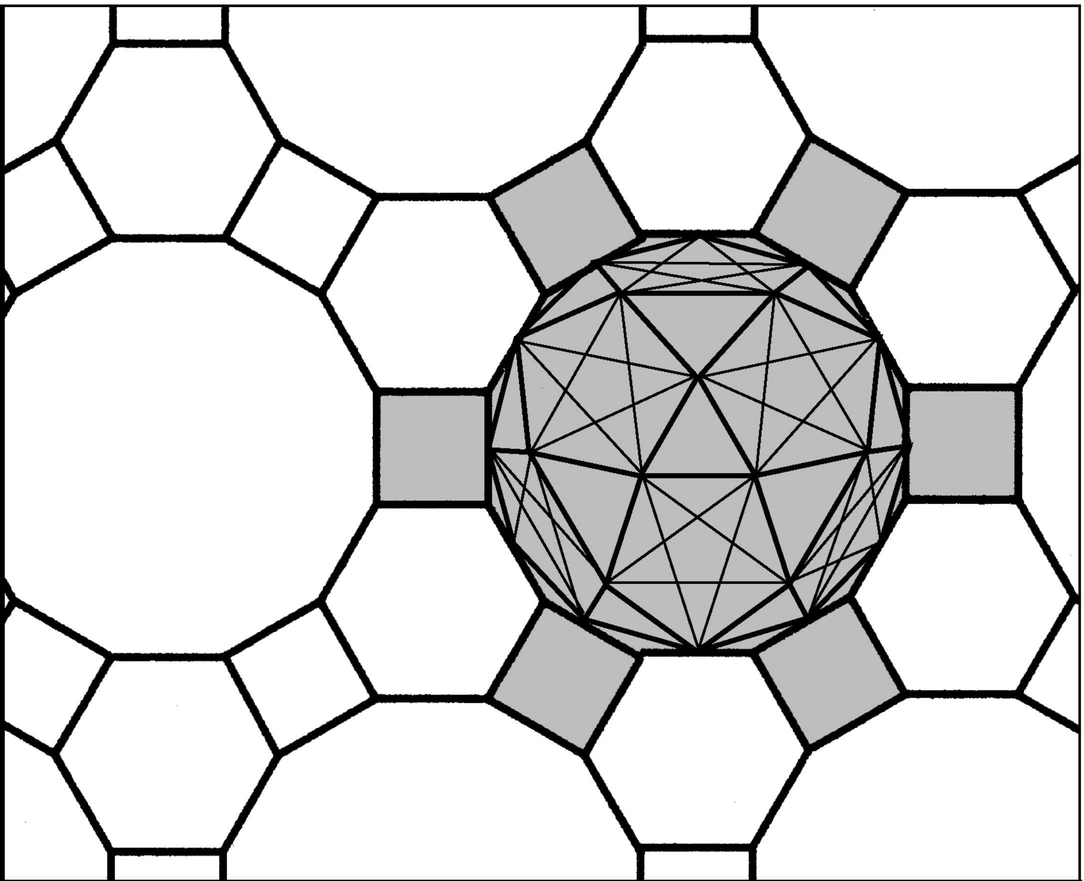
Workbook 50



Tutor's Guide

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

HELM gratefully acknowledges the valuable support of colleagues at the following universities and colleges involved in the critical reading, trialling, enhancement and revision of the learning materials:
Aston, Bournemouth & Poole College, Cambridge, City, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes Institute of Applied Technology, Harper Adams, Hertfordshire, Leicester, Liverpool, London Metropolitan, Moray College, Northumbria, Nottingham, Nottingham Trent, Oxford Brookes, Plymouth, Portsmouth, Queens Belfast, Robert Gordon, Royal Forest of Dean College, Salford, Sligo Institute of Technology, Southampton, Southampton Institute, Surrey, Teesside, Ulster, University of Wales Institute Cardiff, West Kingsway College (London), West Notts College.

