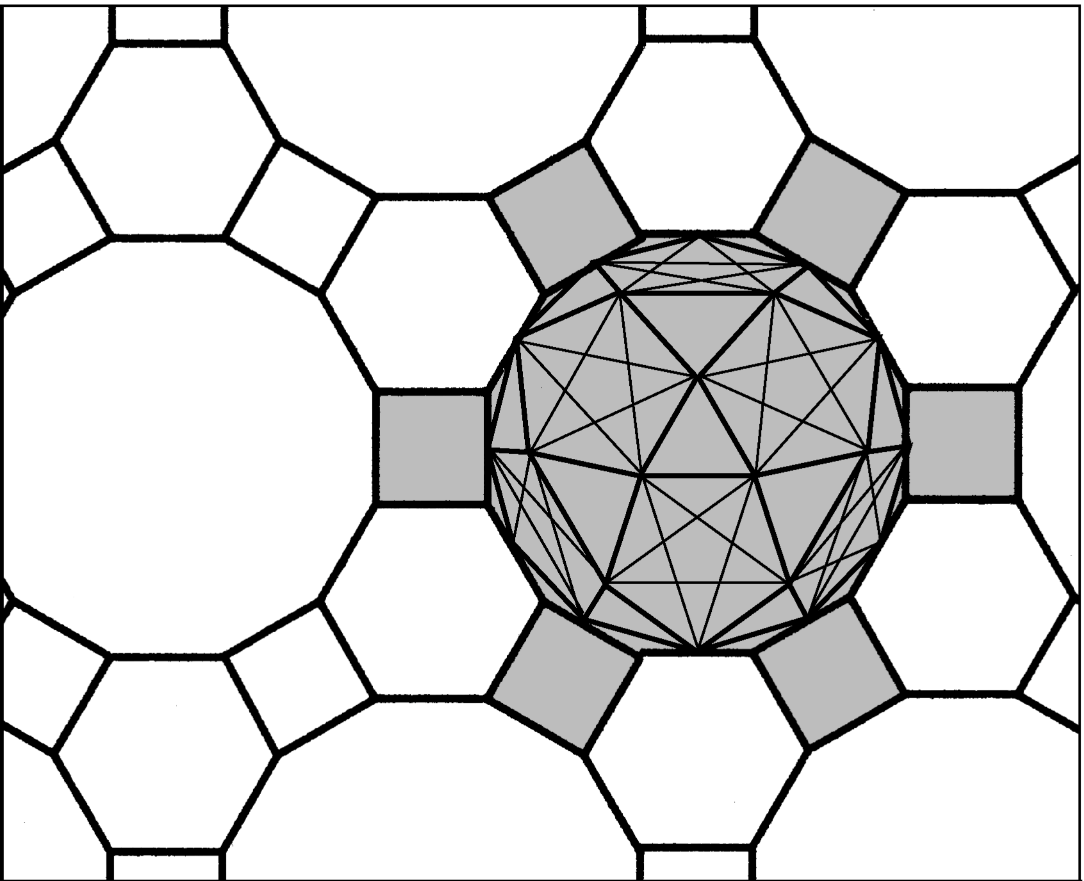
Workbook 49



Student's Guide

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

HELM gratefully acknowledges the valuable support of colleagues at the following universities and colleges involved in the critical reading, trialling, enhancement and revision of the learning materials:
Aston, Bournemouth & Poole College, Cambridge, City, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes Institute of Applied Technology, Harper Adams, Hertfordshire, Leicester, Liverpool, London Metropolitan, Moray College, Northumbria, Nottingham, Nottingham Trent, Oxford Brookes, Plymouth, Portsmouth, Queens Belfast, Robert Gordon, Royal Forest of Dean College, Salford, Sligo Institute of Technology, Southampton, Southampton Institute, Surrey, Teesside, Ulster, University of Wales Institute Cardiff, West Kingsway College (London), West Notts College.

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HELM Workbooks List

1	Basic Algebra	26	Functions of a Complex Variable
2	Basic Functions	27	Multiple Integration
3	Equations, Inequalities & Partial Fractions	28	Differential Vector Calculus
4	Trigonometry	29	Integral Vector Calculus
5	Functions and Modelling	30	Introduction to Numerical Methods
6	Exponential and Logarithmic Functions	31	Numerical Methods of Approximation
7	Matrices	32	Numerical Initial Value Problems
8	Matrix Solution of Equations	33	Numerical Boundary Value Problems
9	Vectors	34	Modelling Motion
10	Complex Numbers	35	Sets and Probability
11	Differentiation	36	Descriptive Statistics
12	Applications of Differentiation	37	Discrete Probability Distributions
13	Integration	38	Continuous Probability Distributions
14	Applications of Integration 1	39	The Normal Distribution
15	Applications of Integration 2	40	Sampling Distributions and Estimation
16	Sequences and Series	41	Hypothesis Testing
17	Conics and Polar Coordinates	42	Goodness of Fit and Contingency Tables
18	Functions of Several Variables	43	Regression and Correlation
19	Differential Equations	44	Analysis of Variance
20	Laplace Transforms	45	Non-parametric Statistics
21	z-Transforms	46	Reliability and Quality Control
22	Eigenvalues and Eigenvectors	47	Mathematics and Physics Miscellany
23	Fourier Series	48	Engineering Case Study
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Contents 49

