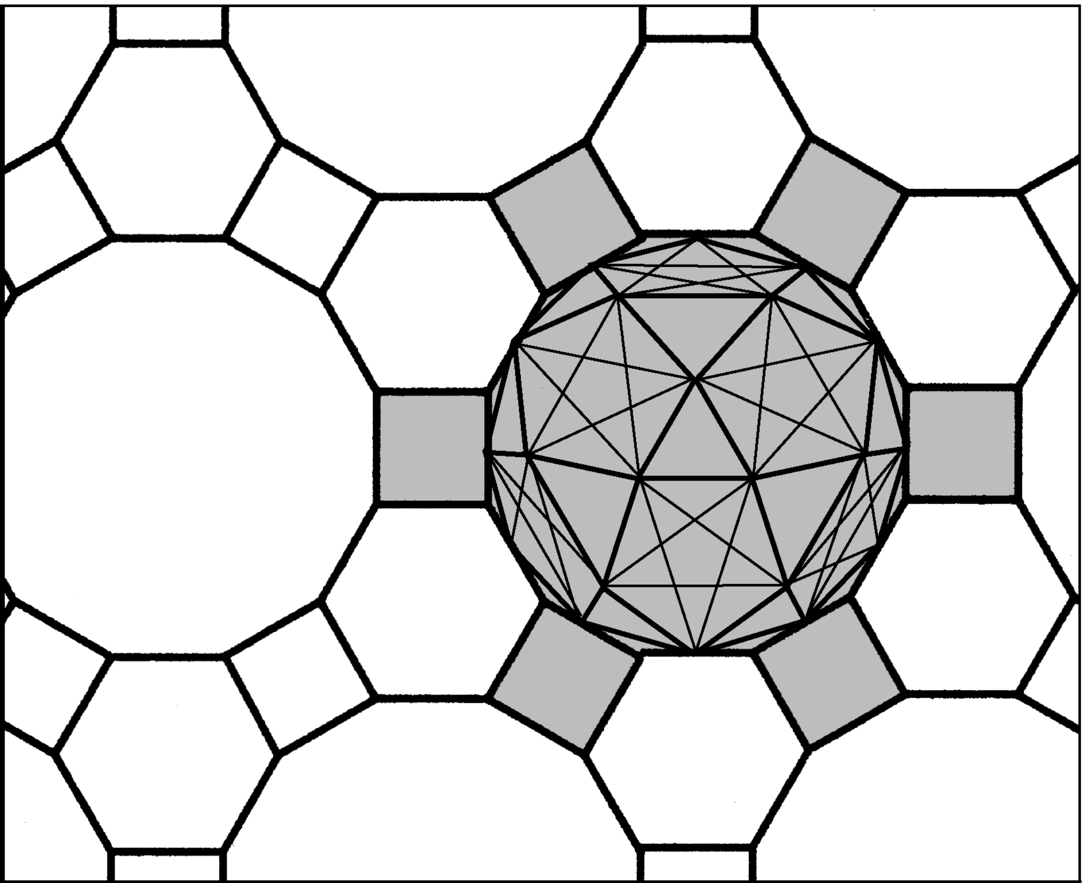
Workbook 15



Applications of Integration 2

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

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HELM Contacts:
Post: HELM, Mathematics Education Centre, Loughborough University, Loughborough, LE11 3TU.
Email: helm@lboro.ac.uk *Web:* http://helm.lboro.ac.uk

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Contents 15

Applications of Integration 2

Learning outcomes

In this Workbook you will learn to interpret an integral as the limit of a sum. You will learn how to apply this approach to the meaning of an integral to calculate important attributes of a curve: the area under the curve, the length of a curve segment, the volume and surface area obtained when a segment of a curve is rotated about an axis. Other quantities of interest which can also be calculated using integration is the position of the centre of mass of a plane lamina and the moment of inertia of a lamina about an axis. You will also learn how to determine the mean value of an integal.

Integration of Vectors

15.1

Introduction

The area known as vector calculus is used to model mathematically a vast range of engineering phenomena including electrostatics, electromagnetic fields, air flow around aircraft and heat flow in nuclear reactors. In this Section we introduce briefly the integral calculus of vectors.

Prerequisites

Before starting this Section you should ...

- have a knowledge of vectors, in Cartesian form
- be able to calculate the scalar product of two vectors
- be able to calculate the vector product of two vectors
- be able to integrate scalar functions

Learning Outcomes

On completion you should be able to ...

- integrate vectors

1. Integration of vectors

If a vector depends upon time t, it is often necessary to integrate it with respect to time. Recall that $\underline{i}$, $\underline{j}$ and $\underline{k}$ are constant vectors and must be treated thus in any integration. Hence the integral,

$$\underline{I} = \int (f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k})\, dt$$

is evaluated as three scalar integrals i.e. $\underline{I} = \left(\int f(t)\, dt\right)\underline{i} + \left(\int g(t)\, dt\right)\underline{j} + \left(\int h(t)\, dt\right)\underline{k}$

Example 1

If $\underline{r} = 3t\underline{i} + t^2\underline{j} + (1+2t)\underline{k}$, evaluate $\displaystyle\int_0^1 \underline{r}\, dt$.

Solution

$$\int_0^1 \underline{r}\, dt = \left(\int_0^1 3t\, dt\right)\underline{i} + \left(\int_0^1 t^2\, dt\right)\underline{j} + \left(\int_0^1 (1+2t)\, dt\right)\underline{k}$$

$$= \left[\frac{3t^2}{2}\right]_0^1 \underline{i} + \left[\frac{t^3}{3}\right]_0^1 \underline{j} + \left[t+t^2\right]_0^1 \underline{k} = \frac{3}{2}\underline{i} + \frac{1}{3}\underline{j} + 2\underline{k}$$

Trajectories

To simplify the modelling of the path of a body projected from a fixed point we usually ignore any air resistance and effects due to the wind. Once this initial model is understood other variables and effects can be introduced into the model.

A particle is projected from a point O with velocity $\underline{u}$ and an angle θ above the horizontal as shown in Figure 1.

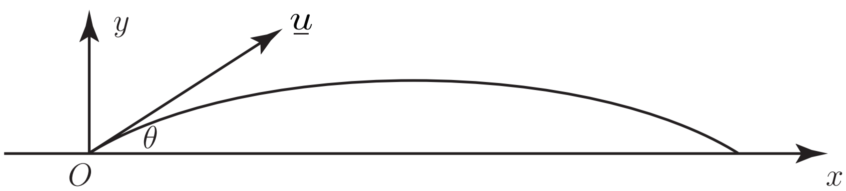


Figure 1

The only force acting on the particle in flight is gravity acting downwards, so if m is the mass of the projectile and taking axes as shown, the force due to gravity is $-mg\underline{j}$. Now using Newton's second law (rate of change of momentum is equal to the applied force) we have

$$\frac{d(m\underline{v})}{dt} = -mg\underline{j}$$

Cancelling the common factor m and integrating we have

$\underline{v}(t) = -gt\underline{j} + \underline{c}$ where $\underline{c}$ is a **constant vector**.