HELM Contacts:
Post: HELM, Mathematics Education Centre, Loughborough University, Loughborough, LE11 3TU.
Email: helm@lboro.ac.uk *Web:* http://helm.lboro.ac.uk

HELM Workbooks List

1	Basic Algebra	26	Functions of a Complex Variable
2	Basic Functions	27	Multiple Integration
3	Equations, Inequalities & Partial Fractions	28	Differential Vector Calculus
4	Trigonometry	29	Integral Vector Calculus
5	Functions and Modelling	30	Introduction to Numerical Methods
6	Exponential and Logarithmic Functions	31	Numerical Methods of Approximation
7	Matrices	32	Numerical Initial Value Problems
8	Matrix Solution of Equations	33	Numerical Boundary Value Problems
9	Vectors	34	Modelling Motion
10	Complex Numbers	35	Sets and Probability
11	Differentiation	36	Descriptive Statistics
12	Applications of Differentiation	37	Discrete Probability Distributions
13	Integration	38	Continuous Probability Distributions
14	Applications of Integration 1	39	The Normal Distribution
15	Applications of Integration 2	40	Sampling Distributions and Estimation
16	Sequences and Series	41	Hypothesis Testing
17	Conics and Polar Coordinates	42	Goodness of Fit and Contingency Tables
18	Functions of Several Variables	43	Regression and Correlation
19	Differential Equations	44	Analysis of Variance
20	Laplace Transforms	45	Non-parametric Statistics
21	z-Transforms	46	Reliability and Quality Control
22	Eigenvalues and Eigenvectors	47	Mathematics and Physics Miscellany
23	Fourier Series	48	Engineering Case Study
24	Fourier Transforms	49	Student's Guide
25	Partial Differential Equations	50	Tutor's Guide

Production of this 2015 edition, containing corrections and minor revisions of the 2008 edition, was funded by the **sigma** Network.

sigma Σ
network for excellence in mathematics and statistics support

Contents 50

Tutor's Guide

Introduction to HELM 50.1

The HELM project (Helping Engineers Learn Mathematics) was supported by a £250,000 HEFCE FDTL4 grant for the period Oct 2002-Sept 2005. A HEFCE - funded Transferability Study was undertaken October 2005-September 2006 encouraging the wider uptake of the use of the HELM materials.

1. The HELM project

The HELM team comprised staff at Loughborough University and four consortium partners in other English universities: Hull, Manchester, Reading and Sunderland. The project's aims were to considerably enhance, extend and thoroughly test Loughborough's original Open Learning materials. These were to be achieved mainly by the writing of additional Workbooks and incorporating engineering examples and case studies closely related to the mathematics presented, enhancing the question data-banks, upgrading the Interactive Learning segments and adding some more for basic mathematics topics, and promoting widespread trialling.

The HELM project's output consisted of Workbooks, Interactive Learning segments, a Computer Aided Assessment regime which is used to help 'drive the student learning' and a report on possible modes of usage of this flexible material.

The Workbooks may be integrated into existing engineering degree programmes either by selecting isolated stand-alone units to complement other materials or by creating a complete scheme of work for a semester or year or two years by selecting from the large set of Workbooks available. These may be used to support lectures or for independent learning.

HELM's emphasis is on flexibility - the work can be undertaken as private study, distance learning or can be teacher-led, or a combination, according to the learning style and competence of the student and the approach of the particular lecturer.

2. HELM project Workbooks

50 Workbooks are available which comprise:

- 46 Student Workbooks (listed in 50.4) written specifically with the typical engineering student in mind containing mathematical and statistical topics, worked examples, tasks and related engineering examples.
- A Workbook containing an introduction to dimensional analysis, supplementary mathematical topics and physics case studies.
- A Workbook containing Engineering Case Studies ranging over many engineering disciplines.
- A Students' Guide
- A Tutor's Guide (this document)

The main project materials are the Workbooks which are subdivided into manageable Sections. As far as possible, each Section is designed to be a self-contained piece of work that can be attempted by the student in a few hours. In general, a whole Workbook typically represents 2 to 3 weeks' work. Each Workbook Section begins with statements of pre-requisites and the desired learning outcomes.