Student's Guide

Introduction

49.1

The HELM project (Helping Engineers Learn Mathematics) was supported by a £250,000 HEFCE FDTL4 grant for the period Oct 2002-Sept 2005. A HEFCE - funded Transferability Study was undertaken October 2005-September 2006 encouraging the wider uptake of the use of the HELM materials.

1. The HELM project

The HELM team comprised staff at Loughborough University and four consortium partners in other English universities: Hull, Manchester, Reading and Sunderland. The project's aims were to considerably enhance, extend and thoroughly test Loughborough's original Open Learning materials. These were to be achieved mainly by the writing of additional Workbooks and incorporating engineering examples and case studies closely related to the mathematics presented, enhancing the question databanks, upgrading the Interactive Learning segments and adding some more for basic mathematics topics, and promoting widespread trialling.

The HELM project's output consisted of Workbooks, Interactive Learning segments, a Computer Aided Assessment regime which is used to help 'drive the student learning' and a report on possible modes of usage of this flexible material.

The Workbooks may be integrated into existing engineering degree programmes either by selecting isolated stand-alone units to complement other materials or by creating a complete scheme of work for a semester or year or two years by selecting from the large set of Workbooks available. These may be used to support lectures or for independent learning.

HELM's emphasis is on flexibility - the work can be undertaken as private study, distance learning or can be teacher-led, or a combination, according to the learning style and competence of the student and the approach of the particular lecturer.

2. HELM project Workbooks

50 Workbooks are available which comprise:

- 46 Student Workbooks (listed in 50.4) written specifically with the typical engineering student in mind containing mathematical and statistical topics, worked examples, tasks and related engineering examples.
- A Workbook containing an introduction to dimensional analysis, supplementary mathematical topics and physics case studies.
- A Workbook containing Engineering Case Studies ranging over many engineering disciplines.
- A Students' Guide
- A Tutor's Guide (this document)

The main project materials are the Workbooks which are subdivided into manageable Sections. As far as possible, each Section is designed to be a self-contained piece of work that can be attempted by the student in a few hours. In general, a whole Workbook typically represents 2 to 3 weeks' work. Each Workbook Section begins with statements of pre-requisites and the desired learning outcomes.

The Workbooks include (a) worked examples, (b) tasks for students to undertake with space for students to attempt the questions, and, often, intermediate results provided to guide them through problems in stages, and (c) exercises where normally only the answer is given.

It is often possible for the lecturer to select certain Sections from a Workbook and omit other Sections, possibly reducing the reproduction costs and, more importantly, better tailoring the materials to the needs of a specific group.

With funding from **sigma** the workbooks were updated during 2014 and republished Spring 2015, and are now available to all Higher Education Institutes worldwide.

3. HELM project Interactive Learning Segments

These are now outdated and unavailable.

4. HELM project Assessment Regime

In formal educational environments assessment is normally an integral part of learning, and this was recognised by the HELM project. Students need encouragement and confirmation that progress is being made. The HELM assessment strategy was based on using Computer-Aided Assessment (CAA) to encourage self-assessment, which many students neglect, to verify that the appropriate skills have been learned. The project's philosophy was that assessment should be at the heart of any learning and teaching strategy and Loughborough University's own implementation of HELM makes extensive use of CAA to drive the students' learning.

In the past HELM provided an integrated web-delivered CAA regime based on Questionmark Perception for both self-testing and formal assessment, with around 5000 questions; most having a page of specific feedback. These are now largely superseded and so no longer available to other institutions.

HELM Workbooks 49.2

1. List of Workbooks

No.	Title	Pages
1	Basic Algebra	89
2	Basic Functions	75
3	Equations, Inequalities & Partial Fractions	71
4	Trigonometry	77
5	Functions and Modelling	49
6	Exponential and Logarithmic Functions	73
7	Matrices	50
8	Matrix Solution of Equations	32
9	Vectors	66
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24	Fourier Transforms	37
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27	Multiple Integration	83
28	Differential Vector Calculus	53
29	Integral Vector Calculus	80
30	Introduction to Numerical Methods	64
31	Numerical Methods of Approximation	86
32	Numerical Initial Value Problems	80
33	Numerical Boundary Value Problems	36
34	Modelling Motion	63
35	Sets and Probability	53
36	Descriptive Statistics	51

No.	Title	Pages
37	Discrete Probability Distributions	60
38	Continuous Probability Distributions	27
39	The Normal Distribution	40
40	Sampling Distributions and Estimation	22
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2. Nomenclature used for problems

- **Examples** are problems with fully worked solutions.
- **Engineering Examples** (found in most Mathematics Workbooks but not the Statistics Workbooks) are problems with an engineering context having fully worked solutions.
- **Tasks** are problems with spaces for the student's working, followed by fully worked solutions. Many Tasks are often broken up into stages with the answer to a stage given before the next stage is reached. [Note: Some tutors may provide workbooks without these worked solutions.]
- **Exercises** are problems for the student to do without spaces provided for the student's working. In general they do not have fully worked solutions, merely answers, but exceptions are: Numerical Workbooks 30-33 and Statistics Workbooks 35-46 which do have fully worked solutions.