However, velocity is the rate of change of position: $\underline{v}(t) = \dfrac{d\underline{r}}{dt}$ so

$$\frac{d\underline{r}}{dt} = -gt\underline{j} + \underline{c}$$

Integrating once more:

$$\underline{r}(t) = -\frac{1}{2}gt^2\underline{j} + \underline{c}t + \underline{d} \text{ where } \underline{d} \text{ is another constant vector.}$$

The values of these constant vectors may be determined by using the **initial conditions** in this problem: when $t = 0$ then $\underline{r} = \underline{0}$ and $\underline{v} = \underline{u}$. Imposing these initial conditions gives

$\underline{d} = \underline{0}$ and $\underline{c} = u\cos\theta\underline{i} + u\sin\theta\underline{j}$ where u is the magnitude of $\underline{u}$. This gives

$$\underline{r}(t) = ut\cos\theta\underline{i} + (ut\sin\theta - \frac{1}{2}gt^2)\underline{j}.$$

The interested reader might try to show why the path of the particle is a parabola.

Exercises

1. Given $\underline{r} = 3\sin t\ \underline{i} - \cos t\ \underline{j} + (2-t)\underline{k}$, evaluate $\displaystyle\int_0^{\pi} \underline{r}\,dt$.

2. Given $\underline{v} = \underline{i} - 3\underline{j} + \underline{k}$, evaluate:

   (a) $\displaystyle\int_0^1 \underline{v}\,dt$, (b) $\displaystyle\int_0^2 \underline{v}\,dt$

3. The vector, $\underline{a}$, is defined by $\underline{a} = t^2\underline{i} + e^{-t}\underline{j} + t\underline{k}$. Evaluate

   (a) $\displaystyle\int_0^1 \underline{a}\,dt$, (b) $\displaystyle\int_2^3 \underline{a}\,dt$, (c) $\displaystyle\int_1^4 \underline{a}\,dt$

4. Let $\underline{a}$ and $\underline{b}$ be two three-dimensional vectors. Is the following result true?

   $$\int_{t_1}^{t_2} \underline{a}\,dt \times \int_{t_1}^{t_2} \underline{b}\,dt = \int_{t_1}^{t_2} \underline{a}\times\underline{b}\,dt$$

   where $\times$ denotes the vector product.

Answers

1. $6\underline{i} + 1.348\underline{k}$

2. (a) $\underline{i} - 3\underline{j} + \underline{k}$ (b) $2\underline{i} - 6\underline{j} + 2\underline{k}$

3. (a) $0.333\underline{i} + 0.632\underline{j} + 0.5\underline{k}$ (b) $6.333\underline{i} + 0.0855\underline{j} + 2.5\underline{k}$ (c) $21\underline{i} + 0.3496\underline{j} + 7.5\underline{k}$

4. No.

Calculating Centres of Mass

15.2

Introduction

In this Section we show how the idea of integration as the limit of a sum can be used to find the centre of mass of an object such as a thin plate (like a sheet of metal). Such a plate is also known as a **lamina**. An understanding of the term **moment** is necessary and so this concept is introduced.

Prerequisites

Before starting this Section you should ...

- understand integration as the limit of a sum
- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- calculate the position of the centre of mass of a variety of simple plane shapes

1. The centre of mass of a collection of point masses

Suppose we have a collection of masses located at a number of known points along a line. The **centre of mass** is the point where, for many purposes, all the mass can be assumed to be located.

For example, if two objects each of mass m are placed at distances 1 and 2 units from a point O, as shown in Figure 2a, then the total mass, $2m$, might be assumed to be concentrated at distance 1.5 units as shown in Figure 2b. This is the point where we could imagine placing a pivot to achieve a perfectly balanced system.

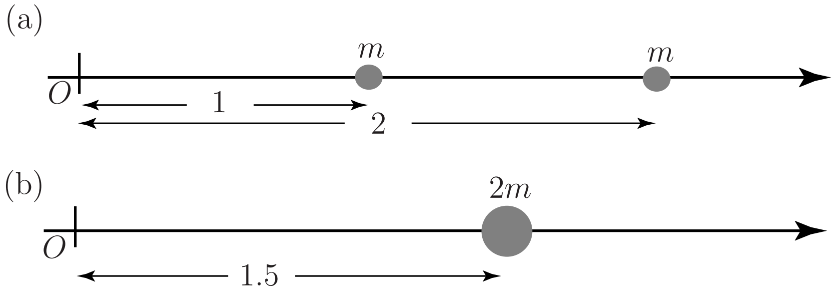


Figure 2: Equivalent position of the centre of mass of the objects in (a) is shown in (b)

To think of this another way, if a pivot is placed at the origin O, as on a see-saw, then the two masses at $x = 1$ and $x = 2$ together have the same **turning effect** or **moment** as a single mass $2m$ located at $x = 1.5$. This is illustrated in Figure 3.

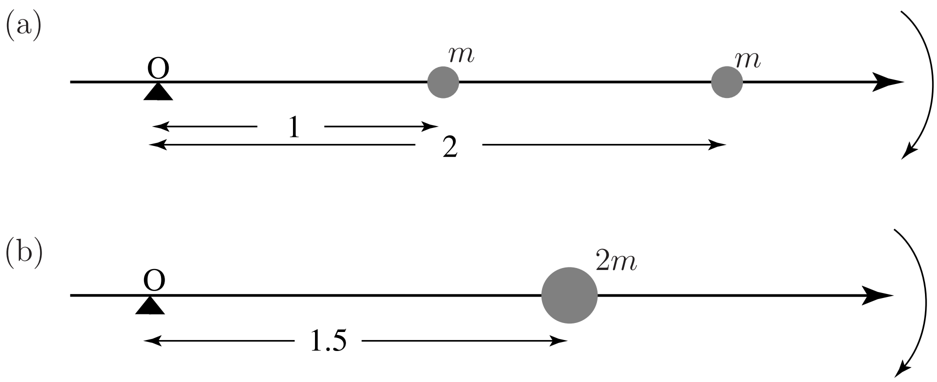


Figure 3: The single object of mass $2m$ has the same turning effect as the two objects each of mass m

Before we can calculate the position of the centre of mass of a collection of masses it is important to define the term 'moment' more precisely. Given a mass M located a distance d from O, as shown in Figure 4, its moment about O is defined to be

moment $= M \times d$

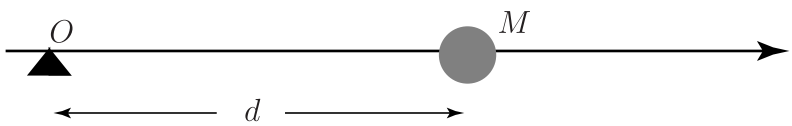
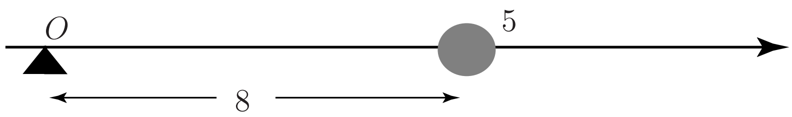