The Workbooks include (a) worked examples, (b) tasks for students to undertake with space for students to attempt the questions, and, often, intermediate results provided to guide them through problems in stages, and (c) exercises where normally only the answer is given.

It is often possible for the lecturer to select certain Sections from a Workbook and omit other Sections, possibly reducing the reproduction costs and, more importantly, better tailoring the materials to the needs of a specific group.

With funding from **sigma** the workbooks were updated during 2014 and republished Spring 2015, and are now available to all Higher Education Institutes worldwide.

3. HELM project Interactive Learning Segments

These are now outdated and unavailable.

4. HELM project Assessment Regime

In formal educational environments assessment is normally an integral part of learning, and this was recognised by the HELM project. Students need encouragement and confirmation that progress is being made. The HELM assessment strategy was based on using Computer-Aided Assessment (CAA) to encourage self-assessment, which many students neglect, to verify that the appropriate skills have been learned. The project's philosophy was that assessment should be at the heart of any learning and teaching strategy and Loughborough University's own implementation of HELM makes extensive use of CAA to drive the students' learning.

In the past HELM provided an integrated web-delivered CAA regime based on Questionmark Perception for both self-testing and formal assessment, with around 5000 questions; most having a page of specific feedback. These are now largely superseded and so no longer available to other institutions.

HELM Consortium, Triallist Institutions and Individual Contributors

50.2

HELM learning resources were produced primarily by a consortium of writers and developers at five universities:

Hull, Loughborough, Manchester, Reading, Sunderland.

The HELM consortium gratefully acknowledges the valuable support of many colleagues at their institutions and at the following institutions involved in additional writing, critical reading, trialling and revising of the learning materials:

Universites
Aston
Cambridge
City
Glamorgan
Glasgow
Glasgow Caledonian
Hertfordshire
Leicester
Liverpool
London Metropolitan
Newcastle
Northumbria
Newcastle
Nottingham
Nottingham Trent
Oxford Brookes
Plymouth
Queen's Belfast
Robert Gordon
Southampton
Southampton Solent
Surrey
Teesside
Ulster
University of Wales Institute Cardiff

Other HE/FE Institutions
Bournemouth & Poole College
Glenrothes Institute of Applied Technology
Harper Adams University College
Moray College
Royal Forest of Dean College
Sligo Institute of Technology
Westminster Kingsway College
West Notts College

Individual Consortium Contributors

Loughborough University

Writing team

Dr Tony Croft

Dr David Green

Dr Rhian Green

Dr Martin Harrison

Stephen Lee

Dr Peter Lewis

Dr Leslie Mustoe

David Pidcock

Prof Chris Rielly

Dr Joe Ward

Advisors

Dr Nigel Beacham

Dr Sarah Carpenter

Dr Adam Crawford

Myles Danson

Lesley Davis

Dr Aruna Palipana

Clare Trott

Dr Andy Wilson

Administration and Production

Dr David Green

Dr Martin Harrison

Louise Kitching

Sophie McAloon

Dr Aruna Palipana

David Pidcock

Helen Sherwood

Dr Joe Ward

Clare Wright

Finance

Dr David Green

University of Hull

Writing team

Prof Keith Attenborough

Dr Mary Attenborough

Dr Patrice Boulanger

Dr John Killingbeck

Don Maskell

University of Manchester

Writing team

Mr Max Caley

Dr Paul JW Bolton

Dr Alison M Durham

Mr Kamlesh Ghodasara

Dr Colin Steele

Advisors

Dr David Apsley

Dr Adrian Bell

Dr Shuguang Li

Dr John Parkinson

Prof. Stephen Williamson

University of Reading

Writing team

Dr Peter Chamberlain

Dr David Stirling

University of Sunderland

Writing team

Dr Malcolm Farrow (later Newcastle Univ)

Walter Middleton

Contributors from other institutions

The following is a list of all those at other institutions who have helped in the development of the HELM materials by contributing examples and text, critically reading, providing feedback from trialling, pointing out errors, and offering general advice and guidance.