3. Notation used

In general HELM uses italic serif font letters (e.g. $f(x)$) to represent functions, variables and constants. However, as exceptions HELM Workbooks use the following non-italic sans-serif letters:

Mathematics

e for the exponential constant and for the exponential function (primarily use in introductory Workbook 6, elsewhere e is often used)

i where $\mathsf{i}^2 = -1$

ln for natural logarithm

Statistics

E for Expectation

P for Probability

V for Variance

M for Median

Complex numbers

HELM uses i rather than j to represent $\sqrt{(-1)}$ so $\mathrm{i}^2 = -1$, although there are one or two exceptions to this (in Workbook 48: Engineering Case Studies).

Vectors

HELM uses underlining of vectors rather than using bold e.g. $\underline{a}$

HELM uses $\underline{\hat{n}}$ for the unit normal vector but does not put the $\hat{}$ on the basic unit vectors in the x, y and z directions which have the standard symbols $\underline{i}, \underline{j}, \underline{k}$.

Identities

Although HELM introduces and uses the identity symbol '$\equiv$ ' extensively in Workbook 1: Basic Algebra and in Workbook 4: Trigonometry it is not normally used elsewhere and the more normal '$=$' is used except where emphasis seems advisable. (HELM is therefore not consistent.)

4. Description of HELM Workbook layout

On the following three pages are explanatory pages concerning Workbook Layout.

Description of HELM Workbook layout

Complex Arithmetic 10.1

Introduction

Complex numbers are used in many areas of engineering and science. In this Section we define what a complex number is and explore how two such numbers may be combined together by adding, subtracting, multiplying and dividing. We also show how to find 'complex roots' of polynomial equations.

A **complex number** is a generalisation of an ordinary real number. In fact, as we shall see, a complex number is a pair of real numbers ordered in a particular way. Fundamental to the study of complex numbers is the symbol i with the strange looking property $i^2 = -1$. Apart from this property complex numbers follow the usual rules of number algebra.

Workbook introduction.

What you should learn.

What you should know before you start.

Prerequisites

Before starting this Section you should . . .

- be able to add, subtract, multiply and divide real numbers
- be able to combine algebraic fractions together
- understand what a polynomial is
- have a knowledge of trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- combine complex numbers together
- find the modulus and conjugate of a complex number
- obtain complex solutions to polynomial equations

Key Points.
Take especial note of these.

Key Point 1

The symbol i is such that

$$i^2 = -1$$

Using the normal rules of algebra it follows that

$$i^3 = i^2 \times i = -i \qquad i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$$

and so on.

Task for you to try with space for your working.
Answer presented after solution box.

Task

If $z = -2 + i$ and $w = 3 + 2i$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

Your solution

(a)

Answer

$z + 2w = 4 + 5i$

Your solution

(b) Hint: you should find that $z - w = -5 - i$

Answer

$|z - w| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$

Your solution

(c)

Answer

$zw = -6 + 3i - 4i + 2i^2 = -8 - i$

Worked example.
Solution with explanation follows in box.

Example 2

Find $\frac{z}{w}$ if $z = 2 - 3\text{i}$ and $w = 2 + \text{i}$.

Solution

$$\frac{z}{w} = \frac{2-3\text{i}}{2+\text{i}} = \frac{(2-3\text{i})\times(2-\text{i})}{(2+\text{i})\times(2-\text{i})} \quad \text{rationalising}$$

$$= \frac{4-3+\text{i}(-6-2)}{4+1} \quad \text{multiplying out}$$

$$= \frac{1}{5} - \frac{8}{5}\text{i} \quad \text{dividing through}$$

Exercise for you to do.
Answers follow in box (usually no detailed solution).

Exercises

1. Find the roots of the equation $x^2 + 2x + 2 = 0$.
2. If i is one root of the cubic equation $x^3 + 2x^2 + x + 2 = 0$ find the two other roots.
3. Find the complex number z if $2z + z^* + 3\text{i} + 2 = 0$.
4. If $z = \cos\theta + \text{i}\sin\theta$ show that $\frac{z}{z^*} = \cos 2\theta + \text{i}\sin 2\theta$.

