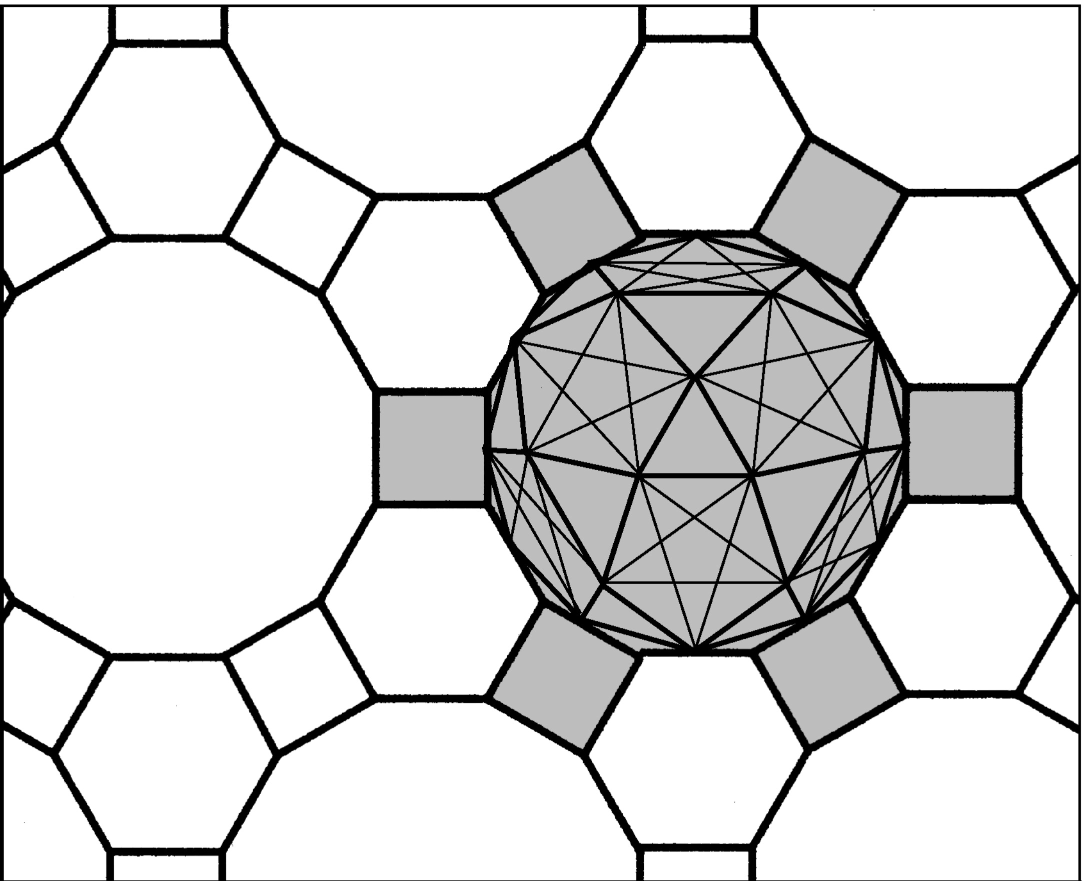
Workbook 17



Conics and Polar Coordinates

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

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Conics and Polar Coordinates

Learning outcomes

In this Workbook you will learn about some of the most important curves in the whole of mathematics - the conic sections: the ellipse, the parabola and the hyperbola. You will learn how to recognise these curves and how to describe them in Cartesian and in polar form. In the final block you will learn how to describe cruves using a parametric approach and, in particular, how the conic sections are described in parametric form.

Conic Sections

17.1

Introduction

The **conic sections** (or **conics**) - the **ellipse**, the **parabola** and the **hyperbola** - play an important role both in mathematics and in the application of mathematics to engineering. In this Section we look in detail at the equations of the conics in both standard form and general form.

Although there are various ways that can be used to define a conic, we concentrate in this Section on defining conics using Cartesian coordinates (x, y). However, at the end of this Section we examine an alternative way to obtain the conics.

Prerequisites

Before starting this Section you should . . .

- be able to factorise simple algebraic expressions
- be able to change the subject in simple algebraic equations
- be able to complete the square in quadratic expressions

Learning Outcomes

On completion you should be able to . . .

- understand how conics are obtained as curves of intersection of a double-cone with a plane
- state the standard form of the equations of the ellipse, the parabola and the hyperbola
- classify quadratic expressions in $x,\ y$ in terms of conics

1. The ellipse, parabola and hyperbola

Mathematicians, engineers and scientists encounter numerous functions in their work: polynomials, trigonometric and hyperbolic functions amongst them. However, throughout the history of science one group of functions, the conics, arise time and time again not only in the development of mathematical theory but also in practical applications. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC.

Essentially, the conics form that class of curves which are obtained when a double cone is intersected by a plane. There are three main types: the **ellipse**, the **parabola** and the **hyperbola**. From the ellipse we obtain the **circle** as a special case, and from the hyperbola we obtain the **rectangular hyperbola** as a special case. These curves are illustrated in the Figures 1 and 2.

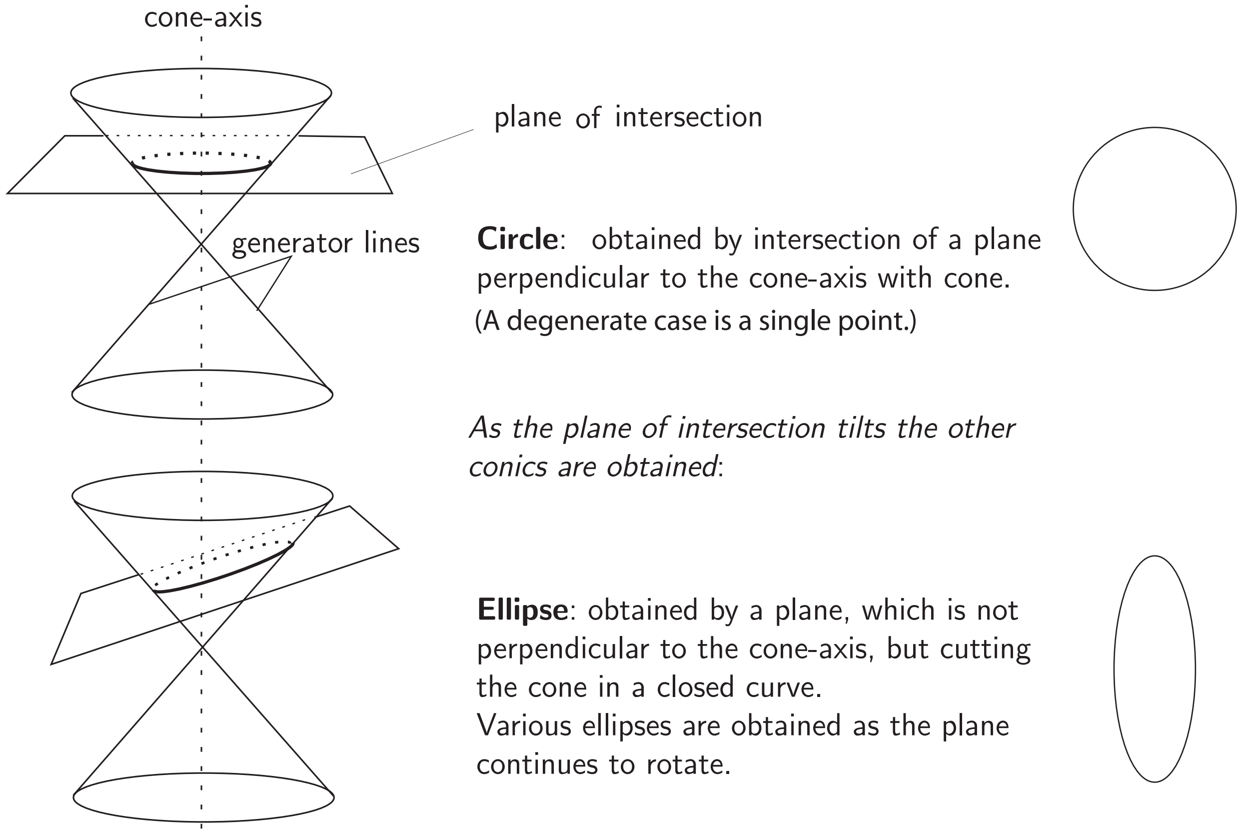
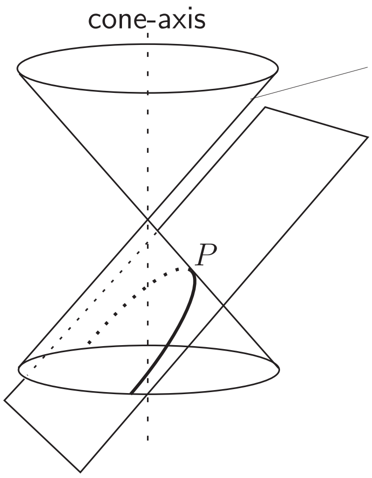
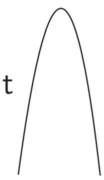
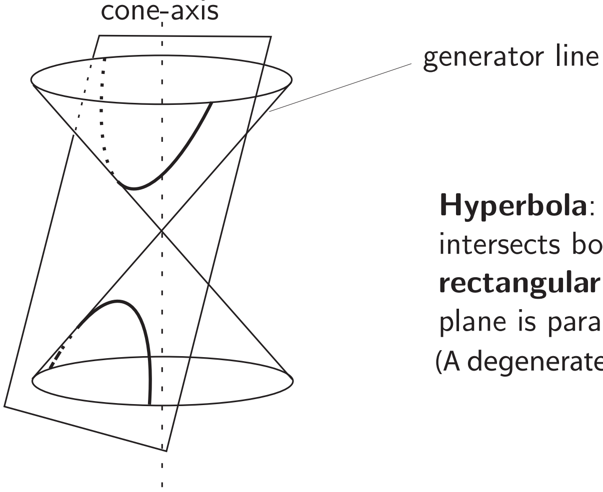


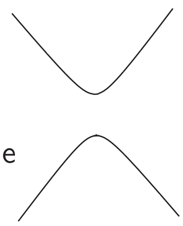
Figure 1: Circle and ellipse





Parabola: obtained when the plane is parallel to the generator of the cone. Different parabolas are obtained as the point P moves along a generator.





Hyperbola: obtained when the plane intersects both parts of the cone. The **rectangular hyperbola** is obtained when the plane is parallel to the cone-axis.
(A degenerate case is two straight lines.)

Figure 2: Parabola and hyperbola

The ellipse

We are all aware that the paths followed by the planets around the sun are elliptical. However, more generally the ellipse occurs in many areas of engineering. The standard form of an ellipse is shown in Figure 3.