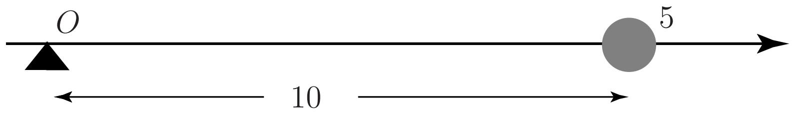
Figure 4: The moment of the mass M about O is $M \times d$

In words, the moment of the mass about O, is the mass multiplied by its distance from O. The units of moment will therefore be kg m if the mass is measured in kilogrammes and the distance in metres. (N.B. Unless specified otherwise these will be the units we shall always use.)

Task

Calculate the moment of the mass about O in each of the following cases.

(a)


(b)


Your solution

(a) (b)

Answer

(a) 40 kg m (b) 50 kg m

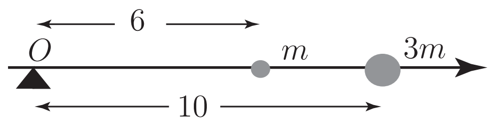
Intuition tells us that a large moment corresponds to a large turning effect. A mass placed 8 metres from the origin has a smaller turning effect than the same mass placed 10 metres from the origin. This is, of course, why it is easier to rock a see-saw by pushing it at a point further from the pivot. Our intuition also tells us the side of the pivot on which the masses are placed is important. Those placed to the left of the pivot have a turning effect opposite to those placed to the right.

For a collection of masses the moment of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses. This definition enables us to calculate the position of the centre of mass. It is conventional to label the x coordinate of the centre of mass as $\bar{x}$, pronounced 'x bar'.

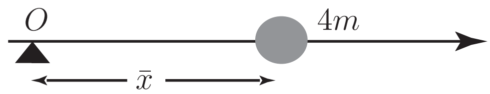
Key Point 1

The **moment** of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses.

Objects of mass m and $3m$ are placed at the locations shown in diagram (a). Find the distance $\bar{x}$ of the centre of mass from the origin O as illustrated in diagram (b).



(a)



(b)

First calculate the sum of the individual moments:

Your solution

Answer

$6 \times m + 10 \times 3m = 36m$

The moment of the total mass about O is $4m \times \bar{x}$.
The moment of the total mass is equal to the sum of the moments of the individual masses. Write down and solve the equation satisfied by $\bar{x}$:

Your solution

Answer

$36m = 4m\bar{x}$, so $\bar{x} = 9$

So the centre of mass is located a distance 9 units along the x-axis. Note that it is closer to the position of the $3m$ mass than to the position of the $1m$ mass (actually in the ratio $3:1$).

Example 2

Obtain an equation for the location of the centre of mass of two objects of masses m_1 and m_2:

(a) located at distances x_1 and x_2 respectively, as shown in Figure 5(a)

(b) positioned on opposite sides of the origin as shown in Figure 5(b)

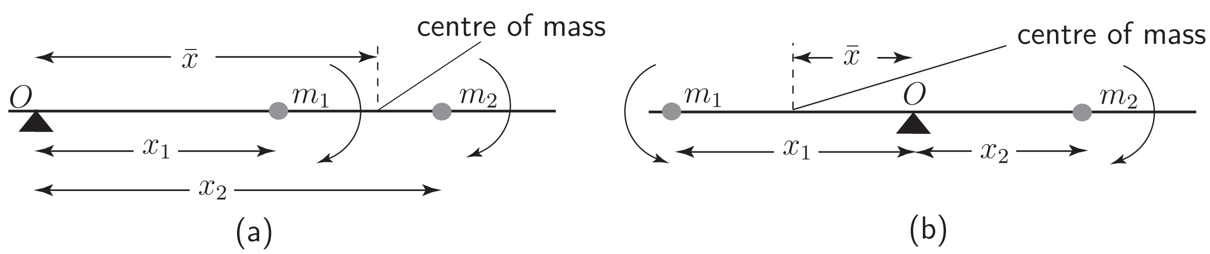


Figure 5

Referring to Figure 5(a) we first write down an expression for the sum of the individual moments:

$$m_1x_1 + m_2x_2$$

The total mass is $m_1 + m_2$ and the moment of the total mass is $(m_1 + m_2) \times \bar{x}$.
The moment of the total mass is equal to the sum of the moments of the individual masses. The equation satisfied by $\bar{x}$ is

$$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 \qquad \text{so} \quad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

For the second case, as depicted in Figure 5(b), the mass m_1 positioned on the left-hand side has a turning effect **opposite** to that of the mass m_2 positioned on the right-hand side. To take account of this difference we use a minus sign when determining the moment of m_1 about the origin. This gives a total moment

$$-(m_1x_1) + (m_2x_2)$$

leading to

$$(-m_1x_1 + m_2x_2) = (m_1 + m_2)\bar{x} \qquad \text{so} \qquad \bar{x} = \frac{-m_1x_1 + m_2x_2}{m_1 + m_2}$$

However, this is precisely what would have been obtained if, when working out the moment of a mass, we use its **coordinate** (which takes account of sign) rather than using its **distance** from the origin.

The formula obtained in the Task can be generalised very easily to deal with the general situation of n masses, $m_1, m_2, \ldots, m_n$ located at **coordinate positions** x_1, $x_2, \ldots x_n$ and is given in Key Point 2.

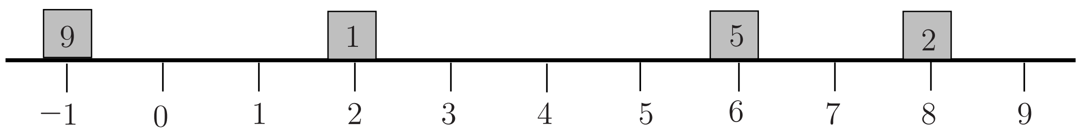
Key Point 2

The **centre of mass** of individual masses $m_1, m_2, \ldots, m_n$ located at positions x_1, $x_2, \ldots x_n$ is

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_ix_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

Task

Calculate the centre of mass of the 4 masses distributed as shown below.



Use Key Point 2 to calculate $\bar{x}$:

Your solution

$\bar{x} =$

Answer

$$\bar{x} = \frac{(9)(-1) + (1)(2) + (5)(6) + (2)(8)}{9 + 1 + 5 + 2} = \frac{39}{17}$$

The centre of mass is located a distance $\frac{39}{17} \approx 2.29$ units along the x-axis from O.