Felix Ampofof

Philip Aston

Rob Beale

Kevin Brace

Pargat Calay

Richard Cameron

Abbi Coman Ricketts

Grace Corcoran

Bob Crane

Gavin Cutler

Mansel Davies

Ian Drumm

Mark Gerrard

David Graham

Jim Grimbleby

Christopher Haines

Stephen Hibberd

Alison Holmes

Ken Houston

Peter Hudson

Sudhir Jain

Sanowar Khan

Joseph Kyle

David Livie

Peter Long

David Lowe

Brian Lowe

Brian Malyan

David Malvern

John Martin

Sye Murray

Shepard Ndlovu

Chris O'Hagan

Paul Oman

Steve Outram

Norman Parkes

Sarah Parsons

Lynn Pevy

Alexei Piunovsky

Margaret Platt

Julie Renton

Mick Scrivens

Elaine Smith

Alan Stevens

Richard Steward

Ian Taylor

Stephen Thorns

James Vickers

Juliette White

James Wisdom

Hilary Wood

Lindsay Wood

HELM Transferability Project (HELMet)

50.3

1. Introduction

During the course of the HELM Project some 30 institutions implemented one or more of the three aspects of the HELM Resources (Workbooks, Interactive Learning Segments, CAA questions) into at least one of their programmes.

As CAL and CAA are no longer supported, discussions here focus on the HELM Workbooks alone.

2. Use of Workbooks

2.1 Workbooks as lecture notes (hard copy format)

In the cases where HELM Workbooks were used to replace existing lecture notes (20% of implementations), students were issued hard copies of the Workbooks relevant to their modules. Workbooks were either issued in full at the beginning of each topic or subsections were issued lecture by lecture. The lecturers choosing to issue relevant sections lecture by lecture felt that this gave them the advantage of controlling what the students had before them in the class and allowing the lecturer to give very focussed direction to the students on what they should be doing during a particular time period. However, issuing complete Workbooks eased any complications in reproduction and simplified the situation when students had been absent, as they simply knew which Workbook to collect rather than having to identify particular subsections. In all cases, the materials were made available prior to the lecturer beginning the topic, allowing students to preview the material in advance of the lectures.

Lecturers, using the Workbooks as their core notes essentially employed one of two approaches to their lectures.

Some staff, particularly those teaching the material for the first time or for the first time to a particular group, developed lectures, often Powerpoint based, which reflected the content of the Workbooks exactly, using the same or very similar examples. Students found this reassuring; they felt safe that they did not have to take notes, therefore being able to focus on understanding, and they knew exactly where they were in the material enabling them to easily follow the lecture, and could make annotations to their Workbooks in the appropriate places. However, some students did feel that this made the lecture have less value as they could simply study for themselves at home and that in such lectures they did not have the opportunity to work through the examples themselves to reinforce their learning.

Other staff lectured in parallel to the content of the Workbook, using their own examples and developing some of the theory using their own existing notes. Students were kept appraised of which sections of the Workbooks were being addressed and were directed to work through specific examples and exercises to follow up the material covered within the lectures. Students appreciated that this allowed them to see more examples than just those contained in the Workbooks, but they did tend to feel that in these circumstances they needed to make more extensive notes. It is helpful to students that when this second approach is chosen that notation used by the lecturer is kept consistent with that in the Workbooks even where this is not the lecturer's preferred method - lecturers are more able to adapt than students struggling to comprehend new mathematical concepts.