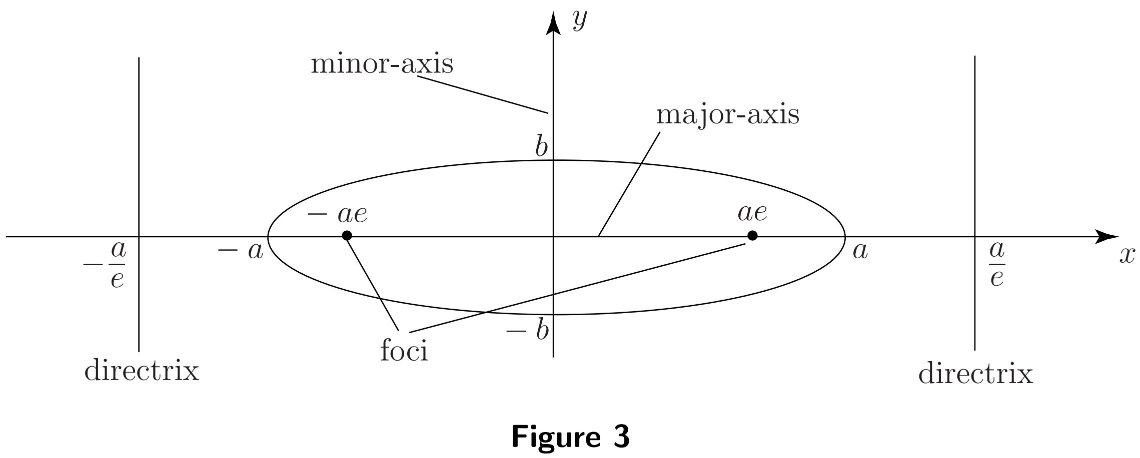


Figure 3

If $a > b$ (as in Figure 1) then the x-axis is called the **major-axis** and the y-axis is called the **minor-axis**. On the other hand if $b > a$ then the y-axis is called the major-axis and the x-axis is then the minor-axis. Two points, inside the ellipse are of importance; these are the **foci**. If $a > b$ these are located at coordinate positions $\pm ae$ (or at $\pm be$ if $b > a$) on the major-axis, with e, called the **eccentricity**, given by

$$e^2 = 1 - \frac{b^2}{a^2} \qquad (b < a) \qquad \text{or by} \qquad e^2 = 1 - \frac{a^2}{b^2} \qquad (a < b)$$

The foci of an ellipse have the property that if light rays are emitted from one focus then on reflection at the elliptic curve they pass through at the other focus.

Key Point 1

The standard Cartesian equation of the **ellipse** with its centre at the origin is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

This ellipse has intercepts on the x-axis at $x = \pm a$ and on the y-axis at $\pm b$. The curve is also symmetrical about both axes. The curve reduces to a circle in the special case in which $a = b$.

Example 1

(a) Sketch the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ (b) Find the eccentricity e

(c) Locate the positions of the foci.

Solution

(a) We can calculate the values of y as x changes from 0 to 2:

x	0	0.30	0.60	0.90	1.20	1.50	1.80	2
y	3	2.97	2.86	2.68	2.40	1.98	1.31	0

From this table of values, and using the symmetry of the curve, a sketch can be drawn (see Figure 4). Here $b = 3$ and $a = 2$ so the y-axis is the major axis and the x-axis is the minor axis.

Here $b = 3$ and $a = 2$ so the y-axis is the major axis and the x-axis is the minor axis.

(b) $e^2 = 1 - a^2/b^2 = 1 - 4/9 = 5/9 \qquad \therefore \quad e = \sqrt{5}/3$

(c) Since $b > a$ and $be = \sqrt{5}$, the foci are located at $\pm\sqrt{5}$ on the y-axis.

Solution (contd.)

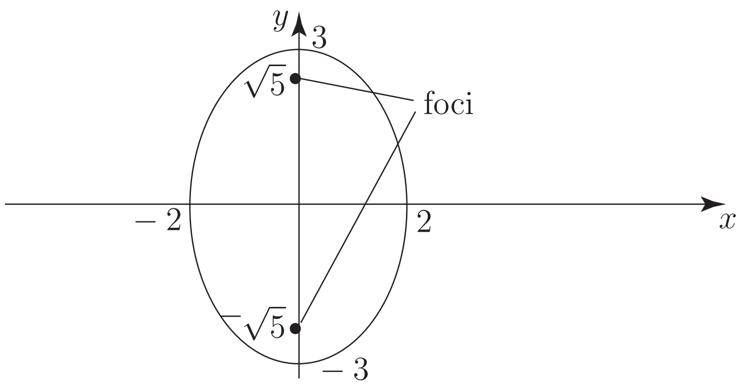


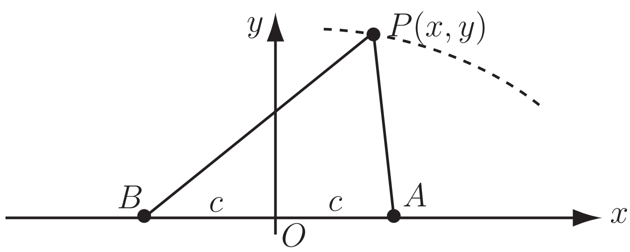
Figure 4

Key Point 1 gives the equation of the ellipse with its centre at the origin. If the centre of the ellipse has coordinates $(\alpha,\ \beta)$ and still has its axes parallel to the x- and y-axes the standard equation becomes

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1.$$

Task

Consider the points A and B with Cartesian coordinates $(c,0)$ and $(-c,0)$ respectively. A curve has the property that for every point P on it the sum of the distances PA and PB is a constant (which we will call $2a$). Derive the Cartesian form of the equation of the curve and show that it is an ellipse.



Your solution

Answer
We use Pythagoras's theorem to work out the distances PA and PB:

Let $R_1 = PB = [(x+c)^2 + y^2]^{1/2}$ and let $R_2 = PA = [(c-x)^2 + y^2]^{1/2}$

We now take the given equation $R_1 + R_2 = 2a$ and multiply both sides by $R_1 - R_2$. The quantity $R_1^2 - R_2^2$ on the left is calculated to be $4cx$, and $2a(R_1 - R_2)$ is on the right. We thus obtain a pair of equations: $R_1 + R_2 = 2a$ and $R_1 - R_2 = \dfrac{2cx}{a}$

Adding these equations together gives $R_1 = a + \dfrac{cx}{a}$ and squaring this equation gives

$$x^2 + c^2 + 2cx + y^2 = a^2 + \frac{c^2x^2}{a^2} + 2cx$$

Simplifying: $x^2(1 - \dfrac{c^2}{a^2}) + y^2 = a^2 - c^2$ whence $\dfrac{x^2}{a^2} + \dfrac{y^2}{(a^2 - c^2)} = 1$

This is the standard equation of an ellipse if we set $b^2 = a^2 - c^2$, which is the traditional equation which relates the two semi-axis lengths a and b to the distance c of the foci from the centre of the ellipse.

The foci A and B have **optical** properties; a beam of light travelling from A along AP and undergoing a mirror reflection from the ellipse at P will return along the path PB to the other focus B.

The circle

The circle is a special case of the ellipse; it occurs when $a = b = r$ so the equation becomes

$\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1$ or, more commonly $x^2 + y^2 = r^2$

Here, the **centre** of the circle is located at the origin $(0,0)$ and the **radius** of the circle is r. If the centre of the circle at a point (α, β) then the equation takes the form:

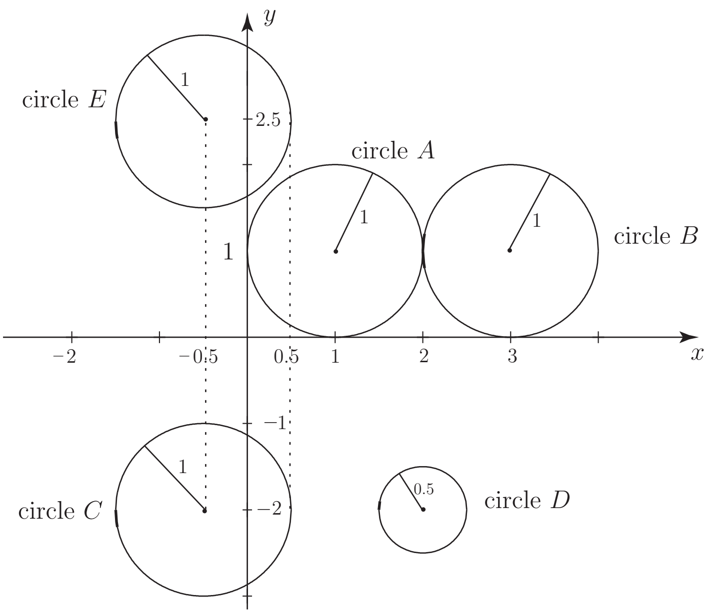
$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

Key Point 2

The equation of a circle with centre at (α, β) and radius r is $(x-\alpha)^2 + (y-\beta)^2 = r^2$

Task

Write down the equations of the five circles (A to E) below:



Your solution

Answer

$A \quad (x-1)^2 + (y-1)^2 = 1$

$B \quad (x-3)^2 + (y-1)^2 = 1$

$C \quad (x+0.5)^2 + (y+2)^2 = 1$

$D \quad (x-2)^2 + (y+2)^2 = (0.5)^2$

$E \quad (x+0.5)^2 + (y-2.5)^2 = 1$

Example 2

Show that the expression

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

represents the equation of a circle. Find its centre and radius.

Solution

We shall see later how to recognise this as the equation of a circle simply by examination of the coefficients of the quadratic terms x^2, y^2 and xy. However, in the present example we will use the process of completing the square, for x and for y, to show that the expression can be written in standard form.

Now $\quad x^2 + y^2 - 2x + 6y + 6 \equiv x^2 - 2x + y^2 + 6y + 6.$

Also,

$$x^2 - 2x \equiv (x-1)^2 - 1 \quad \text{and} \quad y^2 + 6y \equiv (y+3)^2 - 9.$$

Hence we can write

$$x^2 + y^2 - 2x + 6y + 6 \equiv (x-1)^2 - 1 + (y+3)^2 - 9 + 6 = 0$$

or, taking the free constants to the right-hand side:

$$(x-1)^2 + (y+3)^2 = 4.$$

By comparing this with the standard form we conclude this represents the equation of a circle with centre at $(1, -3)$ and radius 2.

Task

Find the centre and radius of each of the following circles:

(a) $x^2 + y^2 - 4x - 6y = -12$ (b) $2x^2 + 2y^2 + 4x + 1 = 0$

Your solution

Answer

(a) centre: $(2, 3)$ radius 1 (b) centre: $(-1, 0)$ radius $\sqrt{2}/2$.

Engineering Example 1

A circle-cutting machine

Introduction

A cutting machine creates circular holes in a piece of sheet-metal by starting at the centre of the circle and cutting its way outwards until a hole of the correct radius exists. However, prior to cutting, the circle is characterised by three points on its circumference, rather than by its centre and radius. Therefore, it is necessary to be able to find the centre and radius of a circle given three points that it passes through.

Problem in words

Given three points on the circumference of a circle, find its centre and radius

(a) for three general points

(b) (i) for $(-6,5)$, $(-3,6)$ and $(2,1)$ (ii) for $(-0.7,0.6)$, $(5.9,1.4)$ and $(0.8,-2.8)$

where coordinates are in cm.