Distribution of particles in a plane

If the particles are distributed in a plane then the position of the centre of mass can be calculated in a similar way.

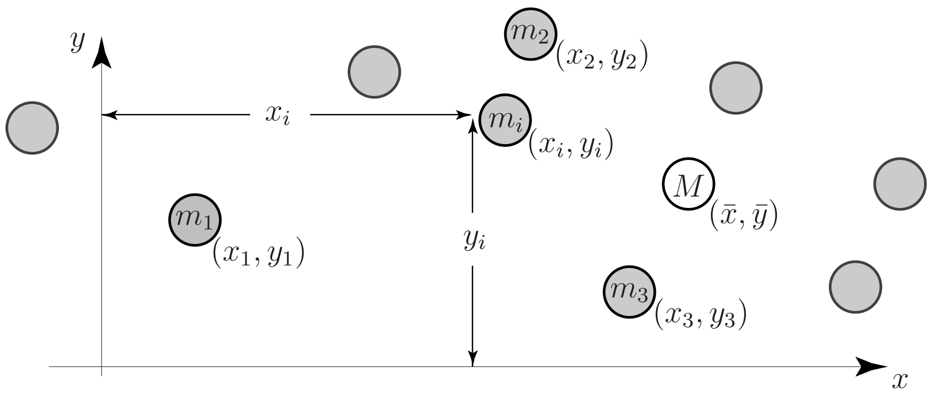


Figure 6: These masses are distributed throughout the xy plane

Now we must consider the moments of the individual masses taken about the x-axis **and** about the y-axis. For example, in Figure 6, mass m_i has a moment $m_i y_i$ about the x-axis and a moment $m_i x_i$ about the y-axis. Now we define the centre of mass at that point $(\bar{x}, \bar{y})$ such that the total mass $M = m_1 + m_2 + \ldots m_n$ placed at this point would have the same moment about each of the axes as the sum of the individual moments of the particles about these axes.

Key Point 3

The **centre of mass** of $m_1, m_2, \ldots, m_n$ located at (x_1, y_1), $(x_2, y_2), \ldots (x_n, y_n)$ has coordinates $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i}, \qquad \bar{y} = \frac{\displaystyle\sum_{i=1}^{n} m_i y_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

Task

Masses of 5 kg, 3 kg and 9 kg are located at the points with coordinates $(-1, 1)$, $(4, 3)$, and $(8, 7)$ respectively. Find the coordinates of their centre of mass.

Use Key Point 3:

Your solution

$\bar{x} =$

$\bar{y} =$

Answer

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{3} m_i x_i}{\displaystyle\sum_{i=1}^{3} m_i} = \frac{5(-1) + 3(4) + 9(8)}{5 + 3 + 9} = \frac{79}{17} \approx 4.65$$

$$\bar{y} = \frac{5(1) + 3(3) + 9(7)}{17} = 4.53.$$

Hence the centre of mass is located at the point $(4.65, 4.53)$.

Exercises

1. Find the x coordinate of the centre of mass of 5 identical masses placed at $x = 2$, $x = 5$, $x = 7$, $x = 9$, $x = 12$.

2. Derive the formula for $\bar{y}$ given in Key Point 3.

Answer 1. $\bar{x} = 7$

2. Finding the centre of mass of a plane uniform lamina

In the previous Section we calculated the centre of mass of several individual point masses. We are now interested in finding the centre of mass of a thin sheet of material, such as a plane sheet of metal, called a **lamina**. The mass is not now located at individual points. Rather, it is distributed continuously over the lamina. In what follows we assume that the mass is distributed **uniformly** over the lamina and you will see how integration as the limit of a sum is used to find the centre of mass.

Figure 6 shows a lamina where the centre of mass has been marked at point G with coordinates $(\bar{x}, \bar{y})$. If the total mass of the lamina is M then the moments about the y- and x-axes are respectively $M\bar{x}$ and $M\bar{y}$. Our approach to locating the position of G, i.e. finding $\bar{x}$ and $\bar{y}$, is to divide the lamina into many small pieces, find the mass of each piece, and calculate the moment of each piece about the axes. The sum of the moments of the individual pieces about the y-axis must then be equal to $M\bar{x}$ and the sum of the moments of the individual pieces about the x-axis must equal $M\bar{y}$.

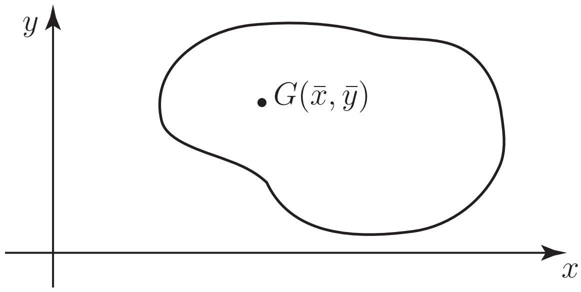


Figure 6: The centre of mass of the lamina is located at $G(\bar{x}, \bar{y})$

There are no formulae which can be memorized for finding the centre of mass of a lamina because of the wide variety of possible shapes. Instead you should be familiar with the general technique for deriving the centre of mass.

An important preliminary concept is 'mass per unit area' which we now introduce.

Mass per unit area

Suppose we have a uniform lamina and select a piece of the lamina which has area equal to **one unit**. Let ρ stand for the mass of such a piece. Then ρ is called the **mass per unit area**. The mass of any other piece can be expressed in terms of ρ. For example, an area of 2 units must have mass 2ρ, an area of 3 units must have mass 3ρ, and so on. Any portion of the lamina which has area A has mass ρA.

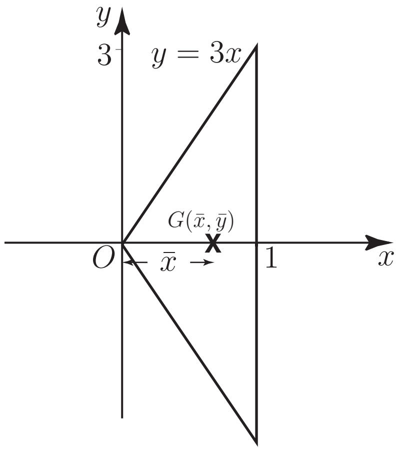
Key Point 4

If a lamina has mass per unit area, ρ, then the mass of part of the lamina having area A is $A\rho$.

We will investigate the calculation of centre of mass through the following Tasks.

Task

Consider the plane sheet, or lamina, shown below. Find the location of its centre of mass $(\bar{x}, \bar{y})$. (By symmetry the centre of mass of this lamina lies on the x-axis.)



(a) First inspect the figure and note the symmetry of the lamina. Purely from the symmetry, what must be the y coordinate, $\bar{y}$, of the centre of mass ?