2.2 Workbooks as supplementary linked material for whole modules

The largest subgroup (50% of implementations) used the Workbooks as supplementary notes, linking them explicitly to the content of the lectures.

In 60% of these cases, hard copies of the Workbooks were issued to students at lectures or tutorials thus providing an incentive to attend. In the other 40% of these cases, electronic links to the Workbooks sited on the institutions' websites were given.

Students were directed to use the material to help them understand the content of their lectures and to provide additional examples and exercises to use during tutorials and for private study. It was found that few students made the effort to work through the HELM material if they viewed it as an optional (albeit very useful) extra. Some students also expressed a lack of conviction about working through the Workbooks in class, being content to attend tutorials to collect the Workbooks and taking them home unopened.

Some success has been achieved using a peer tutoring scheme where the Workbooks are used as the subject matter. This provided a focus for the sessions and excellent support for students struggling with their mathematics and strong backup for their mentors.

Nevertheless, interviewed students did, in all cases, appreciate having the materials and felt that when they were preparing for examinations they would be very useful.

2.3 Workbooks as supplementary linked material for parts of modules

Of those institutions using the Workbooks as supplementary materials, around 20% restricted use of the materials to a small subset of the module content. They tended to use the most fundamental Workbooks (1: Basic Algebra, 2: Basic Functions, 3: Equations, Inequalities & Partial Fractions) as they wanted to establish the foundations with what would be regarded as fairly weak mathematicians. For more advanced mathematical topics they continued with their existing notes. They found that using the materials for this group allowed the students to be able to spend time honing their skills before moving on to more challenging topics.

2.4 Workbooks as unlinked supplementary material

Some 30% of implementations used the Workbooks as supplementary materials and made the Workbooks available to students but did not specifically attach them to particular modules or courses. In most cases the Workbooks were made available online for students to download and (if required) print off themselves. Links to the Workbooks were provided by lecturers and/or through support centres.

In the support centres, hard copies were sometimes obtainable on request or at least there were reference copies available. It is difficult to measure the rate of uptake in these circumstances. However, having this range of materials available as part of a support service for students is seen as valuable by students and staff alike.

It was noticeable that **mature students** and those with **special needs** (typically dyslexics) were **very appreciative** of the Workbooks and did make extensive use of them, no matter which usage mode was employed, citing being able to work at their own pace and being confident that they had a complete set of notes as large positives.

3 Tutorial assistants

Many institutions use postgraduates for tutoring and the HELM Workbooks provide a sure foundation for them. (It is well documented that staff often fail to keep their tutorial assistants adequately informed of their lecture notes and chosen methods, and students seeking help often have woefully inaccurate notes.)

HELM Workbook Structure and Notation 50.4

1. List of Workbooks

No.	Title	Pages
1	Basic Algebra	89
2	Basic Functions	75
3	Equations, Inequalities & Partial Fractions	71
4	Trigonometry	77
5	Functions and Modelling	49
6	Exponential and Logarithmic Functions	73
7	Matrices	50
8	Matrix Solution of Equations	32
9	Vectors	66
10	Complex Numbers	34
11	Differentiation	58
12	Applications of Differentiation	63
13	Integration	62
14	Applications of Integration 1	34
15	Applications of Integration 2	31
16	Sequences and Series	51
17	Conics and Polar Coordinates	43
18	Functions of Several Variables	40
19	Differential Equations	70
20	Laplace Transforms	72
21	z-Transforms	96
22	Eigenvalues and Eigenvectors	53
23	Fourier Series	73
24	Fourier Transforms	37
25	Partial Differential Equations	42
26	Functions of a Complex Variable	58
27	Multiple Integration	83
28	Differential Vector Calculus	53
29	Integral Vector Calculus	80
30	Introduction to Numerical Methods	64
31	Numerical Methods of Approximation	86
32	Numerical Initial Value Problems	80
33	Numerical Boundary Value Problems	36
34	Modelling Motion	63