Mathematical statement of problem

A circle passes through the three points. Find the centre (x_0, y_0) and radius R of this circle when the three circumferential points are

(a) (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(b) (i) $(-6,5)$, $(-3,6)$ and $(2,1)$

(ii) $(-0.7,0.6)$, $(5.9,1.4)$ and $(0.8,-2.8)$

Measurements are in centimetres; give answers correct to 2 decimal places.

Mathematical analysis

(a) The equation of a circle with centre at (x_0, y_0) and radius R is

$$(x-x_0)^2+(y-y_0)^2=R^2$$

and, if this passes through the 3 points (x_1, y_1), (x_2, y_2) and (x_3, y_3) then

$$\begin{aligned}(x_1-x_0)^2+(y_1-y_0)^2 &= R^2 &\quad(1)\\ (x_2-x_0)^2+(y_2-y_0)^2 &= R^2 &\quad(2)\\ (x_3-x_0)^2+(y_3-y_0)^2 &= R^2 &\quad(3)\end{aligned}$$

Eliminating the R^2 term between (1) and (2) gives

$$(x_1-x_0)^2+(y_1-y_0)^2=(x_2-x_0)^2+(y_2-y_0)^2$$

so that

$$x_1^2-2x_0x_1+y_1^2-2y_0y_1=x_2^2-2x_0x_2+y_2^2-2y_0y_2 \qquad (4)$$

Similarly, eliminating R^2 between (1) and (3) gives

$$x_1^2 - 2x_0x_1 + y_1^2 - 2y_0y_1 = x_3^2 - 2x_0x_3 + y_3^2 - 2y_0y_3 \tag{5}$$

Re-arranging (4) and (5) gives a system of two equations in x_0 and y_0.

$$2(x_2 - x_1)x_0 + 2(y_2 - y_1)y_0 = x_2^2 + y_2^2 - x_1^2 - y_1^2 \tag{6}$$

$$2(x_3 - x_1)x_0 + 2(y_3 - y_1)y_0 = x_3^2 + y_3^2 - x_1^2 - y_1^2 \tag{7}$$

Multiplying (6) by $(y_3 - y_1)$, and multiplying (7) by $(y_2 - y_1)$, subtracting and re-arranging gives

$$x_0 = \frac{1}{2}\left(\frac{(y_3 - y_1)(x_2^2 + y_2^2) + (y_1 - y_2)(x_3^2 + y_3^2) + (y_2 - y_3)(x_1^2 + y_1^2)}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1}\right) \tag{8}$$

while a similar procedure gives

$$y_0 = \frac{1}{2}\left(\frac{(x_1 - x_3)(x_2^2 + y_2^2) + (x_2 - x_1)(x_3^2 + y_3^2) + (x_3 - x_2)(x_1^2 + y_1^2)}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1}\right) \tag{9}$$

Knowing x_0 and y_0, the radius R can be found from

$$R = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \tag{10}$$

(or alternatively using x_2 and y_2 (or x_3 and y_3) as appropriate).

Equations (8), (9) and (10) can now be used to analyse the two particular circles above.

(i) Here $x_1 = -6$ cm, $y_1 = 5$ cm, $x_2 = -3$ cm, $y_2 = 6$ cm, $x_3 = 2$ cm and $y_3 = 1$ cm, so that

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 = -3 - 12 + 10 + 6 - 36 + 15 = -20$$

and

$$x_1^2 + y_1^2 = 61 \qquad x_2^2 + y_2^2 = 45 \qquad x_3^2 + y_3^2 = 5$$

From (8)

$$x_0 = \frac{1}{2}\left(\frac{-4 \times 45 + (-1) \times 5 + 5 \times 61}{-20}\right) = \frac{-180 - 5 + 305}{-40} = -3$$

while (9) gives

$$y_0 = \frac{1}{2}\left(\frac{-8 \times 45 + 3 \times 5 + 5 \times 61}{-20}\right)$$
$$= \frac{-360 + 15 + 305}{-40} = 1$$

The radius can be found from (10)

$$R = \sqrt{(-6 - (-3))^2 + (5 - 1)^2} = \sqrt{25} = 5$$

so that the circle has centre at $(-3, 1)$ and a radius of 5 cm.

(ii) Now $x_1 = -0.7$ cm, $y_1 = 0.6$ cm, $x_2 = 5.9$ cm, $y_2 = 1.4$ cm, $x_3 = 0.8$ cm and $y_3 = -2.8$ cm, so that

$$x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1 = -16.52-1.12+0.48-1.96-0.98-3.54 = -23.64$$

and

$$x_1^2 + y_1^2 = 0.85 \qquad x_2^2 + y_2^2 = 36.77 \qquad x_3^2 + y_3^2 = 8.48$$

so from (8)

$$x_0 = \frac{1}{2}\left(\frac{-125.018 - 6.784 + 3.57}{-23.64}\right) = \frac{-128.232}{-47.28} = 2.7121827$$

and from (9)

$$y_0 = \frac{1}{2}\left(\frac{-55.155 + 55.968 - 4.335}{-23.64}\right) = \frac{-3.522}{-47.28} = 0.0744924$$

and from (10)

$$R = \sqrt{(-0.7 - 2.7121827)^2 + (0.6 - 0.0744924)^2} = \sqrt{11.9191490} = 3.4524121$$

so that, to 2 d.p., the circle has centre at $(2.71, 0.07)$ and a radius of 3.45 cm.

Mathematical comment

Note that the expression

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1$$

appears in the denominator for both x_0 and y_0. If this expression is equal to zero, the calculation will break down. Geometrically, this corresponds to the three points being in a straight line so that no circle can be drawn, or not all points being distinct so no unique circle is defined.

Engineering Example 2

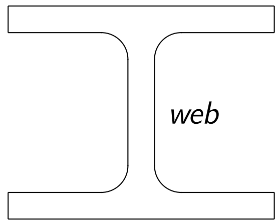
The web-flange junction

Introduction

In problems of torsion, the **torsion constant**, J, which is a function of the shape and structure of the element under consideration, is an important quantity.

A common beam section is the **thick I-section** shown here, for which the torsion constant is given by

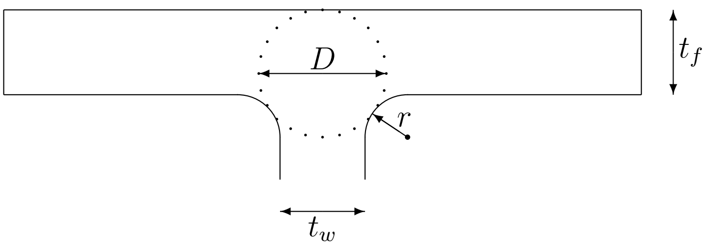
$$J = 2J_1 + J_2 + 2\alpha D^4$$



where the J_1 and J_2 terms refer to the flanges and web respectively, and the D^4 term refers to the web-flange junction. In fact

$$\alpha = \min\left[\frac{t_f}{t_w}, \frac{t_w}{t_f}\right]\left(0.15 + 0.1\,\frac{r}{t_f}\right)$$

where t_f and t_w are the thicknesses of the flange and web respectively, and r is the radius of the concave circle element between the flange and the web. D is the diameter of the circle of the web-flange junction.



As D occurs in the form D^4, the torsion constant is very sensitive to it. Calculation of D is therefore a crucial part of the calculation of J.

Problem in words

Find D, the diameter of the circle within the web–flange junction as a function of the other dimensions of the structural element.

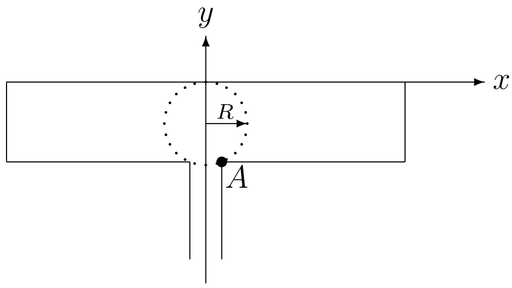
Mathematical statement of problem

(a) Find D, the diameter of the circle, in terms of t_f and t_w (the thicknesses of the flange and the web respectively) in the case where $r = 0$. When $t_f = 3$cm and $t_w = 2$cm, find D.

(b) For $r \neq 0$, find D in terms of t_f, t_w and r. In the special case where $t_f = 3$ cm, $t_w = 2$ cm and $r = 0.4$ cm, find D.

Mathematical analysis

(a) Consider a co-ordinate system based on the midpoint of the outer surface of the flange.



The centre of the circle will lie at $(0, -R)$ where R is the radius of the circle, i.e. $R = D/2$. The equation of the circle is

$$x^2 + (y + R)^2 = R^2 \qquad (1)$$

In addition, the circle passes through the 'corner' at point A $(t_w/2, -t_f)$, so

$$\left(\frac{t_w}{2}\right)^2 + (-t_f + R)^2 = R^2 \qquad (2)$$

On expanding

$$\frac{t_w^2}{4} + t_f^2 - 2Rt_f + R^2 = R^2$$

giving

$$2Rt_f = \frac{t_w^2}{4} + t_f^2 \qquad \Rightarrow \qquad R = \frac{(t_w^2/4) + t_f^2}{2t_f} = \frac{t_w^2}{8t_f} + \frac{t_f}{2}$$

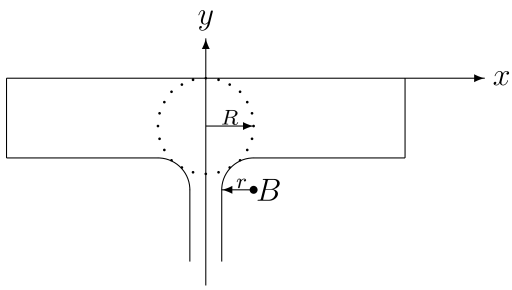
so that

$$D = 2R = \frac{t_w^2}{4t_f} + t_f \qquad (3)$$

Setting $t_f = 3$ cm, $t_w = 2$ cm gives

$$D = \frac{2^2}{4 \times 3} + 3 = 3.33 \text{ cm}$$

(b) Again using a co-ordinate system based on the mid-point of the outer surface of the flange, consider now the case $r \neq 0$.