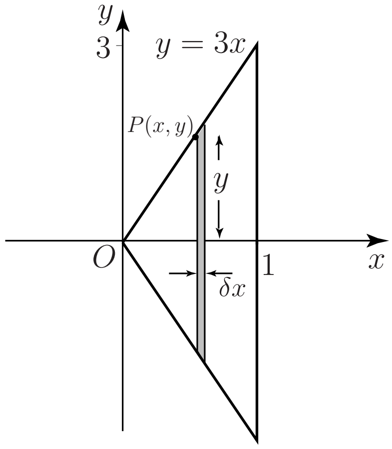
Your solution

Answer

$\bar{y} = 0$ since the centre of mass must lie on the x-axis

(b) Let ρ stand for the mass per unit area of the lamina. The total area is 3 units. The total mass is therefore 3ρ. Its moment about the y-axis is $3\rho\bar{x}$.

To find $\bar{x}$ first divide the lamina into a large number of thin vertical slices. In the figure below a typical slice has been highlighted. Note that the slice has been drawn from the point P on the line $y = 3x$. The point P has coordinates (x, y). The thickness of the slice is δx.



A typical slice of this sheet has been shade.

Assuming that the slice is rectangular in shape, write down its area:

Your solution

Answer

$2y\delta x$

(c) Writing ρ as the mass per unit area, write down the mass of the slice:

Your solution

Answer

$(2y\delta x)\rho$

(d) The centre of mass of this slice lies on the x-axis. So the slice can be assumed to be a point mass, $2y\rho\delta x$, located a distance x from O.

Write down the moment of the mass of the slice about the y-axis:

Your solution

Answer

$(2y\delta x)\rho x$

(e) By adding up contributions from all such slices in the lamina we obtain the sum of the moments of the individual masses:

$$\sum_{x=0}^{x=1} 2\rho xy\delta x$$

The limits on the sum are chosen so that all slices are included.

Write down the integral defined by letting $\delta x \to 0$:

Your solution

Answer

$$\int_{x=0}^{x=1} 2\rho xy\, dx$$

(f) Noting that $y = 3x$, express the integrand in terms of x and evaluate it:

Your solution

Answer

$$\int_0^1 6\rho x^2\, dx = \Big[2\rho x^3\Big]_0^1 = 2\rho$$

(g) Calculate $\bar{x}$ and hence find the centre of mass of the lamina:

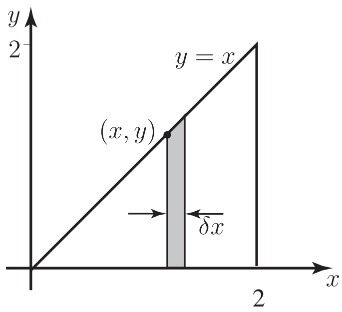
Your solution

Answer

This must equal the moment of the total mass acting at the centre of mass so $3\rho\bar{x} = 2\rho$ giving $\bar{x} = \dfrac{2}{3}$. Now the coordinates of the centre of mass are thus $(\frac{2}{3}, 0)$.

Task

Find the centre of mass of the plane lamina shown below.



The coordinates of $\bar{x}$ and $\bar{y}$ must be calculated separately.

Stage 1: To calculate $\bar{x}$

(a) Let ρ equal the mass per unit area. Write down the total area, the total mass, and its moment about the y-axis:

Your solution

Answer

$2, 2\rho, 2\rho\bar{x}$

(b) To calculate $\bar{x}$ the lamina is divided into thin slices; a typical slice is shown in the figure above. We assume that the shaded slice is rectangular, which is a reasonable approximation.

Write down the height of the typical strip shown in the figure, its area, and its mass:

Your solution

Answer

y, $y\delta x$, $\quad (y\delta x)\rho$

(c) Write down the moment about the y-axis of the typical strip:

Your solution

Answer

$(y\delta x)\rho x$

(d) The sum of the moments of all strips is

$$\sum_{x=0}^{x=2} \rho xy\delta x$$

Write down the integral which follows as $\delta x \to 0$:

Your solution

Answer

$$\int_0^2 \rho xy\, dx$$

(e) In this example, $y = x$ because the line $y = x$ defines the upper limit of each strip (and hence its height). Substitute this value for y in the integral, and evaluate it:

Your solution

Answer

$$\int_0^2 \rho x^2\, dx = \frac{8}{3}\rho$$

(f) Equating the sum of individual moments and the total moment gives $2\rho\bar{x} = \dfrac{8}{3}\rho$. Deduce $\bar{x}$:

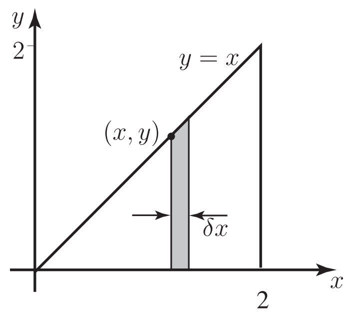
Your solution

Answer

$\bar{x} = \frac{4}{3}$

We will illustrate two alternative ways of calculating $\bar{y}$.

Stage 2: **To calculate $\bar{y}$ using vertical strips**



(a) Referring to the figure again, which we repeat here, the centre of mass of the slice must lie half way along its length, that is its y coordinate is $\dfrac{y}{2}$. Assume that all the mass of the slice, $y\rho\delta x$, acts at this point. Then its moment about the x-axis is $y\rho\delta x\,\dfrac{y}{2}$. Adding contributions from all slices gives the sum

$$\sum_{x=0}^{x=2} \frac{y^2\rho}{2}\delta x$$

(b) Write down the integral which is defined as $\delta x \to 0$:

Your solution

Answer

$$\int_{x=0}^{2} \frac{\rho y^2}{2}\,dx$$

(c) We can write the above as

$$\rho\int_{x=0}^{2} \frac{y^2}{2}\,dx \quad \text{and in this example } y = x\text{, so the integral becomes}$$

$$\rho\int_{x=0}^{2} \frac{x^2}{2}\,dx$$

Evaluate this.

Your solution

Answer

$$\frac{4\rho}{3}$$

(d) This is the sum of the individual moments about the x-axis and must equal the moment of the total mass about the x-axis which has already been found as $2\rho\bar{y}$. Therefore

$$2\rho\bar{y} = \frac{4\rho}{3} \qquad \text{from which} \quad \bar{y} = \frac{2}{3}$$

(e) Finally deduce $\bar{y}$ and state the coordinates of the centre of mass:

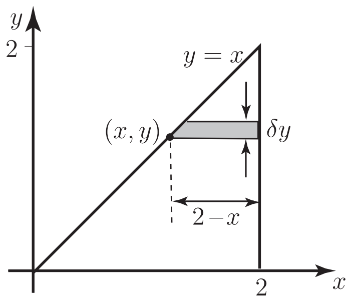
Your solution

Answer

$\bar{y} = \dfrac{2}{3}$ and the coordinates of the centre of mass are $(\dfrac{4}{3}, \dfrac{2}{3})$

Stage 3: **To calculate $\bar{y}$ using horizontal strips**

(a) This time the lamina is divided into a number of horizontal slices; a typical slice is shown below.