No.	Title	Pages
35	Sets and Probability	53
36	Descriptive Statistics	51
37	Discrete Probability Distributions	60
38	Continuous Probability Distributions	27
39	The Normal Distribution	40
40	Sampling Distributions and Estimation	22
41	Hypothesis Testing	42
42	Goodness of Fit and Contingency Tables	24
43	Regression and Correlation	32
44	Analysis of Variance	57
45	Non-parametric Statistics	36
46	Reliability and Quality Control	38
47	Mathematics and Physics Miscellany	69
48	Engineering Case Studies	97
49	Student's Guide	31
50	Tutor's Guide	75

2. Nature of the Workbooks

The 50 HELM Workbooks comprise:

- 33 Mathematics Workbooks (1 to 33) written specifically with the typical engineering student in mind containing mathematical topics, worked examples, tasks, exercises and related engineering examples.
- 1 Workbook (34) emphasising the mathematical modelling of motion, with worked examples, tasks, exercises and related engineering examples.
- 12 Statistics Workbooks (35 to 46) written specifically with the typical engineering student in mind containing statistical topics, worked examples with an emphasis on engineering contexts, tasks and exercises.
- 1 Workbook (47) containing Sections on Dimensional analysis, Mathematical explorations and 11 Physics case studies.
- 1 Workbook (48) containing 20 Engineering Case Studies ranging over many engineering disciplines.
- 1 Student's Guide (49) containing helpful advice, various indexes and extensive facts and formulae sheets for mathematics and mechanics.
- 1 Tutor's Guide (50) relating success stories and challenges and encapsulating good practice derived from trialling in a variety of institutions with their individual contexts and cultures.

The Workbooks are subdivided into manageable Sections. As far as possible, each Section is designed to be a self-contained piece of work that can be attempted by the student in a few hours. In general, a whole Workbook represents about 2 to 3 weeks' work. Each Section begins with statements of prerequisites and desired learning outcomes.

The Tasks include space for students to attempt the questions; many Tasks guide the student through problems in stages.

It is often feasible for the lecturer to select certain Sections from a Workbook and omit others, reducing the reproduction costs if using hardcopy and better tailoring the materials to the needs of a specific group.

3. Notation Used

Fonts

In general HELM uses italic serif font letters to represent functions, variables and constants. However, as exceptions HELM Workbooks use the following non-italic sans-serif letters:

Mathematics

e for the exponential constant and for the exponential function (primarily use in introductory Workbook 6, elsewhere e is often used)

i where $\mathsf{i}^2 = -1$

ln for natural logarithm

Statistics

E for Expectation   P for Probability   V for Variance   M for Median

Complex numbers

HELM uses i rather than j to represent $\sqrt{(-1)}$ so $\mathsf{i}^2 = -1$, although there are one or two exceptions to this (in Workbook 48: Engineering Case Studies).

Vectors

HELM uses underlining of vectors rather than using bold e.g. $\underline{a}$

HELM uses $\underline{\hat{n}}$ for the unit normal vector but does not put the ˆ on the basic unit vectors in the x, y and z directions which have the standard symbols $\underline{i}, \underline{j}, \underline{k}$.

Identities

Although HELM introduces and uses the identity symbol '$\equiv$' extensively in Workbook 1: Basic Algebra and in Workbook 4: Trigonometry it is not normally used elsewhere and the more normal '$=$' is used except where emphasis seems advisable. (HELM is therefore not consistent.)

Issues and Notes for Tutors

50.5

1. The use of units in applied problems

Problems in engineering almost invariably involve physical quantities, for example distance, mass, time, current, measured in a variety of units: metres, miles, amps, litres, etc. When using mathematics to solve applied problems we often are a little slipshod in our approach to the way unknown variables and associated units are introduced and used.