Point B $(t_w/2 + r, -t_f - r)$ lies, not on the circle described by (1), but on the slightly larger circle with the same centre, and radius $R + r$. The equation of this circle is

$$x^2 + (y + R)^2 = (R + r)^2 \tag{4}$$

Putting the co-ordinates of point B into equation (4) gives

$$\left(\frac{t_w}{2} + r\right)^2 + (-t_f - r + R)^2 = (R + r)^2 \tag{5}$$

which, on expanding gives

$$\frac{t_w^2}{4} + t_w r + r^2 + t_f^2 + r^2 + R^2 + 2t_f r - 2t_f R - 2rR = R^2 + 2rR + r^2$$

Cancelling and gathering terms gives

$$\begin{aligned} \frac{t_w^2}{4} + t_w r + r^2 + t_f^2 + 2t_f r &= 4rR + 2t_f R \\ &= 2R\,(2r + t_f) \end{aligned}$$

so that

$$2R = D = \frac{(t_w^2/4) + t_w r + r^2 + t_f^2 + 2t_f r}{(2r + t_f)}$$

so $$D = \frac{t_w^2 + 4t_w r + 4r^2 + 4t_f^2 + 8t_f r}{(8r + 4t_f)} \tag{6}$$

Now putting $t_f = 3$ cm, $t_w = 2$ cm and $r = 0.4$ cm makes

$$D = \frac{2^2 + (4 \times 2 \times 0.4) + (4 \times 0.4^2) + (4 \times 3^2) + (8 \times 3 \times 0.4)}{(8 \times 0.4) + (4 \times 3)} = \frac{53.44}{15.2} = 3.52 \text{ cm}$$

Interpretation

Note that setting $r = 0$ in Equation (6) recovers the special case of $r = 0$ given by equation (3). The value of D is now available to be used in calculations of the torsion constant, J.

The parabola

The standard form of the parabola is shown in Figure 5. Here the x-axis is the line of symmetry of the parabola.

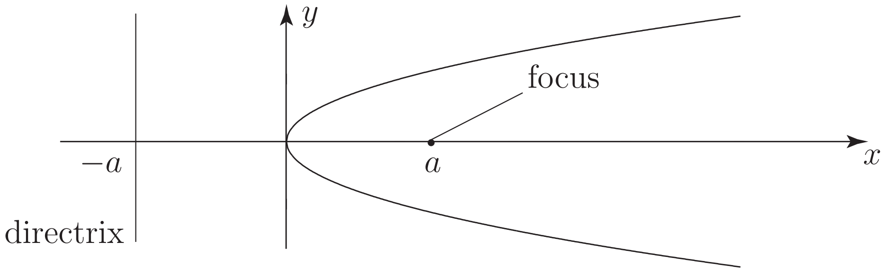


Figure 5

Key Point 3

The standard equation of the **parabola** with focus at $(a, 0)$ is

$$y^2 = 4ax$$

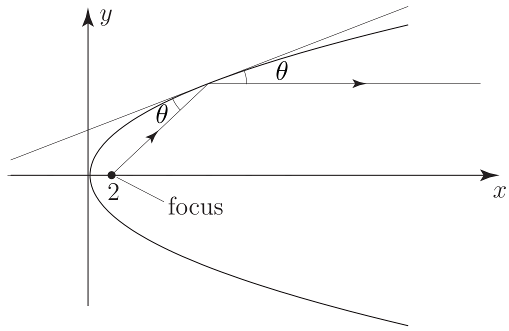
It can be shown that light rays parallel to the x-axis will, on reflection from the parabolic curve, come together at the focus. This is an important property and is used in the construction of some kinds of telescopes, satellite dishes and car headlights.

Task

Sketch the curve $y^2 = 8x$. Find the position of the focus and confirm its light-focusing property.

Your solution

Answer
This is a standard parabola ($y^2 = 4ax$) with $a = 2$. Thus the focus is located at coordinate position $(2, 0)$.



If your sketch is sufficiently accurate you should find that light-rays (lines) parallel to the x-axis when reflected off the parabolic surface pass through the focus. (Draw a tangent at the point of reflection and ensure that the angle of incidence (θ say) is the same as the angle of reflection.)

By changing the equation of the parabola slightly we can change the position of the parabola along the x-axis. See Figure 6.

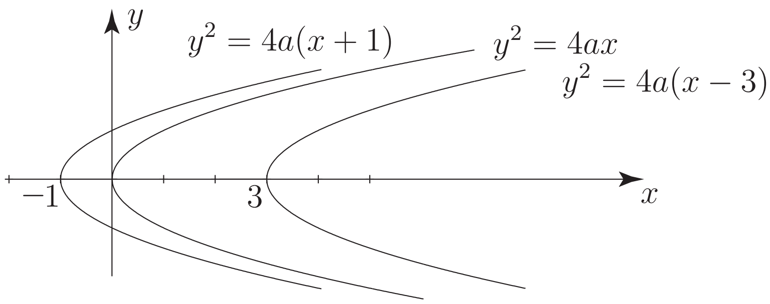


Figure 6: Parabola $y = 4a(x - b)$ with vertex at $x = b$

We can also have parabolas where the y-axis is the line of symmetry (see Figure 7). In this case the standard equation is

$$x^2 = 4ay \qquad \text{or} \qquad y = \frac{x^2}{4a}$$

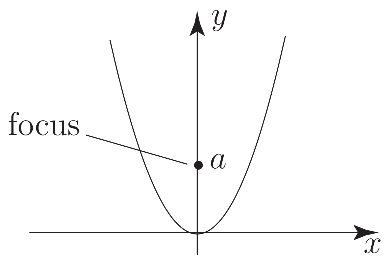


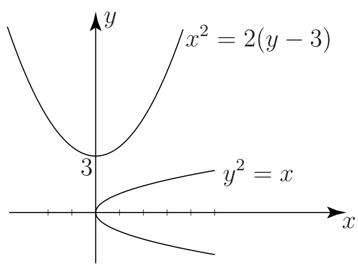
Figure 7

Task

Sketch the curves $y^2 = x$ and $x^2 = 2(y-3)$.

Your solution

Answer



The focus of the parabola $y^2 = 4a(x-b)$ is located at coordinate position $(a+b,\ 0)$. Changing the value of a changes the convexity of the parabola (see Figure 8).

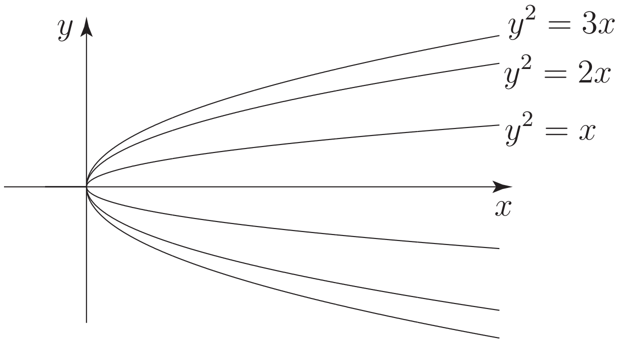


Figure 8

The hyperbola

The standard form of the hyperbola is shown in Figure 9(a).
This has standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The eccentricity, e, is defined by

$$e^2 = 1 + \frac{b^2}{a^2} \qquad (e > 1)$$

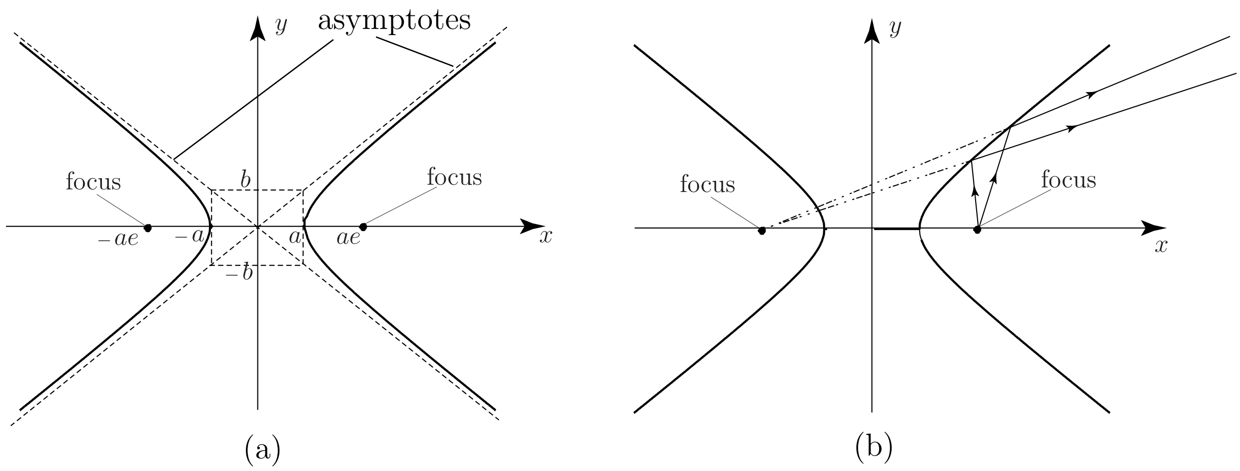


Figure 9

Note the change in sign compared to the equivalent expressions for the ellipse. The lines $y = \pm\dfrac{b}{a}x$ are **asymptotes** to the hyperbola (these are the lines to which each branch of the hyperbola approach as $x \to \pm\infty$).

If light is emitted from one focus then on hitting the hyperbolic curve it is reflected in such a way as to appear to be coming from the *other* focus. See Figure 9(b). The hyperbola has fewer uses in applications than the other conic sections and so we will not dwell here on its properties.

Key Point 4

The standard equation of the hyperbola with foci at $(\pm ae, 0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{with eccentricity } e \text{ given by} \qquad e^2 = 1 + \frac{b^2}{a^2} \quad (e > 1)$$

General conics

The conics we have considered above - the ellipse, the parabola and the hyperbola - have all been presented in standard form:- their axes are parallel to either the x- or y-axis. However, conics may be rotated to any angle with respect to the axes: they clearly remain conics, but what equations do they have?

It can be shown that the equation of **any conic**, can be described by the quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are constants.

If not all of A, B, C are zero (and F is a suitable number) the graph of this equation is

(i) an ellipse if $B^2 < 4AC$ (circle if $A = C$ **and** $B = 0$)

(ii) a parabola if $B^2 = 4AC$

(iii) a hyperbola if $B^2 > 4AC$

Example 3

Classify each of the following equations as ellipse, parabola or hyperbola:

(a) $x^2 + 2xy + 3y^2 + x - 1 = 0$
(b) $x^2 + 2xy + y^2 - 3y + 7 = 0$
(c) $2x^2 + xy + 2y^2 - 2x + 3y = 6$
(d) $3x^2 + 2x - 5y + 3y^2 - 10 = 0$

Solution

(a) Here $A = 1,\ B = 2,\ C = 3$ $\therefore$ $B^2 < 4AC$. This is an ellipse.

(b) Here $A = 1,\ B = 2,\ C = 1$ $\therefore$ $B^2 = 4AC$. This is a parabola.

(c) Here $A = 2,\ B = 1,\ C = 2$ $\therefore$ $B^2 < 4AC$ also $A = C$ but $B \neq 0$. This is an ellipse.

(d) Here $A = 3,\ B = 0,\ C = 3$ $\therefore$ $B^2 < 4AC$. Also $A = C$ and $B = 0$. This is a circle.