A typical horizontal slice is shaded.

The length of the typical slice shown is $2 - x$.

Write down its area, its mass and its moment **about the x-axis**:

Your solution

Answer

$(2 - x)\delta y, \qquad \rho(2 - x)\delta y,\ \rho(2 - x)y\delta y$

(b) Write down the expression for the sum of all such moments and the corresponding integral as $\delta y \to 0$.

Your solution

Answer

$$\sum_{y=0}^{y=2} \rho(2 - x)y\delta y, \qquad \int_0^2 \rho(2 - x)y\, dy$$

(c) Now, since $y = x$ the integral can be written entirely in terms of y as

$$\int_0^2 \rho(2-y)y\,dy$$

Evaluate the integral and hence find $\bar{y}$:

Your solution

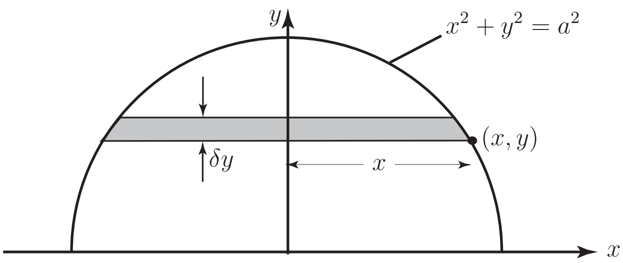
Answer

$\dfrac{4\rho}{3}$; As before the total mass is 2ρ, and its moment about the x-axis is $2\rho\bar{y}$. Hence

$2\rho\bar{y} = \dfrac{4\rho}{3}$ from which $\bar{y} = \dfrac{2}{3}$ which was the result obtained before in Stage 2.

Task

Find the position of the centre of mass of a uniform semi-circular lamina of radius a, shown below.



A typical horizontal strip is shaded.

The equation of a circle centre the origin, and of radius a is $x^2 + y^2 = a^2$.

By symmetry $\bar{x} = 0$. However it is necessary to calculate $\bar{y}$.

(a) The lamina is divided into a number of horizontal strips and a typical strip is shown. Assume that each strip is rectangular. Writing the mass per unit area as ρ, state the area and the mass of the strip:

Your solution

Answer

$2x\delta y, \qquad 2x\rho\delta y$

(b) Write down the moment of the mass about the x-axis:

Your solution

Answer

$2x\rho y\delta y$

(c) Write down the expression representing the sum of the moments of all strips and the corresponding integral obtained as $\delta y \to 0$:

Your solution

Answer

$$\sum_{y=0}^{y=a} 2x\rho y\delta y, \qquad \int_0^a 2x\rho y\, dy$$

(d) Now since $x^2 + y^2 = a^2$ we have $x = \sqrt{a^2 - y^2}$ and the integral becomes:

$$\int_0^a 2\rho y\sqrt{a^2 - y^2}\, dy$$

Evaluate this integral by making the substitution $u^2 = a^2 - y^2$ to obtain the total moment.

Your solution

Answer

$$\frac{2\rho a^3}{3}$$

(e) The total area is half that of a circle of radius a, that is $\frac{1}{2}\pi a^2$. The total mass is $\frac{1}{2}\pi a^2\rho$ and its moment is $\frac{1}{2}\pi a^2\rho\bar{y}$.

Deduce $\bar{y}$:

Your solution

Answer

$$\frac{1}{2}\pi a^2\rho\bar{y} = \frac{2\rho a^3}{3} \qquad \text{from which} \qquad \bar{y} = \frac{4a}{3\pi}$$

Engineering Example 1

Suspended cable

Introduction

A cable of constant line density is suspended between two vertical poles of equal height such that it takes the shape of a curve, $y = 6\cosh(x/6)$. The origin of the curve is a point mid-way between the feet of the poles and y is the height above the ground. If the cable is 600 metres long show that the distance between the poles is 55 metres to the nearest metre. Find the height of the centre of mass of the cable above the ground to the nearest metre.

Mathematical statement of the problem

We can draw a picture of the cable as in Figure 7 where A and B denote the end points.

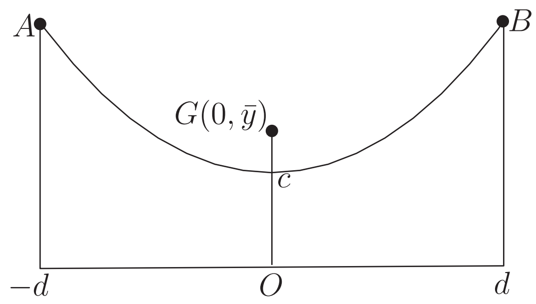


Figure 7

For the first part of this problem we use the result found in HELM 14 that the distance along a curve $y = f(x)$ from $x = a$ to $x = b$ is given by $s = \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

where in this case we are given $y = 6\cosh\left(\dfrac{x}{6}\right)$ and therefore $\dfrac{dy}{dx} = \sinh\left(\dfrac{x}{6}\right)$.

If we take the distance between the poles to be $2d$ then the values of x in this integration go from $-d$ to $+d$. So we need to find d such that:

$$600 = \int_{-d}^{d} \sqrt{1 + \left(\sinh\left(\frac{x}{6}\right)\right)^2}\, dx. \qquad (1)$$

For the second part of this problem we need to find the centre of mass of the cable. From the symmetry of the problem we know that the centre of mass must lie on the y-axis. To find the height of the centre of mass we need to take each section of the cable and consider the moment about the x-axis through the origin. A section of the cable has mass $\rho\delta s$ where ρ is the line density of the cable and δs is the length of a small section of the cable.

so the moment about the x-axis will be $\displaystyle\sum_{x=-d}^{x=d} \rho y \delta s$

taking the limit as $\delta s \to 0$ and using the fact that $\delta s = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, \delta x$

we get that the moment about the x-axis to be $\rho \int_{-d}^{d} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

This must equal the moment of a single point mass, equal to the total mass of the cable, placed at its centre of mass. As the length of the cable is 600 metres then the mass of the cable is 600ρ and we have

$$600\rho\bar{y} = \rho\int_{-d}^{d} y\, \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Dividing both sides of this equation by ρ we get:

$$600\bar{y} = \int_{-d}^{d} y\, \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

where we have already established the value of d from Equation (1) so we can solve this equation to find $\bar{y}$.