Answers 1. $x = -1 \pm \text{i}$ 2. $-\text{i},\ -2$ 3. $-\frac{2}{3} - 3\text{i}$

General Advice to Students Studying Mathematics

49.3

1. Communication with the lecturer or tutor

When your lecturer or tutor writes something that you cannot understand, says something which you don't hear clearly, or provides notes which seem unintelligible or wrong, don't be reluctant to query it! Almost certainly you won't be the only one with this problem. Help yourself and the rest of the class. You will also be doing the lecturer or tutor a favour. Furthermore, ask the question as soon as you reasonably can. Waiting until the end of class can be very frustrating for all concerned!

2. Reading instructions

It seems human nature not to want to read instructions properly (if at all) when faced with a practical task. This even applies to mathematics problem sheets, to coursework and to examination papers. Careful reading of instructions is especially important in mathematics, otherwise you can finish up giving the right answer to the wrong problem and so gaining little or no credit when credit is really due. Miscopying the question is easily done in mathematics and can have dire consequences. It is easy to turn a simple problem into a fiendishly difficult one by doing that - and not only losing credit for that question but also wasting a lot of time (which may well indirectly lead to further loss of credit).

3. Handwriting

If your handwriting is not clear your tutor will have difficulty reading your work when trying to help you, and when marking your work may misread what you intended or get frustrated and lose patience and so not award the mark that the work merits. It has even been known for students to find it hard to read their own writing a few days later!

What are your particular idiosyncrasies in handwriting, which lead to misreading? Be aware and avoid them when it really matters!

Here are some possibilities for confusion (but there many others!)

- $+$ and t
- 0 and o and O (zero and lower and upper case letter 'oh')
- 1 and l and / and I and i ('one' ; letter 'ell' ; 'slash' or 'solidus'; letters "I" and "i")
- 2 and z
- j and y and g
- $\times$ and x (times sign and letter ' x')

Clarifying what you mean by use of brackets is discussed later, but here is an example where you either must write very clearly or resort to brackets to avoid ambiguity:

What do you mean by $\sqrt{3}/2$? Is it $(\sqrt{3})/2$ or $\sqrt{(3/2)}$? You can express whichever you mean more clearly by writing it as either $\dfrac{\sqrt{3}}{2}$ or $\sqrt{\dfrac{3}{2}}$, or by using brackets.

4. Calculators

Although calculators are much better at doing calculations than students they do not always give the right answer.

One of the commonest error with calculators is **forgetting to switch between degrees and radians**. Radians are invariably used in calculus and it is sensible to keep your calculator in this mode. (It is only if x is in radians that the derivative of $\sin(x)$ is $\cos(x)$, for example.)

Another error arises when using graphics facilities. Some graphic calculators only display the right half of the graph $y = x^{1/2}$ if the general root key ($\sqrt[x]{y}$) is used but will give both halves if there is a special cube root button ($\sqrt[3]{\ }$) which is used.

(The explanation lies in the fact that the general root key ($\sqrt[x]{y}$) uses logarithms during the computational process and, since the log of a negative number is not defined, the negative part is "lost".)

5. Brackets (aka parentheses)

Omitting pairs of brackets can lead to faulty algebraic manipulations and incorrect numerical computations.

Expanding $-2\times(p-q)$ should lead to $-2p+2q$ but if (through laziness) it is expressed as $-2\times p-q$ then the outcome is likely to be $-2p-q$ or maybe $-2p-2q$.

Expressing $-3(x+1)^2$ as $-3 \times x^2 + 2x + 1$ is a recipe for disaster leading to $-3x^2 + 2x + 1$ instead of $-3x^2 - 6x - 3$.

(Incidentally, a more subtle error is the belief that a **minus sign means a negative number**. This is not true if x is a negative number, of course.)

Writing fractions can be a problem. For instance, if you write "$2/7y$" do you mean "$(2/7)y$" or "$2/(7y)$"? To be safe you can insert brackets in such an expression or write it clearly as either $\frac{2}{7}y$ or $\frac{2}{7y}$ as appropriate.

In integration, too, problems can easily arise:

$$2\int(4x^3 + 4x - 3)dx = 2 \times x^4 + 2x^2 - 3x + \text{ constant } = 2x^4 + 2x^2 - 3x + C \qquad \text{WRONG!}$$

It should be

$$2\int(4x^3 + 4x - 3)dx = 2(x^4 + 2x^2 - 3x) + C = 2x^4 + 4x^2 - 6x + C \qquad \text{RIGHT!}$$

In general, **if in any doubt put in brackets**. This nearly always works.

6. BODMAS to the rescue!

Order of operations

Common mathematical practice is to perform particular mathematical operations in certain orders. Such conventions reduce the number of brackets needed. For example, it is understood that "$4x+3$" means "$(4x+3)$", and never "$4(x+3)$" In general multiplication is performed **before** (has precedence over) addition. This priority can be reversed by inserting brackets if necessary. It is essential to use the correct order (precedence) of these operations in arithmetic and algebra.