Task

Classify each of the following conics:

(a) $x^2 - 2xy - 3y^2 + x - 1 = 0$

(b) $2x^2 + xy - y^2 - 2x + 3y = 0$

(c) $4x^2 - y + 3 = 0$

(d) $-x^2 - xy - y^2 + 3x = 0$

(e) $2x^2 + 2y^2 - x + 3y = 7$

Your solution

Answer

(a)	$A = 1, B = -2, C = -3$	$B^2 > 4AC$	$\therefore$	hyperbola
(b)	$A = 2, B = 1, C = -1$	$B^2 > 4AC$	$\therefore$	hyperbola
(c)	$A = 4, B = 0, C = 0$	$B^2 = 4AC$	$\therefore$	parabola
(d)	$A = -1, B = -1, C = -1$	$B^2 < 4AC,\ A = C,\ \ B \neq 0$	$\therefore$	ellipse
(e)	$A = 2,\ B = 0,\ C = 2$	$B^2 < 4AC,\ \ A = C$ and $B = 0$	$\therefore$	circle

Exercises

1. The equation $9x^2 + 4y^2 - 36x + 24y - 1 = 0$ represents an ellipse. Find its centre, the semi-major and semi-minor axes and the coordinate positions of the foci.

2. Find the equation of a circle of radius 3 which has its centre at $(-1, 2.2)$

3. Find the centre and radius of the circle $x^2 + y^2 - 2x - 2y - 5 = 0$

4. Find the position of the focus of the parabola $y^2 - x + 3 = 0$

5. Classify each of the following conics

   (a) $x^2 + 2x - y - 3 = 0$

   (b) $8x^2 + 12xy + 17y^2 - 20 = 0$

   (c) $x^2 + xy - 1 = 0$

   (d) $4x^2 - y^2 - 4y = 0$

   (e) $6x^2 + 9y^2 - 24x - 54y + 51 = 0$

6. An asteroid has an elliptical orbit around the Sun. The major axis is of length 5×10^8 km. If the distance between the foci is 4×10^8 km find the equation of the orbit.

Answers

1. centre: $(2, -3)$, semi-major 3, semi-minor 2, foci: $(2, -3 \pm \sqrt{5})$

2. $(x + 1)^2 + (y - 2.2)^2 = 9$

3. centre: $(1, 1)$ radius $\sqrt{7}$

4. $y^2 = (x - 3) \qquad \therefore \qquad a = 1, \quad b = -3$. Hence focus is at coordinate position $(4, 0)$.

5. (a) parabola with vertex $(-1, -4)$

   (b) ellipse

   (c) hyperbola

   (d) hyperbola

   (e) ellipse with centre $(2, 3)$

6. $9x^2 + 25y^2 = 5.625 \times 10^7$

Polar Coordinates 17.2

Introduction

In this Section we extend the use of polar coordinates. These were first introduced in HELM 2.8. They were also used in the discussion on complex numbers in HELM 10.2. We shall examine the application of polars to the description of curves, particularly conics. Some curves, spirals for example, which are very difficult to describe in terms of Cartesian coordinates (x, y) are relatively easily defined in polars $[r, \theta]$.

Prerequisites

Before starting this Section you should . . .

- be familiar with Cartesian coordinates
- be familiar with trigonometric functions and how to manipulate then
- be able to simplify algebraic expressions and manipulate algebraic fractions

Learning Outcomes

On completion you should be able to . . .

- understand how Cartesian coordinates and polar coordinates are related
- find the polar form of a curve given in Cartesian form
- recognise some conics given in polar form

1. Polar Coordinates

In this Section we consider the application of polar coordinates to the description of curves; in particular, to conics.

If the Cartesian coordinates of a point P are (x, y) then P can be located on a Cartesian plane as indicated in Figure 10.

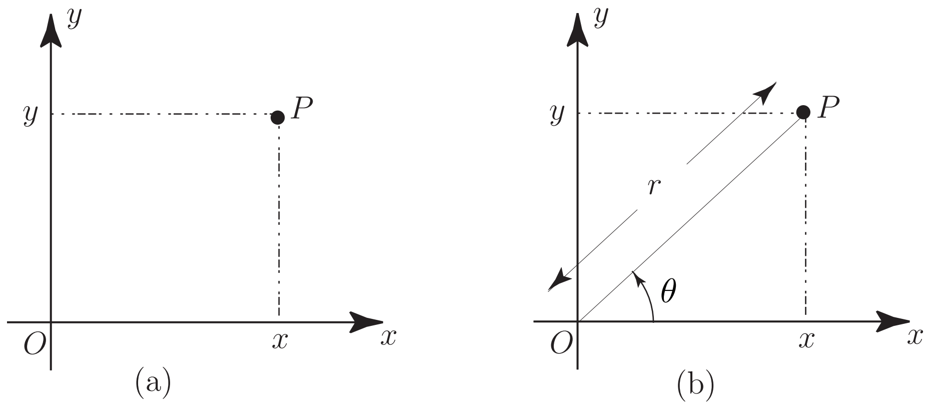


Figure 10

However, the same point P can be located by using polar coordinates r, θ where r is the distance of P from the origin and θ is the angle, measured anti-clockwise, that the line OP makes when measured from the positive x-direction. See Figure 10(b). In this Section we shall denote the polar coordinates of a point by using square brackets.

From Figure 10 it is clear that Cartesian and polar coordinates are directly related. The relations are noted in the following Key Point.

Key Point 5

If (x, y) are the Cartesian coordinates and $[r, \theta]$ the polar coordinates of a point P then

$$x = r\cos\theta \qquad y = r\sin\theta$$

and, equivalently,

$$r = +\sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

From these relations we see that it is a straightforward matter to calculate (x, y) given $[r, \theta]$. However, some care is needed (particularly with the determination of θ) if we want to calculate $[r, \theta]$ from (x, y).

Example 4

On a Cartesian plane locate points P, Q, R, S which have their locations specified by polar coordinates $[2, \frac{\pi}{2}]$, $[2, 3\frac{\pi}{2}]$, $[3, \frac{\pi}{6}]$, $[\sqrt{2}, \pi]$ respectively.

Solution

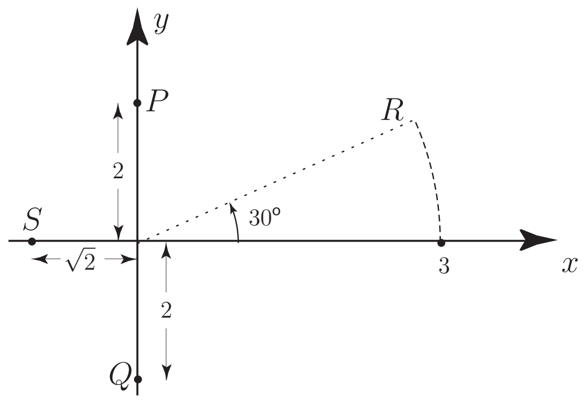


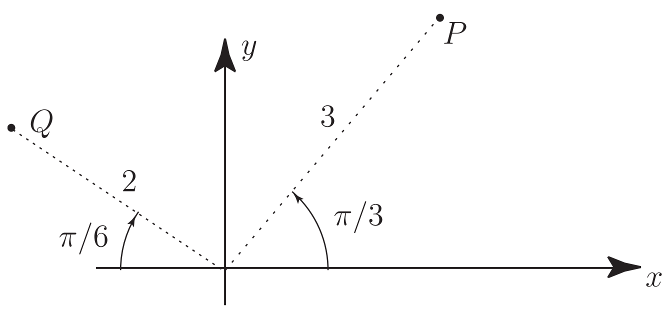
Figure 11

Task

Two points P, Q have polar coordinates $[3, \frac{\pi}{3}]$ and $[2, \frac{5\pi}{6}]$ respectively. By locating these points on a Cartesian plane find their equivalent Cartesian coordinates.

Your solution

Answer



$P: \ (3\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3}) \equiv (\frac{3}{2}, \frac{3\sqrt{3}}{2})$

$Q: \ (-2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}) \equiv (\frac{-2\sqrt{3}}{2}, 1)$

The polar coordinates of a point are not unique. So, the polar coordinates $[a, \theta]$ and $[a, \phi]$ **represent the same point** in the Cartesian plane provided θ and ϕ differ by an integer multiple of 2π. See Figure 12.

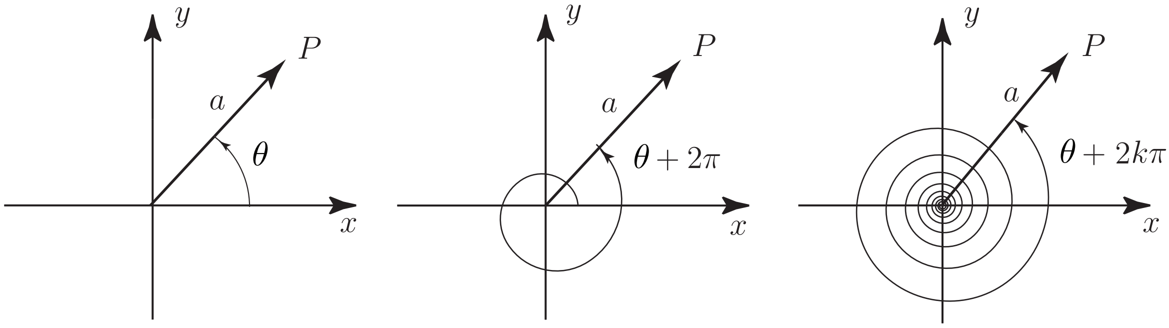


Figure 12

For example, the polar coordinates $[2, \dfrac{\pi}{3}]$, $[2, \dfrac{7\pi}{3}]$, $[2, -\dfrac{5\pi}{3}]$ all represent the same point in the Cartesian plane.

Key Point 6

By convention, we measure the positive angle θ in an **anti-clockwise direction**.

The angle $-\theta$ is interpreted as the angle θ measured in a **clockwise direction**.

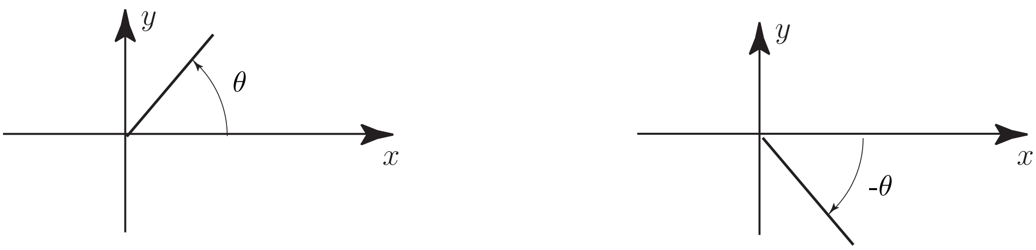


Figure 13

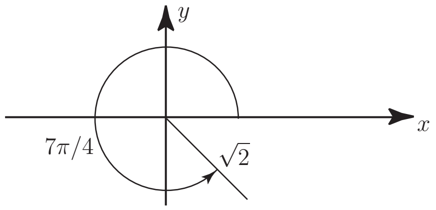
Exercises

1. The Cartesian coordinates of P, Q are $(1, -1)$ and $(-1, \sqrt{3})$. What are their equivalent polar coordinates?

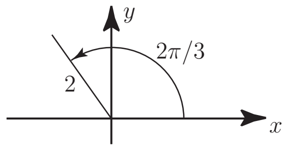
2. Locate the points P, Q, R with polar coordinates $[1, \dfrac{\pi}{3}]$, $[2, \dfrac{7\pi}{3}]$, $[2, \dfrac{10\pi}{3}]$. What do you notice?