Mathematical analysis

We need to find d so that $600 = \int_{-d}^{d} \sqrt{1+\left(\sinh\left(\frac{x}{6}\right)\right)^2}\, dx$

Rearranging the hyperbolic identity $\cosh^2(u) - \sinh^2(u) \equiv 1$ we obtain $\sqrt{1+(\sinh(u))^2} = \cosh(u)$

so the integral becomes $\int_{-d}^{d} \cosh\left(\frac{x}{6}\right)\, dx = \left[6\ \sinh\left(\frac{x}{6}\right)\right]_{-d}^{+d} = 6\left(\sinh\left(\frac{d}{6}\right) - \sinh\left(-\frac{d}{6}\right)\right)$

so

$$12\sinh\left(\frac{d}{6}\right) = 600 \text{ and } d = 6\sinh^{-1}(50).$$

Using the log identity for the $\sinh^{-1}$ function:

$$\sinh^{-1}(x) \equiv \ln(x+\sqrt{x^2+1})$$

we find that $d = 27.63$ m so the distance between the poles is 55 m to the nearest metre.

To find the height of the centre of mass above the ground we use

$$600\bar{y} = \int_{-d}^{d} y\, \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Substituting $y = 6\cosh\left(\frac{x}{6}\right)$ and therefore $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\sinh\left(\frac{x}{6}\right)\right)^2} = \cosh\left(\frac{x}{6}\right)$ we get

$$\int_{-d}^{d} 6\cosh\left(\frac{x}{6}\right)\cosh\left(\frac{x}{6}\right) dx = \int_{-d}^{d} 6\cosh^2\left(\frac{x}{6}\right) dx$$

From the hyperbolic identities we know that $\cosh^2(x) \equiv \frac{1}{2}(\cosh(2x)+1)$

so this integral becomes $\int_{-d}^{d} 3\left(\cosh\left(\frac{x}{3}\right)+1\right) dx = \left[9\sinh\left(\frac{x}{3}\right)+3x\right]_{-d}^{+d} = 18\sinh\left(\frac{d}{3}\right)+6d$

So we have that $600\bar{y} = 18\sinh\left(\frac{d}{3}\right)+6d$

From the first part of this problem we found that $d = 27.63$ so substituting for d we find $\bar{y} = 150$ metres to the nearest metre.

Interpretation
We have found that the two vertical poles holding the cable have a distance between them of 55 metres and the height of the centre of mass of the cable above the ground is 150 metres.

Exercise

Find the centre of mass of a lamina bounded by $y^2 = 4x$, for $0 \leq x \leq 9$.

Answer $(\frac{27}{5}, 0)$.

Moment of Inertia

15.3

Introduction

In this Section we show how integration is used to calculate moments of inertia. These are essential for an understanding of the dynamics of rotating bodies such as flywheels.

Prerequisites

Before starting this Section you should ...

- understand integration as the limit of a sum
- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- calculate the moment of inertia of a number of simple plane bodies

1. Introduction

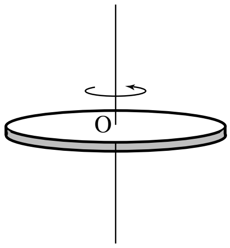


Figure 8: A lamina rotating about an axis through O

Figure 8 shows a lamina which is allowed to rotate about an axis perpendicular to the plane of the lamina and through O. The **moment of inertia** about this axis is a measure of how difficult it is to rotate the lamina. It plays the same role for rotating bodies that the mass of an object plays when dealing with motion in a line. An object with large mass needs a large force to achieve a given acceleration. Similarly, an object with large moment of inertia needs a large turning force to achieve a given angular acceleration. Thus knowledge of the moments of inertia of laminas and of solid bodies is essential for understanding their rotational properties.

In this Section we show how the idea of integration as the limit of a sum can be used to find the moment of inertia of a lamina.

2. Calculating the moment of inertia

Suppose a lamina is divided into a large number of small pieces or **elements**. A typical element is shown in Figure 9.

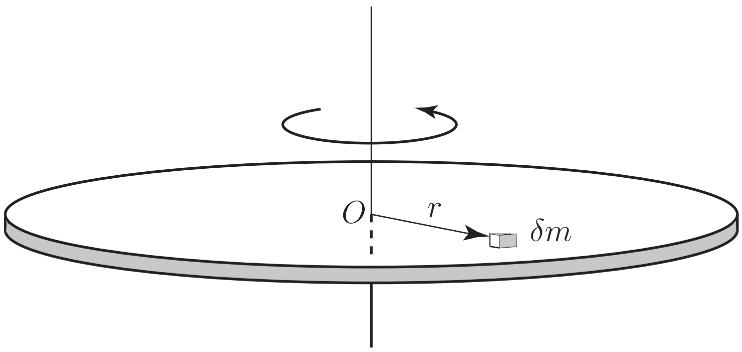


Figure 9: The moment of inertia of the small element is $\delta m\, r^2$

The element has mass δm, and is located a distance r from the axis through O. The moment of inertia of this small piece about the given axis is defined to be $\delta m\, r^2$, that is, the mass multiplied by the square of its distance from the axis of rotation. To find the total moment of inertia we sum the individual contributions to give

$$\sum r^2 \delta m$$

where the sum must be taken in such a way that all parts of the lamina are included. As $\delta m \to 0$ we obtain the following integral as the definition of moment of inertia:

Key Point 5

moment of inertia $I = \int r^2\,dm$

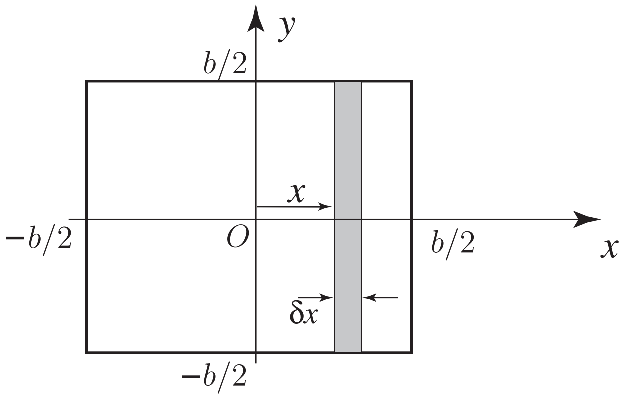
where the limits of integration are chosen so that the entire lamina is included.

The unit of moment of inertia is kg m^2.

We shall illustrate how the moment of inertia is actually calculated in practice, in the following Tasks.

Task

Calculate the moment of inertia about the y-axis of the square lamina of mass M and width b, shown below. (The moment of inertia about the y-axis is a measure of the resistance to rotation around this axis.)



A square lamina rotating about the y-axis.

Let the mass per unit area of the lamina be ρ. Then, because its total area is b^2, its total mass M is $b^2\rho$. Imagine that the lamina has been divided into a large number of thin vertical strips. A typical strip is shown in the figure above. The strips are chosen in this way because each point on a particular strip is approximately the same distance from the axis of rotation (the y-axis).

