^{-2} and the other values are as in part (c).

(f) Find an equation in powers of n relating I_E, (intensity magnitude at point E) W_A, W_B, n and m.

(g) By plotting this function for $I_E = 0.015$ W m^{-2} , $m = 6$ m, $W_A = 1.9$ W, $W_B = 4.1$ W, find the corresponding values for n.

Mathematical analysis

(a) The combined sound intensity magnitude I_C W m^{-2} is given by the sum of the intensity magnitudes due to each source at C. Because of symmetry of the position of C with respect to A and B, write $|\overrightarrow{AC}| = |\overrightarrow{BC}| = r$, then

$$I_C = I_A + I_B = \frac{W_A}{4\pi r_A^2} + \frac{W_B}{4\pi r_B^2} = \frac{W_A + W_B}{4\pi r^2}$$

Using Pythagoras' theorem,

$$r^2 = \left(\frac{m}{2}\right)^2 + p^2 \quad \text{hence} \quad I_C = \frac{W_A + W_B}{4\pi((m/2)^2 + p^2)} = \frac{W_A + W_B}{\pi(m^2 + 4p^2)}$$

(b) Making p the subject of the last formula,

$$p = \pm\frac{1}{2}\sqrt{\left(\frac{W_A + W_B}{\pi I_C}\right) - m^2}$$

The result that there are two possible values of p is a consequence of the symmetry of the sound field about the line joining the two sources. The positive value gives the required location of C above the line joining A and B in Figure 11. The negative value gives a symmetrical location 'below' the line.

Note also that if $0 = \dfrac{W_A + W_B}{\pi I_C} - m^2$ or $I_C = \dfrac{(W_A + W_B)}{\pi m^2}$, then $p = 0$, i.e. C would be on the line joining A and B.

(c) Using the given values, $p = 3.86$ m.

(d) Using Pythagoras' theorem again, the distance from A to D is given by $\sqrt{q^2 + m^2}$. So

$$I_D = I_A + I_B = \frac{W_A}{4\pi r_A^2} + \frac{W_B}{4\pi r_B^2} = \frac{W_A}{4\pi(q^2 + m^2)} + \frac{W_B}{4\pi q^2}$$

(e) Multiplying through by $4\pi q^2(q^2+m^2)$ and collecting together like powers of q produces a quartic equation,

$$4\pi I_D q^4 + [4\pi m^2 I_D - (W_A + W_B])q^2 - W_B m^2 = 0.$$

Since the quartic equation contains only even powers of q, it can be regarded as a quadratic equation in q^2 and this can be solved by the standard formula. Hence

$$q^2 = \frac{-[4\pi m^2 I_D - (W_A + W_B)] \pm \sqrt{[4\pi m^2 I_D - (W_A + W_B)]^2 + 16\pi I_D W_B m^2}}{8\pi I_D}$$

Using the given values, $\quad q^2 = \dfrac{-21.14 \pm 29.87}{1.51}$

Since q must be real, the negative result can be ignored. Hence $q \approx 2.40$ m.

(f) Using the same procedure as in (d) and (e),

$$I_E = I_A + I_B = \frac{W_A}{4\pi r_A^2} + \frac{W_B}{4\pi r_B^2} = \frac{W_A}{4\pi(m+n)^2} + \frac{W_B}{4\pi n^2}$$

$$4\pi I_D n^2(m+n)^2 I_E = W_A n^2 + (m+n)^2 W_B = 0$$

A general expression for the distance n at which the intensity at point E is I_E is given by collecting like powers of n and is another quartic equation, i.e.

$$4\pi I_E n^4 + 8\pi I_E m n^3 + [4\pi I_E m^2 - (W_A + W_B)]n^2 - 2mW_B n - m^2 W_B = 0$$

Unfortunately this cannot be treated simply as a quadratic equation in n^2 since there are terms in odd powers of n. One way forward is to plot the curve corresponding to the equation after substituting the given values, another is to use a numerical method such as Newton-Raphson.

(g) Substitution of the given values produces the equation

$$0.1885n^4 + 2.2619n^2 + 0.7858n^2 - 49.2n - 147.6 = 0.$$

The plot of the quartic equation in Figure 12 shows that there are two roots of interest. Use of a numerical method for finding the roots of polynomials gives values of the roots to any desired accuracy i.e. $n \approx 4.876$ m and $n \approx -9.628$ m.

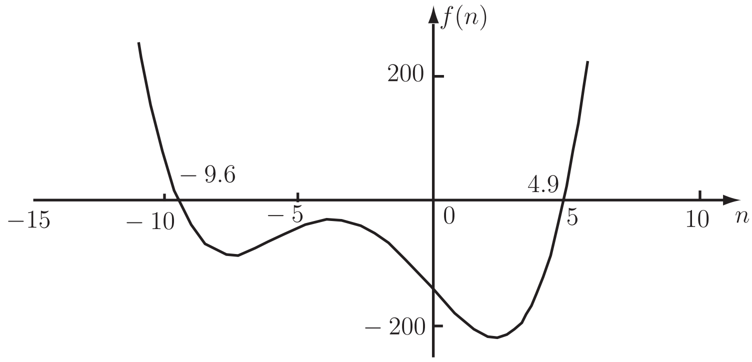


Figure 12

Interpretation

The result for part (g) implies that there are two locations for E along the line joining the two sources where the intensity magnitude will have the given value. One position is about 3.6 m to the left of source A and the other is about 4.9 m to the right of source B.

NOTES

Index for Workbook 5

Workbook 5

HELM

HELM: Helping Engineers Learn Mathematics

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