Answer

1.



$(1, -1) \;\rightarrow\; [\sqrt{2},\;\; 7\pi/4]$



$(-1, \sqrt{3}) \;\rightarrow\; [2,\;\; 2\pi/3]$

2. All these points lie on a straight line through the origin.

2. Simple curves in polar coordinates

We are used to describing the equations of curves in Cartesian variables x, y. Thus $x^2 + y^2 = 1$ represents a circle, centre the origin, and of radius 1, and $y = 2x^2$ is the equation of a parabola whose axis is the y-axis and with vertex located at the origin. (In colloquial terms the vertex is the 'sharp end' of a conic.) We can convert these equations into polar form by using the relations $x = r\cos\theta$, $y = r\sin\theta$.

Example 5

Find the polar coordinate form of

(a) the circle $x^2 + y^2 = 1$ (b) the parabola $y = 2x^2$.

Solution

(a) Using $x = r\cos\theta$, $y = r\sin\theta$ in the expression $x^2 + y^2 = 1$ we have

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 1 \quad \text{or} \quad r^2(\cos^2\theta + \sin^2\theta) = 1$$

giving $r^2 = 1$. We simplify this to $r = 1$ (since $r = -1$ is invalid being a negative distance). Of course we might have guessed this answer since the relation $r = 1$ states that every point on the curve is a constant distance 1 away from the origin.
(b) Repeating the approach used in (a) for $y = 2x^2$ we obtain:

$$r\sin\theta = 2(r\cos\theta)^2 \qquad \text{i.e.} \qquad r\sin\theta - 2r^2\cos^2\theta = 0$$

Therefore $r(\sin\theta - 2r\cos^2\theta) = 0$. Either $r = 0$ (which is a single point, the origin, and is clearly not a parabola) or

$$\sin\theta - 2r\cos^2\theta = 0 \qquad \text{giving, finally} \qquad r = \frac{1}{2}\tan\theta\sec\theta.$$

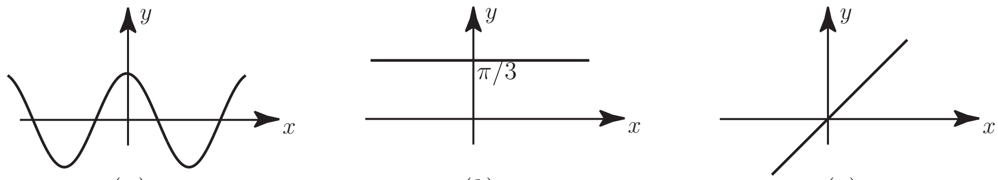
This is the polar equation of this particular parabola, $y = 2x^2$.

Task

Sketch the curves
(a) $y = \cos x$ (b) $y = \dfrac{\pi}{3}$ (c) $y = x$

Your solution

Answer



(a) (b) (c)

Task

Sketch the curve $r = \cos\theta$.

First complete the table of values. Enter values to 2 d.p. and work in radians:

Your solution

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
r							

Answer

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
r	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00

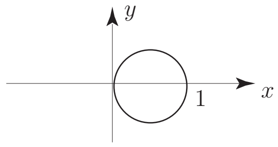
You will see that the values of θ for $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$ give rise to negative values of r (and hence invalid).

Now sketch the curve:

Your solution

Answer

circle: centre $\left(\dfrac{1}{2}, 0\right)$, radius $\dfrac{1}{2}$.



Sketch the curve $\theta = \pi/3$.

Your solution

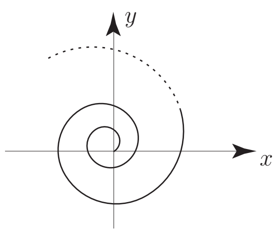
Answer

Radial line passing through the origin at angle $\dfrac{\pi}{3}$ to the positive x-axis.

Sketch the curve $r = \theta$.

Your solution

Answer



3. Standard conics in polar coordinates

In the previous Section we merely stated the standard equations of the conics using Cartesian coordinates. Here we consider an alternative definition of a conic and use this different approach to obtain the equations of the standard conics in polar form. Consider a straight line $x = -d$ (this will be the directrix of the conic) and let e be the eccentricity of the conic (e is a positive real number). It can be shown that the set of points P in the (x, y) plane which satisfy the condition

$$\frac{\text{distance of } P \text{ from origin}}{\text{perpendicular distance from } P \text{ to the line}} = e$$

is a conic with eccentricity e. In particular, it is an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$. See Figure 14.

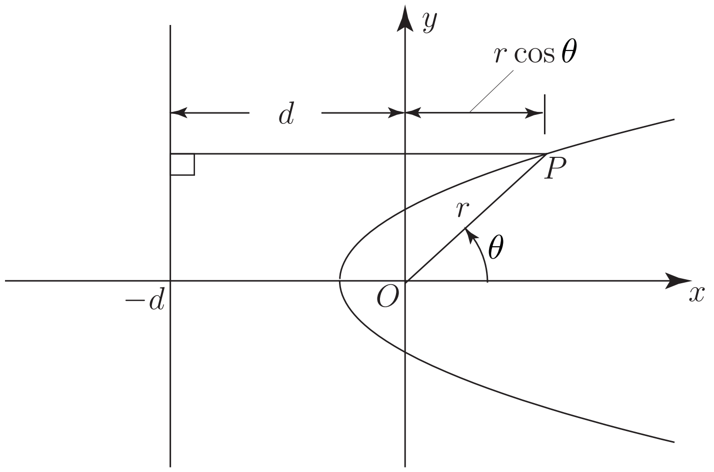


Figure 14

We can obtain the polar coordinate form of this conic in a straightforward manner. If P has polar coordinates $[r, \theta]$ then the relation above gives

$$\frac{r}{d + r\cos\theta} = e \qquad \text{or} \qquad r = e(d + r\cos\theta)$$

Thus, solving for r: $\qquad r = \dfrac{ed}{1 - e\cos\theta}$

This is the equation of the conic.

In all of these conics it can be shown that one of the foci is located at the origin. See Figure 15 in which the pertinent details of the conics are highlighted.

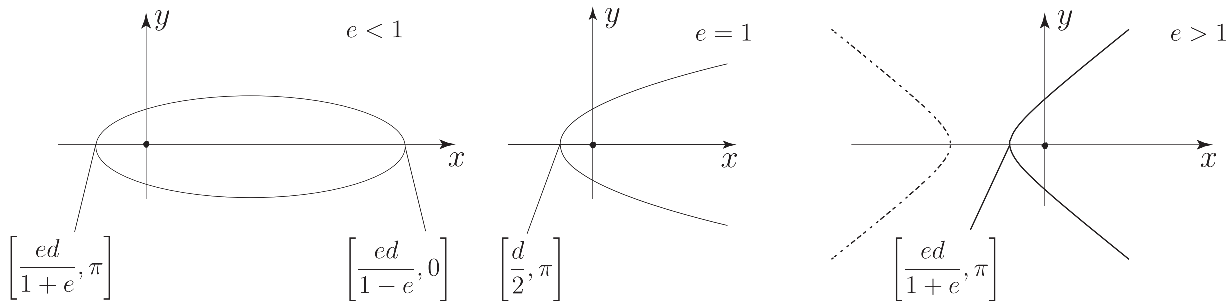


Figure 15

Task

Sketch the ellipse $r = \dfrac{4}{2-\cos\theta}$ and locate the coordinates of its vertices.

Your solution

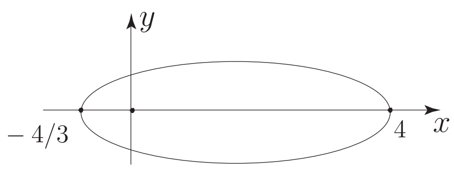
Answer

Here

$$r = \frac{4}{2-\cos\theta} = \frac{2}{1-\frac{1}{2}\cos\theta} \qquad \text{so} \qquad e = \frac{1}{2}$$

Then

$$de = 2 \qquad \frac{de}{1+e} = \frac{2}{\frac{3}{2}} = \frac{4}{3} \qquad \text{and} \qquad \frac{de}{1-e} = \frac{2}{\frac{1}{2}} = 4$$



Exercises

1. Sketch the polar curves (a) $r = \dfrac{1}{1-\cos\theta}$ (b) $r = e^{-\theta}$ (c) $r = \dfrac{6}{3-\cos\theta}$.

2. Find the polar form of the following curves given in Cartesian form:

   (a) $y^2 = 1 + 2x$ (b) $2xy = 1$

3. Find the Cartesian form of the following curves given in polar form

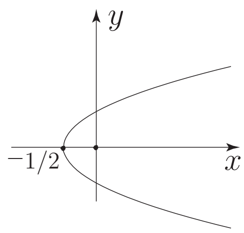
   (a) $r = \dfrac{2}{\sin\theta + 2\cos\theta}$ (b) $r = 3\cos\theta$

   Do you recognise these equations?

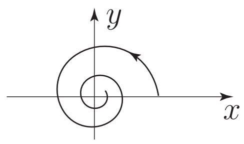
Answers

1.

(a) parabola $e = 1,\ d = 1$

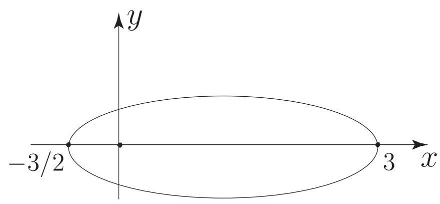


(b) decreasing spiral



(c) $r = \dfrac{2}{1-\frac{1}{3}\cos\theta}$

ellipse since $e = \frac{1}{3} < 1$. Also $de = 2$



2.

(a) $r^2\sin^2\theta = 1 + 2r\cos\theta \quad \therefore \quad r = \dfrac{\cos\theta + 1}{1-\cos^2\theta} = \dfrac{1}{1-\cos\theta}$

(b) $2r^2\cos\theta\sin\theta = 1 \quad \therefore \quad r^2 = \operatorname{cosec} 2\theta$

3.

(a) $r(\sin\theta + 2\cos\theta) = 2 \quad \therefore \quad y + 2x = 2$ which is a straight line

(b) $r = 3\cos\theta \quad \therefore \quad \sqrt{x^2+y^2} = \dfrac{3x}{\sqrt{x^2+y^2}} \quad \therefore \quad x^2 + y^2 = 3x$

in standard form: $\left(x - \dfrac{3}{2}\right)^2 + y^2 = \dfrac{9}{4}$. i.e. a circle, centre $\left(\dfrac{3}{2},\ 0\right)$ with radius $\dfrac{3}{2}$

Parametric Curves

17.3

Introduction

In this Section we examine yet another way of defining curves - the parametric description. We shall see that this is, in some ways, far more useful than either the Cartesian description or the polar form. Although we shall only study planar curves (curves lying in a plane) the parametric description can be easily generalised to the description of spatial curves which twist and turn in three dimensional space.