(a) Referring to the figure, write down the width of each strip:

Your solution

Answer

δx

(b) Write down the area of the strip:

Your solution

Answer

$b\delta x$

(c) With ρ as the mass per unit area write down the mass of the strip:

Your solution

Answer

$\rho b\delta x$

(d) The distance of the strip from the y-axis is x. Write down its moment of inertia :

Your solution

Answer

$mbx^2\delta x$

(e) Adding contributions from all strips gives the expression $\sum \rho bx^2\delta x$ where the sum must be such that the entire lamina is included. As $\delta x \to 0$ the sum defines an integral. Write down this integral:

Your solution

Answer

$$I = \int_{-b/2}^{b/2} \rho bx^2 dx$$

(f) Note that the limits on the integral have been chosen so that the whole lamina is included. Then

$$I = \rho b \int_{-b/2}^{b/2} x^2 dx$$

Evaluate this integral:

Your solution

$I =$

Answer

$I = \rho b \left[\dfrac{x^3}{3} \right]_{-b/2}^{b/2} = \dfrac{\rho b^4}{12}$

(g) Write down an expression for M in terms of b and ρ:

Your solution

Answer

$M = b^2 \rho$

(h) Finally, write an expression for I in terms of M and b:

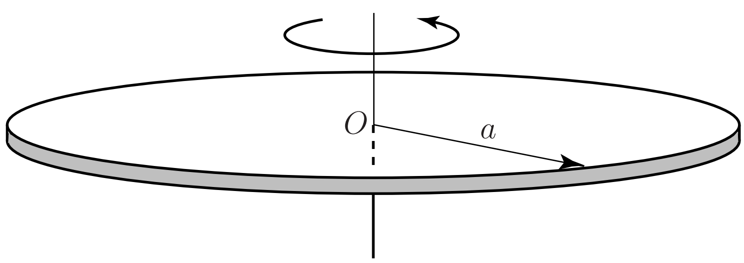
Your solution

Answer

$I = \dfrac{Mb^2}{12}$

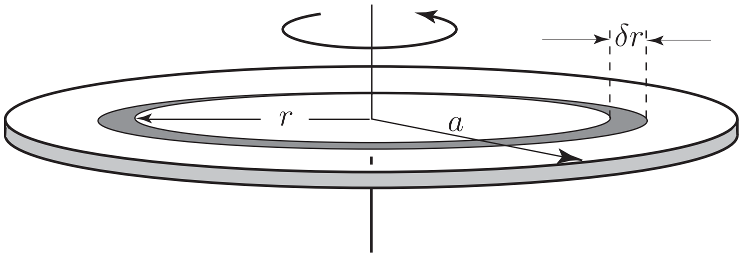
Task

Find the moment of inertia of a circular disc of mass M and radius a about an axis passing through its centre and perpendicular to the disc.



A circular disc rotating about an axis through O.

The figure above shows the disc lying in the plane of the paper. Because of the circular symmetry the disc is divided into concentric rings of width δr. A typical ring is shown below. Note that each point on the ring is approximately the same distance from the axis of rotation.



The lamina is divided into many circular rings.

The ring has radius r and inner circumference $2\pi r$. Imagine cutting the ring and opening it up. Its area will be approximately that of a long thin rectangle of length $2\pi r$ and width δr. Given that ρ is the mass per unit area write down an expression for the mass of the ring:

Your solution

Answer

$2\pi r\rho\delta r$

The moment of inertia of the ring about O is its mass multiplied by the square of its distance from the axis of rotation. This is $(2\pi r\rho\delta r) \times r^2 = 2\pi r^3\rho\delta r$.

The contribution from all rings must be summed. This gives rise to the sum

$$\sum_{r=0}^{r=a} 2\pi r^3\rho\delta r$$

Note the way that the limits have been chosen so that all rings are included in the sum. As $\delta r \to 0$ the limit of the sum defines the integral

$$\int_0^a 2\rho\pi r^3 dr$$

Evaluate this integral to give the moment of inertia I:

Your solution

Answer

$$I = \left[\frac{2\rho\pi r^4}{4}\right]_0^a = \frac{\rho\pi a^4}{2}$$

Write down the radius and area of the whole disc:

Your solution

Answer

$a, \pi a^2$

With ρ as the mass per unit area, write down the mass of the disc M:

Your solution

$M =$

Answer

$M = \pi a^2 \rho$

Finally express I in terms of M and a:

Your solution

$I =$

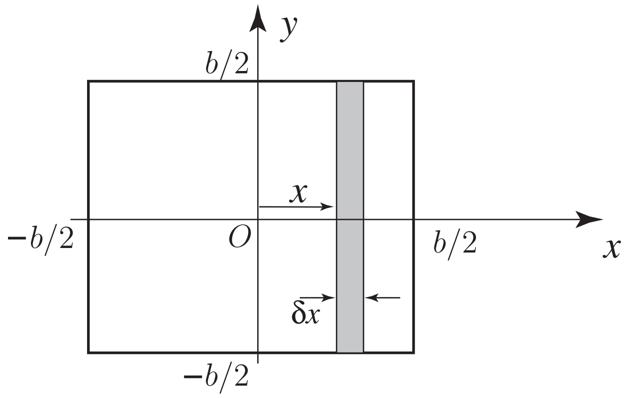
Answer

$I = \dfrac{Ma^2}{2}$

Exercises

1. The moment of inertia about a diameter of a sphere of radius 1 m and mass 1 kg is found by evaluating the integral $\frac{3}{8}\int_{-1}^{1}(1-x^2)^2dx$. Show that this moment of inertia is 0.4 kg m^2.

2. Find the moment of inertia of the square lamina below about one of its sides.



3. Calculate the moment of inertia of a uniform thin rod of mass M and length ℓ about a perpendicular axis of rotation at its end.

4. Calculate the moment of inertia of the rod in Exercise 3 about an axis through its centre and perpendicular to the rod.

5. The **parallel axis theorem** states that the moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centre of mass, plus the mass of the body $\times$ the square of the distance between the two axes. Verify this theorem for the rod in Exercise 3 and Exercise 4.

6. The **perpendicular axis theorem** applies to a lamina lying in the xy plane. It states that the moment of inertia of the lamina about the z-axis is equal to the sum of the moments of inertia about the x- and y-axes. Suppose that a thin circular disc of mass M and radius a lies in the xy plane and the z axis passes through its centre. The moment of inertia of the disc about this axis is $\frac{1}{2}Ma^2$.

   (a) Use this theorem to find the moment of inertia of the disc about the x and y axes.

   (b) Use the parallel axis theorem to find the moment of inertia of the disc about a tangential axis parallel to the plane of the disc.

Answers

2. $\dfrac{Mb^2}{3}$.   3. $\dfrac{1}{3}M\ell^2$.   4. $\dfrac{1}{12}M\ell^2$.

6. (a) The moments of inertia about the x and y axes must be the same by symmetry, and are equal to $0.25\ Ma^2$.

   (b) $1.25\ Ma^2$.

NOTES

Index for Workbook 15

Workbook 15

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