We may ask for the solution of the equation $\sin x = 0.5$ where x, measured in radians, is in the interval $0 \leq x < 2\pi$. But should we say

(a) "Solve the equation $\sin x = 0.5$ where x, measured in degrees, is in the interval $0 \leq x < 360$", or

(b) "Solve the equation $\sin x^\circ = 0.5$ where x is in the interval $0 \leq x < 360$", or

(c) "Solve the equation $\sin x = 0.5$ where x is in the interval $0^\circ \leq x < 360^\circ$"?

Purists may prefer (b) of these options.

Now consider the following problem.

A train is travelling at 80 km h^1 and is 2 km from the next station when it starts to brake so that it comes to a halt at the station. What is the deceleration, assuming it to be uniform during the time that the brakes were applied?

Here is the 'correct' solution.

Let the deceleration be a km h^{-2}.

In 2 km the speed of the train will have decreased by 80 km h^{-1}.

We use the formula $v^2 = u^2 - 2as$, where u km h^{-1} is the initial speed,

v km h^{-1} is the final speed and s km is the distance travelled.

Hence $= \dfrac{u^2 - v^2}{2s} = \dfrac{80 \times 80 - 0 \times 0}{2 \times 2} = \dfrac{80 \times 80}{2 \times 2} = 1600.$

The (uniform) deceleration is therefore 1600 km h^{-2}.

In this approach, a represents the **magnitude** of the deceleration and is a pure number.

A less 'professional' solution now follows.

Let the deceleration be a.

In 2 km the speed of the train will have decreased by 80 km h^{-1}.

We use the formula $v^2 = u^2 - 2as$ where u is the initial speed, v is the final speed and s is the distance travelled.

Hence $a = \dfrac{u^2 - v^2}{2s} = \dfrac{80 \times 80}{2 \times 2} = 1600$ km h^2.

In this second approach, a represents the deceleration itself and so has units. It seems simpler and maybe more natural. However, we then get the inconsistency that the fraction $\dfrac{80 \times 80}{2 \times 2}$, a number, is equated to a deceleration a, which is nonsense.

In the HELM Workbooks we have not always given rigorous solutions or taken the greatest care over the use of units, taking instead a pragmatic line and trying to keep solutions simple.

In truth, in the HELM Workbooks we have not been consistent in our approach, adopting whatever seemed appropriate to the situation.

We hope that the lecturer will understand what is required for a rigorous solution and forgive us these lapses. (Any reports of errors or suggestions for improvement of any HELM resources will always be welcome.)

2. Confidence

We want engineering students to be confident in their mathematics. However, this needs to be justified confidence. Some informative research by Armstrong and Croft (1999) included a diagnostic test of mathematics for engineering undergraduates and a confidence test. The results across 28 different mathematical topics (ranging from arithmetic to integration) were illuminating: There was not a single topic for which the proportion of students expressing a lack of confidence matched or exceeded the proportion of students answering wrongly. In other words there was **overconfidence across all topics**.

It can be concluded that in general any student seeking help needs it and so will many others who may not admit to it! A student telling his tutor that he has no problems should not be assumed to be correct - some gentle probing is always advisable.

Reference: Armstrong P.K. & Croft, A.C., (1999) *Identifying the Learning Needs in Mathematics of Entrants to Undergraduate Engineering Programmes in an English University*, European Journal of Engineering Education, Vol.24, No.1, 59-71.

3. HELM Workbooks and Students with Special Learning Difficulties

The HELM project sought expert advice on the style of the Workbooks. This included some interviews with selected students. In most cases HELM was able to respond to make the Workbooks (based on the earlier Open Learning Project) more accessible to all students.

When tested with various formats, most students with learning difficulties such as dyslexia preferred a sans serif font version and this has been adopted. However, a serif font is used for the mathematical notation. It seems, from past experience that this is preferable since the sans serif mathematical notation is unfamiliar and ambiguous giving, for example a straight x instead of curved, easily confused with a times sign, and the digit one looking like the upper case letter "i" and lower case letter "l" and also like the modulus sign.