What is -4^2 ? It is tempting to think that the expression means $(-4)^2$ which is $+16$ but the mathematical convention is to perform the exponentiation operation before applying the negation operation (represented by the minus sign), and so -4^2 is actually $-(4^2)$, which is -16.

These conventions are encapsulated in the BODMAS rule for deciding the order in which to do mathematical operations. (This is introduced in HELM Workbook 1.)

BODMAS: (Brackets, 'Of', Division, Multiplication, Addition, Subtraction):

1. **B**rackets take highest priority - deal with items inside a pair of brackets first.
2. **O**f is a form of multiplication (e.g. 'half of 10' means $1/2 \times 10$) and comes next.
3. **D**ivision and **M**ultiplication come next and left-to-right order is required (e.g. $4 \div 7x \times k$ is evaluated as $(4 \div 7) \times k$ and not as $4 \div (7x \times k)$).
4. **A**ddition and Subtraction come last (in either order will do but left-to-right is normal).

When faced with several operations at the same level of precedence the left-to-right order is normally used, but it is not essential.

Beware of calculators

Not all calculators follow these conventions in all circumstances, and ambiguities can arise, so you should check what you get for operations such as $4 \div 7 \times 7, 2 - 3^2$ and 3^{2+1}. Inserting brackets will sort out these problems if you are unsure what your calculator will do, or if you want to force it to do something it won't do otherwise.

7. Equality and Identity

The equals sign $(=)$ is often wrongly used as a shorthand symbol for "*gives*" or "*leads to*" or like phrases. For instance, when finding the third derivative of $x^3 + 2x - 3$, some students will write

$$\frac{d^3}{dx^3}(x^3 + 2x - 3) = 3x^2 + 2 = 6x = 6$$

These four expressions are not equal of course.

This practice is more annoying to the tutor than harmful to the student!

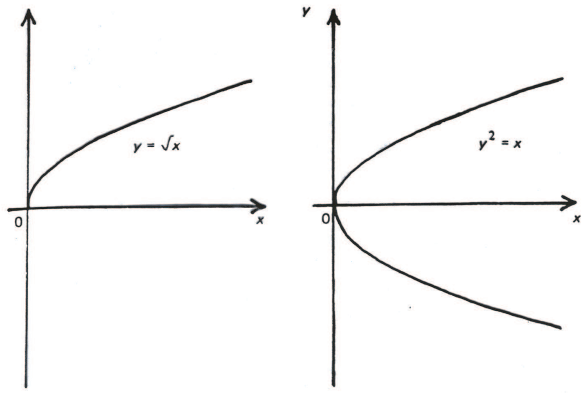
The use of $=$ is commonplace throughout mathematics and hides the distinction between expressions which are true for particular values (e.g. 2x = 2) and those, which are ALWAYS true (e.g. 2x = x + x). The special identity symbol $(\equiv)$ is (or rather can be) used for these: e.g. $2x \equiv x + x$. This symbol has been used sometimes in the HELM Workbooks where emphasis is important (especially in Workbook 1: Basic Algebra and in Workbook 4: Trigonometry) but we *have not done so consistently* - it just isn't the way mathematicians and engineers work! In practice it is nearly always obvious from the context, which is meant.

8. Notational problems

Square root symbol

Every positive number has two real square roots. The expression $\sqrt{n}$ actually means "the **non-negative** square root of n," but many think it can represent either of the square roots of n - i.e., it represents two numbers. This error is actually encouraged by the common practice of referring to $\sqrt{n}$ as "**the** square root of " instead of the more carefully worded "the **positive** square root of ". In fact even that phrase isn't quite correct in all circumstances since it could be zero!

The graphs of $y = \sqrt{x}$ and $y^2 = x$ below illustrate the point:



If you want to refer to both roots then you must use $\pm\sqrt{\ }$, as in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What about $x^{\frac{1}{2}}$? Usually this is taken to mean $\sqrt{x}$ but, particularly in complex number work, it can mean **any** value of the root. So $y = x^{\frac{1}{2}}$ could mean either of the graphs shown above!

Another common error is to replace $\sqrt{1 - \sin^2\theta}$ by $\cos\theta$ (because $1 - \sin^2\theta \equiv \cos^2\theta$). This is wrong because $\cos\theta$ can be negative whereas $\sqrt{\ }$ is never negative, so the result should be expressed as $|\cos\theta|$.

Trigonometric inverses

The expression $\sin^k x$ is interpreted in different ways, depending on the value of k.

$$\sin^3 x \equiv (\sin x)^3 \quad \text{and similarly for cos, tan, sec, cosec and cot}$$

but

$\sin^{-1} x$ means the inverse sine function, sometimes written as arcsin (x), and similarly for cos, tan, sec, cosec and cot.

Note that arcsin $(x) \neq (\sin x)^{-1}$ but cosec $(x) \equiv (\sin x)^{-1}$ because ...

$\sin^{-1}$ is the **inverse function** to sin

$1/\sin$ is the **reciprocal function** of sin, which is called cosec.