Prerequisites

Before starting this Section you should ...

- be familiar with Cartesian coordinates
- be familiar with trigonometric and hyperbolic functions and be able to manipulate them
- be able to differentiate simple functions
- be able to locate turning points and distinguish between maxima and minima.

Learning Outcomes

On completion you should be able to ...

- sketch planar curves given in parametric form
- understand how the same curve can be described using different parameterisations
- recognise some conics given in parametric form

1. Parametric curves

Here we explore the use of a parameter t in the description of curves. We shall see that it has some advantages over the more usual Cartesian description. We start with a simple example.

Example 6

Plot the curve $\underbrace{x = 2\cos t \qquad y = 3\sin t}_{\text{parametric equations of the curve}}$ $\qquad \underbrace{0 \leq t \leq \frac{\pi}{2}}_{\text{parameter range}}$

Solution

The approach to sketching the curve is straightforward. We simply give the parameter t various values as it ranges through $0 \to \dfrac{\pi}{2}$ and, for each value of t, calculate corresponding values of (x, y) which are then plotted on a Cartesian xy plane. The value of t and the corresponding values of x, y are recorded in the following table:

t	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
x	2	1.98	1.90	1.78	1.62	1.41	1.18	0.91	0.62	0.31	0
y	0	0.47	0.93	1.36	1.76	2.12	2.43	2.67	2.85	2.96	3

Plotting the (x, y) coordinates gives the curve in Figure 16.

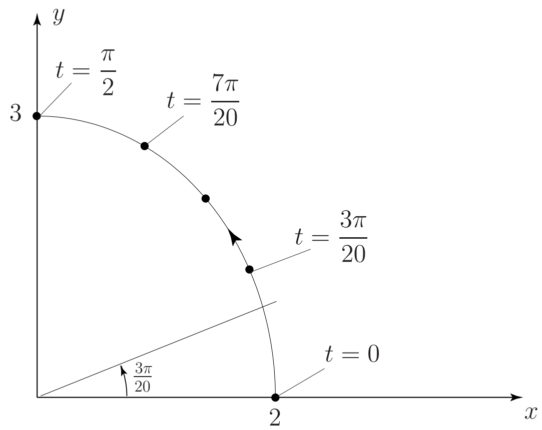


Figure 16

The curve in Figure 16 resembles part of an ellipse. This can be verified by eliminating t from the parametric equations to obtain an expression involving x, y only. If we divide the first parametric equation by 2 and the second by 3, square both and add we obtain

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t \equiv 1 \qquad \text{i.e.} \qquad \frac{x^2}{4} + \frac{y^2}{9} = 1$$

which we easily recognise as an ellipse whose major-axis is the y-axis. Also, as t ranges from $0 \to \dfrac{\pi}{2}$ $x = 2\cos t$ decreases from $2 \to 0$, and $y = 3\sin t$ increases from $0 \to 3$. We conclude that the parametric equations $x = 2\cos t$, $y = 3\sin t$ together with the parametric range $0 \leq t \leq \dfrac{\pi}{2}$ describe that part of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ in the positive quadrant. On the curve in Figure 16 we have used an arrow to indicate the direction that we move along the curve as t **increases** from its initial value 0.

Task

Plot the curve $\quad x = t + 1 \qquad y = 2t^2 - 3 \qquad 0 \leq t \leq 1$

Do you recognise this curve as a conic section?

First construct a table of (x, y) values as t ranges from $0 \to 1$:

Your solution

t	0	0.25	0.5	0.75	1
x					
y					

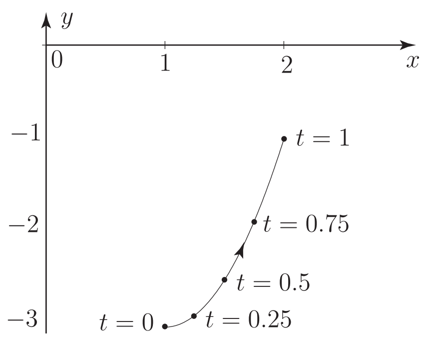
Answer

t	0	0.25	0.5	0.75	1
x	1	1.25	1.5	1.75	2
y	-3	-2.88	-2.5	-1.88	-1

Now plot the points on a Cartesian plane:

Your solution

Answer



Now eliminate the t-variable from $x = t + 1$, $y = 2t^2 - 3$ to obtain the xy form of the curve:

Your solution

Answer

$y = 2x^2 - 4x - 1$ which is the equation of a parabola.

Example 7

Sketch the curve $\quad x = t^2 + 1 \qquad y = 2t^4 - 3 \qquad 0 \leq t \leq 1$

Solution

This is very similar to the previous Task (except for t^4 replacing t^2 in the expression for y and t^2 replacing t in the expression for x). The corresponding table of values is

t	0	0.25	0.5	0.75	1
x	1	1.06	1.25	1.56	2
y	-3	-2.99	-2.88	-2.37	-1

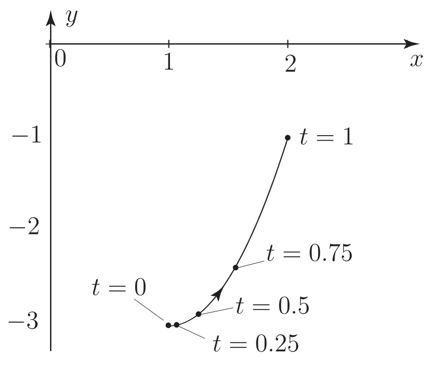


Figure 17

We see that this is **identical** to the curve drawn previously. This is confirmed by eliminating the t-parameter from the expressions defining x, y. Here $t^2 = x - 1$ so $y = 2(x-1)^2 - 3$ which is the same as obtained in the last Task. The main difference is that particular values of t locate (in general) different (x, y) points on the curve for the two parametric representations.

We conclude that a given curve in the xy plane can have many (in fact infinitely many) parametric descriptions.

Task

Show that the two parametric representations below describe the same curve.

(a) $x = \cos t \qquad y = \sin t \qquad 0 \le t \le \dfrac{\pi}{2}$

(b) $x = t \qquad y = \sqrt{1 - t^2} \qquad 0 \le t \le 1$

Eliminate t from the parametric equations in (a):

Your solution

Answer

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

Eliminate t from the parametric equations in (b):

Your solution

Answer

$y = \sqrt{1 - x^2} \qquad \therefore \qquad y^2 = 1 - x^2$ or $x^2 + y^2 = 1$

What do you conclude?

Your solution

Answer

Both parametric descriptions represent (part of) a circle centred at the origin of radius 1.

2. General parametric form

We will assume that any curve in the xy plane may be written in parametric form:

$$\underbrace{x = g(t) \qquad y = h(t)}_{\text{parametric equations of the curve}} \qquad\qquad \underbrace{t_0 \le t \le t_1}_{\text{parameter range}}$$

in which $g(t),\ h(t)$ are **given** functions of t and the parameter t ranges over the values $t_0 \to t_1$. As we give values to t within this range then corresponding values of x, y are calculated from $x = g(t)$, $y = h(t)$ which can then be plotted on an xy plane.

In HELM 12.3, we discovered how to obtain the derivative $\dfrac{dy}{dx}$ from a knowledge of the parametric derivatives $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$. We found

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right) \div \left(\frac{dx}{dt}\right)^3$$

Note that derivatives with respect to the parameter t are often denoted by a dot:

$$\frac{dx}{dt} \equiv \dot{x} \qquad \frac{dy}{dt} \equiv \dot{y} \qquad \frac{d^2x}{dt^2} \equiv \ddot{x} \quad \text{etc}$$

so that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Knowledge of the derivative is sometimes useful in curve sketching.

Example 8

Sketch the curve $\quad x = t^3 + 3t^2 + 2t \qquad y = 3 - 2t - t^2 \qquad -3 \leq t \leq 1.$

Solution

$$x = t^3 + 3t^2 + 2t = t(t+2)(t+1) \qquad y = 3 - 2t - t^2 = -(t+3)(t-1)$$

so that $x = 0$ when $t = 0,\ -1,\ -2$ and $y = 0$ when $t = -3,\ 1$. We calculate the values of x, y at various values of t:

t	-3	-2.50	-2	-1.50	-1	-0.50	0	0.50
x	-6	-1.88	0	0.38	0	-0.38	0	1.88
y	0	1.75	3	3.75	4	3.75	3	1.75

We see that $t = -2$ and $t = 0$ give rise to the **same** coordinate values for (x, y). This represents a **double-point** in the curve which is one where the curve crosses itself. Now

$$\frac{dx}{dt} = 3t^2 + 6t + 2, \quad \frac{dy}{dt} = -2 - 2t \qquad \therefore \qquad \frac{dy}{dx} = \frac{-2(1+t)}{3t^2 + 6t + 2}$$

so there is a turning point when $t = -1$. The reader is urged to calculate $\dfrac{d^2y}{dx^2}$ and to show that this is negative when $t = -1$ (i.e. at $x = 0,\ y = 4$) indicating a maximum when. (The reader should check that vertical tangents occur at $t = -0.43$ and $t = -1.47$, to 2 d.p.)

We can now make a reasonable sketch of the curve:

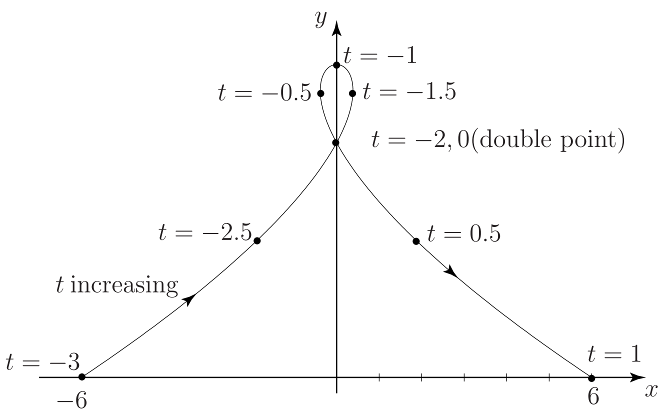


Figure 18

3. Standard forms of conic sections in parametric form

We have seen above that, given a curve in the xy plane, there is no unique way of representing it in parametric form. However, for some commonly occurring curves, particularly the conics, there are accepted standard parametric equations.