9. Checking your work

Human nature seems to lead to most of us being overconfident in our ability to be accurate. In day-to-day life (and indeed in engineering) some imprecision is often acceptable (such as when driving a car, unless in a Formula One race perhaps). But this is not so in mathematics where **absolute** accuracy is demanded. It is vitally important to check your work. (Of course in a timed examination the benefit and disadvantage of checking have to be weighed against each other and will depend upon the circumstances and personal traits.) Ideally you should check by using some alternative method but whether you use the same method or a different one is less important than the act of checking itself.

When solving an algebraic equation (or differential equation), normally an easy way to check the answer is to substitute the result back into original equation, and see if it satisfies the equation. This leads us onto the next more specific situation where checking is very important.

10. Irreversible steps in solving equations

If you apply the same operations to both sides of an equation, the result must be another equation (i.e. the equality must be preserved). The new equation must have all the solutions that the original equation has. **BUT it might also have some new solutions**. This may not seem logical or even possible but unfortunately it is the case when you apply certain operations (which are not reversible).

Reversible operations

1. Multiplying both sides of an equation (except by zero) is reversible: e.g. "multiply both sides by 3": the set of values of x which satisfy $2x^2 = 11x - 5$ is exactly the same as the set of values of x that satisfy $6x^2 = 33x - 15$ [i.e $x = 5$ and $x = \frac{1}{2}$] . (We can simply reverse the operation here by multiplying both sides by $\frac{1}{3}$.)

2. Cubing both sides of an equation is reversible: e.g. the set of values of x which satisfy $x+1 = -3$ is exactly the same as the set of values of x that satisfy $(x+1)^3 = -27$ [i.e. $x = -3$ only]. (We can simply reverse the operation here by cube rooting both sides.)

3. Subtraction is reversible: e.g. "subtract 8 from both sides". The set of values of x which satisfy $2x^2 = 8$ is exactly the same as the set of values of x that satisfy $2x^2 - 8 = 0$. [i.e. $x = 2$ and $x = -2$] (We can simply reverse the operation here by adding 8 to both sides.)

Irreversible operations

Some operations are not reversible, and using them can introduce new solutions (called extraneous solutions) not valid for the original equation.

1. Square rooting is irreversible: e.g. $x = -9$ has only one solution, which is $x = -9$ of course, but after squaring both sides we get $x^2 = 81$, which has two solutions, $x = 9$ and $x = -9$.
2. Multiplication of an equation in variable x by x is irreversible: this always introduces a solution $x = 0$: e.g. $2x^2 = 8$ has two roots 2 and -2 but $2x^3 = 8x$ has three roots 2 and -2 and 0.
3. More generally, multiplication of an equation in variable x by $x - c$ is irreversible: the resulting equation will have the additional new solution $x = c$. [The reason is that multiplying any equation by zero preserves the equality and the factor $x - c$ is zero when $x = c$.]

When any steps taken involve an irreversible operation, then it is essential to check for extraneous roots at the end.

The most common irreversible operation used in solving equations is squaring.

11. Additivity of operations

Many students confuse operations which are additive and those which are not. The normal (wrong) assumption is that the operation will be additive.

An operation f is **additive** if it satisfies $f(x+y) = f(x)+f(y)$ for all x and y. E.g. $2(x+y) = 2x+2y$.

This is **true for some operations**. Examples are:

1. Algebra: $k(p + q) = kp + kq$
2. Differentiation: $d(u + v)/dx = du/dx + dv/dx$
3. Integration: $\displaystyle\int (u + v)dx = \int u\ dx + \int vdx$
4. Laplace transformation: $\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$
5. Matrix (transposition): $(A + B)^T = A^T + B^T$

It is **not true for most** operations. Examples are:

1. Trigonometric identities: e.g. $\sin(x + y) \neq \sin(x) + \sin(y)$
2. Raising to a power: e.g. $(x + y)^2 \neq x^2 + y^2$

3. Taking square root: e.g. $\sqrt{x^2+y^2} \neq \sqrt{x^2} + \sqrt{y^2}$
4. Exponentiation: $\exp(x+y) \neq \exp(x) + \exp(y)$
5. Taking logarithm: $\log(x+y) \neq \log(x) + \log(y)$
6. Matrices (inversion): $(A+B)^{-1} \neq A^{-1} + B^{-1}$

This is a common mistake made by first year undergraduates who have not studied mathematics for some time.

12. Commutativity of operations

Two operations f and g **commute** if you get the same result when you perform them in either order: i.e. $f(g(x)) = g(f(x))$. E.g. if f means "doubling" and g means "trebling" then $f(g(5)) = f(15) = 30$ and $g(f(5)) = g(10) = 30$ so $f(g(5)) = g(f(5))$.