The parabola

The standard parametric equations for a parabola are: $x = at^2 \qquad y = 2at$

Clearly, we have $t = \dfrac{y}{2a}$ and by eliminating t we get $x = a\left(\dfrac{y^2}{4a^2}\right)$ or $y^2 = 4ax$ which we recognise as the standard Cartesian description of a parabola. As an illustration, Figure 19 shows the curve with $a = 2$ and $-1 \leq t \leq 2.3$

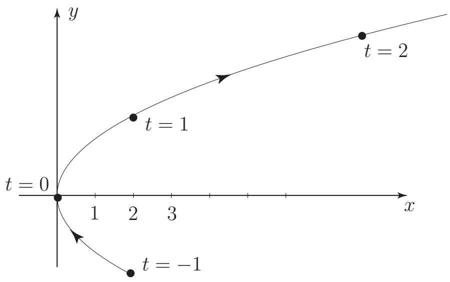


Figure 19

The ellipse

Here, the standard equations are $x = a\cos t \qquad y = b\sin t$

Again, eliminating t (dividing the first equation by a, the second by b, squaring and adding) we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t \equiv 1 \quad \text{or, in more familiar form:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we choose the range for t as $0 \leq t \leq \dfrac{7\pi}{4}$ the following segment of the ellipse is obtained.

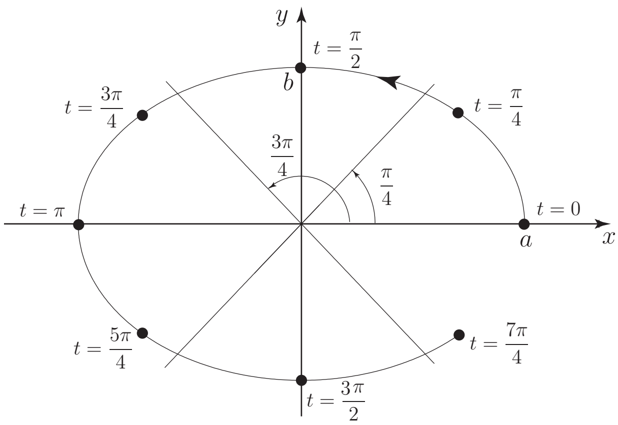


Figure 20

Here we note that (except in the special case when $a = b$, giving a circle) the parameter t is **not** the angle that the radial line makes with the the positive x-axis.

In the study of the orbits of planets and satellites it is often preferable to use plane polar coordinates (r, θ) to treat the problem. In these coordinates an ellipse has an equation of the form $\frac{1}{r} = A + B\cos\theta$, with A and B positive numbers such that $B < A$. Not only is there a difference in the equations on passing from Cartesian to polar coordinates; there is also a change in the origin of coordinates. The polar coordinate equation is using a focus of the ellipse as the origin. In the Cartesian description the foci are two points at $+e$ along the x-axis, where e obeys the equation $e = a - b$, if we assume that $a < b$ i.e. we choose the long axis of the ellipse as the x-axis. This problem gives some practice at algebraic manipulation and also indicates some shortcuts which can be made once the mathematics of the ellipse has been understood.

Example 9

An ellipse is described in plane polar coordinates by the equation

$$\frac{1}{r} = 2 + \cos\theta$$

Convert the equation to Cartesian form. [Hint: remember that $x = r\cos\theta$.]

Solution

Multiplying the given equation by r and then using $x = r\cos\theta$ gives the results

$$1 = 2r + x \qquad \text{so that} \qquad 2r = 1 - x$$

We now square the second equation, remembering that $r^2 = x^2 + y^2$. We now have

$$4(x^2 + y^2) = (1 - x)^2 = 1 + x^2 - 2x \quad \text{so that} \quad 3x^2 + 2x + 4y^2 = 1$$

We now recall the method of completing the square, which allows us to set

$$3x^2 + 2x = 3(x^2 + \frac{2x}{3})^2 - \frac{1}{9})$$

Putting this result into the equation and collecting terms leads to the final result

$$\frac{(x + \frac{1}{3})^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with} \quad a = \frac{2}{3} \text{ and } b = \sqrt{\frac{1}{3}}.$$

This is the standard Cartesian form for the equation of an ellipse **but** we must remember that we started from a polar equation with a focus of the ellipse as origin. The presence of the term $x + \frac{1}{3}$ in the equation above actually tells us that the focus being used as origin was a distance of $\frac{1}{3}$ to the right of the centre of the ellipse at $x = 0$.

The preceding piece of algebra was necessary in order to convince us that the original equation in plane polar coordinates **does** represent an ellipse. However, now that we are convinced of this we can go back and try to extract information in a more speedy way from the equation in its original (r, θ) form.

Solution (contd.)

Try setting $\theta = 0$ and $\theta = \pi$ in the equation

$$\frac{1}{r} = 2 + \cos\theta$$

We find that at $\theta = 0$ we have $r = \dfrac{1}{3}$ while at $\theta = \pi$ we have $r = 1$. These r values correspond to the two ends of the ellipse, so the long axis has a total length $1 + \dfrac{1}{3} = \dfrac{4}{3}$. This tells us that $a = \dfrac{2}{3}$, exactly as found by our longer algebraic derivation. We can further deduce that the focus acting as origin must be at a distance of $\dfrac{1}{3}$ from the centre of the ellipse in order to lead to the two r values at $\theta = 0$ and $\theta = \pi$. If we now use the equation $e = a - b$ mentioned earlier then we find that $\dfrac{1}{9} = \dfrac{4}{9} - b^2$, so that $b = \sqrt{\dfrac{1}{3}}$, as obtained by our lengthy algebra.

The hyperbola

The standard equations are $\quad x = a\cosh t \qquad y = b\sinh t$.

In this case, to eliminate t we use the identity $\cosh^2 t - \sinh^2 t \equiv 1$ giving rise to the equation of the hyperbola in Cartesian form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

In Figure 21 we have chosen a parameter range $-1 \le t \le 2$.

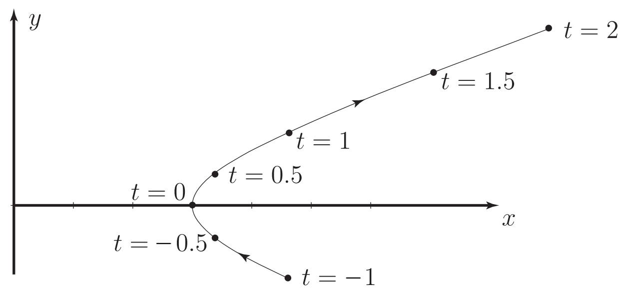


Figure 21

To obtain the complete curve the parameter range $-\infty < t < \infty$ must be used. These parametric equations only give the right-hand branch of the hyperbola. To obtain the left-hand branch we would use $\quad x = -a\cosh t \qquad y = b\sinh t$

Exercises

1. In the following sketch the given parametric curves. Also, eliminate the parameter to give the Cartesian equation in x and y.

   (a) $x = t, \quad y = 2 - t \quad 0 \le t \le 1$ (b) $x = 2 - t, \quad y = t + 1 \quad 0 \le t \le \infty$

   (c) $x = \dfrac{2}{t} \quad y = t - 2 \quad 0 < t < 3$ (d) $x = 3\sin\dfrac{\pi t}{2} \quad y = 4\cos\dfrac{\pi t}{2} \quad -1 \le t \le 0.5$

2. Find the tangent line to the parametric curve $x = t^2 - t \; y - t^2 + t$ at the point where $t = 1$.

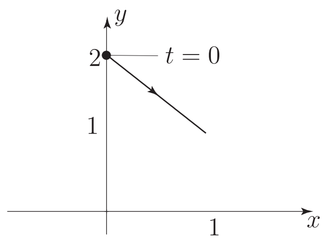
3. For each of the following curves expressed in parametric form obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and use this information to help make a sketch.

   (a) $x = t^2 - 2t, \quad y = t^2 - 4t$

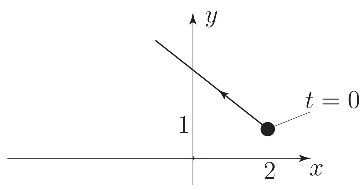
   (b) $x = t^3 - 3t - 2, \quad y = t^2 - t - 2$

Answers

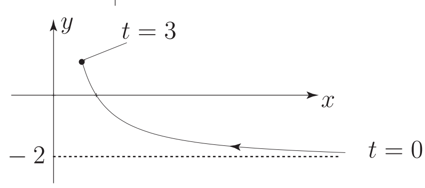
1. (a) $y = 2 - x$



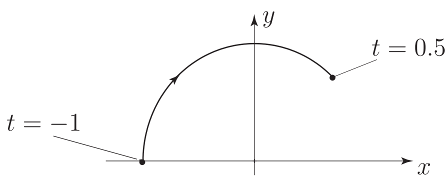
(b) $y = 3 - x$



(c) $y = \dfrac{2}{x} - 2 \qquad \therefore \qquad x(y+2) = 2$



(d) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

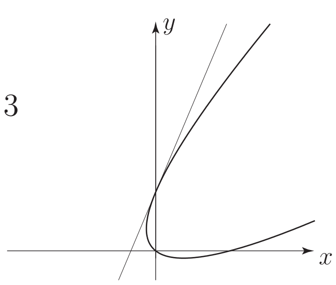


2. $\dfrac{dy}{dt} = 2t + 1 \qquad \dfrac{dx}{dt} = 2t - 1$

$\therefore \qquad \dfrac{dy}{dx} = \dfrac{2t+1}{2t-1} \qquad$ when $t = 1$ then $\dfrac{dy}{dx} = 3$

when $t = 1 \quad x = 0, \quad y = 2$

$\therefore \qquad$ tangent line is $y = 3x + 2$

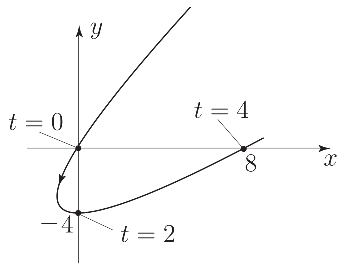


Answer

3. (a) $\dfrac{dy}{dt} = 2t - 4 \qquad \dfrac{dx}{dt} = 2t - 2$

$\dfrac{d^2y}{dt^2} = 2 \qquad \dfrac{d^2x}{dt^2} = 2$

$\dfrac{dy}{dx} = \dfrac{2t-4}{2t-2} = \dfrac{t-2}{t-1} \qquad \dfrac{d^2y}{dx^2} = \dfrac{[(2t-2)-(2t-4)]2}{8(t-1)^3} = \dfrac{1}{2(t-1)^3}$

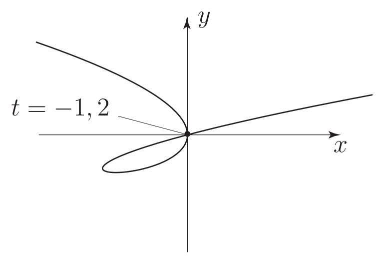


(b) $x = (t-2)(t^2+2t+1) = (t-2)(t+1)^2$
$y = (t+1)(t-2)$

$\dfrac{dy}{dt} = 2t - 1 \qquad \dfrac{dx}{dt} = 3t^2 - 3$

$\dfrac{d^2y}{dt^2} = 2 \qquad \dfrac{d^2x}{dt^2} = 6t$

$\dfrac{d^2y}{dx^2} = \dfrac{[2(3t^2-3)-(2t-1)6t]}{(3t^2-3)^3} = \dfrac{-6t^2+6t-6}{27(t^2-1)^3}$



NOTES

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Workbook 17

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