This is **true for some** combinations of operations. Examples are:

1. Powers and roots of positive numbers: $(\sqrt{x})^3 = \sqrt{(x^3)}$
2. Multiplication by a constant and integration: $2\int u\,dx = \int 2u\ dx$

It is **not true for most** combinations of operations. Examples are:

1. "Doubling" and "Adding 1" $\neq$ "Adding 1" and "Doubling"
2. Powers and addition: $(x+1)^3 \neq x^3 + 1^3$
3. Taking cosine and squaring: $\cos(x^2) \neq \{\cos(x)\}^2$
4. Multiplication and differentiation: $(u \times v)' \neq u' \times v'$
5. Division and integration: $\int (u/v)\,dx \neq \int u\,dx / \int v\,dx$

13. Dimensions and scaling

Dimensional analysis is an important topic for engineers and is treated in Workbook 47. It doesn't tell you if you have the right formula or answer, but it can indicate that something must be wrong. Here are some simple examples:

1. If you're asked to find a length, and your answer is some number of square cms, then you must have made an error somewhere.

2. If you're asked to find an area and your answer is a negative number, then you know you've made an error somewhere UNLESS it is a calculus problem (where an area below the axis may be represented as a negative quantity).

3. The formula for the area, S, of a triangle with sides a, b, c must have dimensions of area so it cannot possibly be either of the following:

   $S = a \times b \times c$ or $S = a + b + c$

   It might in theory be

   $S = (a + b + c)^2$

   which has the right dimensions for area, though that isn't actually correct of course!

   There is in fact a complicated formula involving only a, b, c for S, called Heron's formula:

   $S = \sqrt{\{(a+b+c)(b+c-a)(c+a-b)(a+b-c)/16\}}$.

   You can check that this is dimensionally correct.

Unit Conversion

A related problem is converting from one unit to another. Just because $1\ m = 100\ cm$ does not mean that $1\ m^3 = 100\ cm^3$. Obvious, perhaps, but an easy mistake to make when not concentrating. In fact, of course, there are three dimensions here so the scale factor is 100^3 and $1\ m^3 = 1\,000\,000\ cm^3$.

Scaling error

If the question is a real-world problem, you should ask: "Is my answer sensible?" For instance, if you are given a list of the main components used in the manufacture of a truck and are asked to estimate its unladen weight, and you come up with an answer of 1000 tonnes, then you must have made a mistake either in the calculations or in the units.

14. Some further traps

It is important to remember the following:

(a) Cancelling in fractions

Don't fall into the trap of *partial cancelling*.

This is correct:

$$\frac{(x-1)\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \frac{(x-1)}{(x+3)} \quad \text{(provided } x \neq -2)$$

but this is NOT correct:

$$\frac{(x-1)+\cancel{(x+2)}}{(x+3)\cancel{(x+2)}}=\frac{(x-1)+1}{(x+3)}$$

You only cancel once when the factors in the numerator are multiplied but you must cancel each time when the factors in the numerator are added (or subtracted).

(b) Inequalities

$x \leq 2$ and $x \geq -1$ can be combined to give $-1 \leq x \geq 2$

BUT $x \geq 2$ and $x \leq -1$ cannot be combined to give $2 \geq x \leq -1$, which makes no sense. It is not possible to express these as a single equality (because it would imply $2 \leq -1$!).

(c) Solving equations

$(x-1)(x-2)=0 \Rightarrow x-1=0$ or $x-2=0$ TRUE!

BUT

$(x-1)(x-2)=2 \Rightarrow x-1=2$ or $x-2=2$ FALSE!

It is only with zero right-hand side that such factorisation is valid.

(d) Differentiation

The term $\dfrac{dy}{dx}$ indicates differentiation of the expression y with respect to the variable x . The operation of differentiation itself can be expressed as $\dfrac{d}{dx}$. It is not strictly correct to separate out the dy and the dx as in $\dfrac{dy}{dx}=x^2 \Rightarrow dt = x^2 dx$ but this does work (in solving differential equations), however mathematicians don't like it!

(e) Integration

$\displaystyle\int \frac{1}{x}dx = \ln(x)+c$ NOT CORRECT!

$\displaystyle\int \frac{1}{x}dx = \ln|x|+c$ CORRECT!

15. Stationary Points and Points of Inflection

Most students (and some teachers!) have an imperfect understanding of the definitions of local maximum, local minimum and point of inflection. Simple graphs can be used to illustrate these features.

Of the following three statements only the first two are known with any certainty by most students:

Given a twice differentiable function f for which $f'(a) = 0$

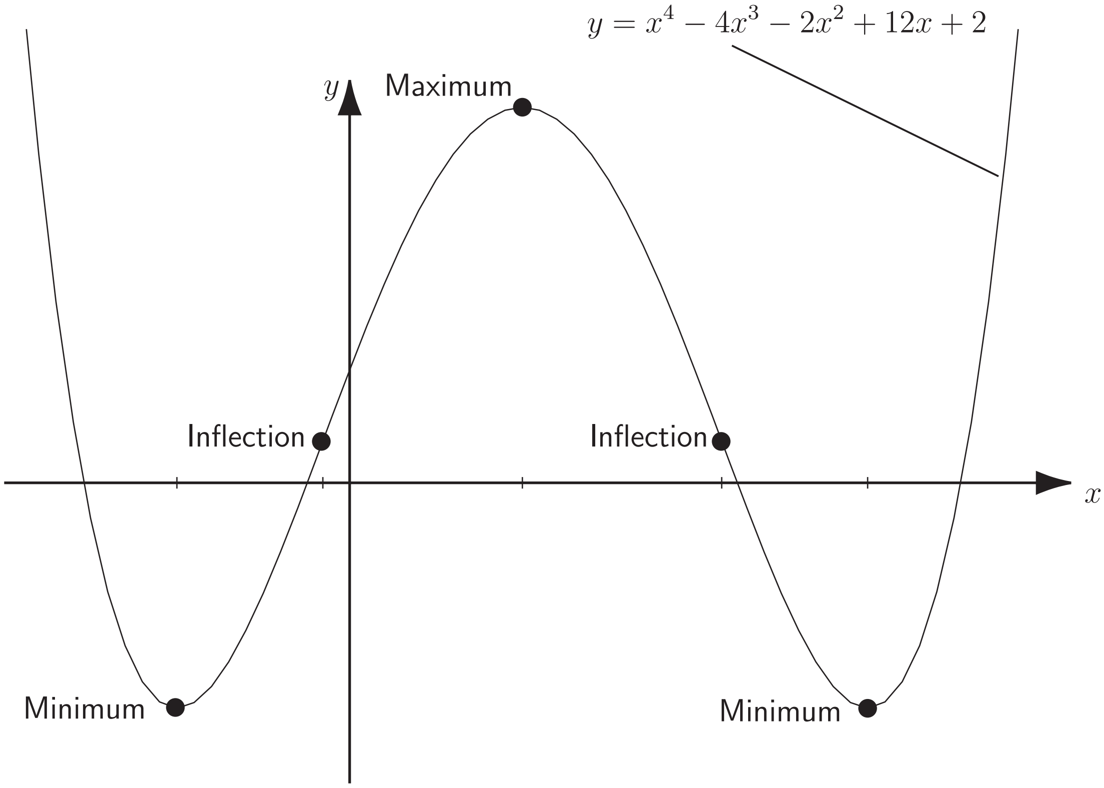
(1) If $f''(a) > 0$, then $f(x)$ has a minimum when $x = a$,

(2) If $f''(a) < 0$, then $f(x)$ has a maximum when $x = a$,

(3) If $f''(a) = 0$, then $f(x)$ has minimum or a maximum or a point of inflection when $x = a$.

Many students think (3) **always** leads to a point of inflection but the graph of $f(x) = x^4$ clearly shows this to be untrue when $x = 0$.

Another misconception is that a point of inflection **requires** $f'(a) = 0$. This is not true as can easily be seen, for example, on the sine curve. This raises another point - for any continuous function there is always a point of inflection between every local minimum and local maximum. The graph below highlights these features.



Maxima and Minima without Calculus

Students all too readily turn to the calculus when needing to find maxima and minima. There are, however, cases when alternative approaches are simpler, quicker or more informative:

Example 1

Find the minimum value of $f(x) \equiv x^2 + 2x + 3$.

Completing the square gives $f(x) = (x+1)^2 + 2$.

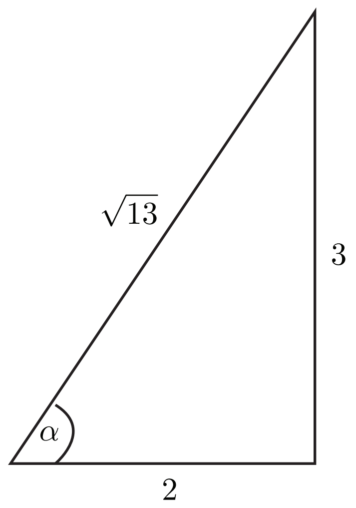
This clearly is a minimum when $x = -1$ and there $f(x)$ has value 2.

Example 2

Find the maximum value of $f(x) \equiv 2\sin(x) + 3\cos(x)$.

Using the trigonometric identity $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ and utilising the triangle in the diagram we have

$$\begin{aligned} f(x) &= \sqrt{13}\left[\frac{2}{\sqrt{13}}\sin x + \frac{3}{\sqrt{13}}\cos x\right] \\ &= \sqrt{13}[\cos\alpha\sin x + \sin\alpha\cos x] \\ &= \sqrt{13}\sin(x+\alpha) \end{aligned}$$



This clearly has a maximum value of $\sqrt{13}$ at $x = \dfrac{\pi}{2} - \alpha$ (for example), which is where $\sin(x+\alpha) = \sin\left(\dfrac{\pi}{2}\right) = 1$.

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NOTES

Workbook 49

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