Although there is an occasional use of italicised words, the preference is to use emboldening, which is clearer.

The main suggestion which was not followed concerns text justification. The recommendation is to use 'ragged right' text. Although fully justified text can cause problems for some students it was decided that HELM's use of very short paragraphs made that a less important consideration and full justification improved the general appearance of the page and emphasised the blocks of text.

4. Student Difficulties

On the next two pages are reproduced two actual engineering students' attempts at an algebra problem and a calculus problem. They demonstrate a range of errors.

These examples can be used as a basis of discussion with tutorial assistants, new staff and others as to how to diagnose and remedy problems experienced by students challenged by mathematical problems.

You will be able to amass your own collection of such examples . . .

An Engineering Student's Errors in Algebra

Below is the work of a first year undergraduate Manufacturing Engineering student.

1. Study the student's work.
2. Note down the different kinds of mistakes made.
3. How would you help this student?

1. $144^{\frac{1}{2}} + 144^{\frac{3}{2}} = 144^{\frac{3}{4}}$

2. (a)

$$\begin{aligned}(x+6)(x+5) &= (6+x)(5+x)\\ &= 6(5+x)+x(5+x)\\ &= 30+6x+5x+x^2\\ &= x^2+11x+30\end{aligned}$$

(b)

$$\begin{aligned}(x+1)(x+2)(x+3) &= (x^2+2x+x+2)x+3\\ &= (x^2+3x+2)x+3\\ &= x^3+3x^2+2x+3x^2+9x+6\\ &= x^3+6x^2+11x+6\end{aligned}$$

3. (a)

$$\begin{aligned}11x+7 &= 0\\ 11x+7-7 &= -7\\ \frac{11x}{11} &= \frac{-7}{11}\\ x &= \frac{-7}{x}\end{aligned}$$

(b)

$$\begin{aligned}x^2+6x-7 &= 0\\ x^2+6x-7+7 &= 7\\ x^2+\frac{6x}{6} &= \frac{7}{6}\\ x^2+x &= \frac{7}{6}\\ x^3 &= \frac{7}{6}\\ x &= \frac{7}{18}\end{aligned}$$

(c)

$$\begin{aligned}x^2+x-8 &= 0\\ x^2+x-8+8 &= 8\\ x^3 &= 8\\ x &= \frac{8}{3}\end{aligned}$$

An Engineering Student's Solution to a Calculus Problem

Below is the work of a first year undergraduate Electrical Engineering student.

1. Study the student's work.
2. How many errors can you see?
3. How would you classify each of them?
4. How could you help this student?

Problem

Given:

$$y = \left(\frac{x+7}{4x-3}\right)^{\frac{1}{3}}, \qquad x > \frac{3}{4}$$

find the gradient at $x = 1$.

Student's Solution

$$\ln y = \frac{\ln(x+7)^{\frac{1}{3}}}{\ln(4x-3)^{\frac{1}{3}}}$$

$$\ln y = \frac{\frac{1}{3}\ln(x+7)}{\frac{1}{3}\ln(4x-3)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\dfrac{\ln(4x-3)}{x+7} - \dfrac{\ln(x+7)}{4x-3}}{(\ln(4x-3))2}$$

$$\frac{dy}{dx} = \left(\frac{x+7}{4x-3}\right)^{\frac{1}{3}} \times \left(\frac{\dfrac{\ln(4x-3)}{x+7} - \dfrac{\ln(x+7)}{4x-3}}{(\ln(4x-3)^2}\right)$$

When $x = 1$

$$\frac{dy}{dx} = \left(\frac{8}{1}\right)^{\frac{1}{3}} \times \left(\frac{\dfrac{\ln 1}{8} - \dfrac{\ln 8}{1}}{(\ln 1)^2}\right)$$

$$= 2 \times \frac{0-0}{0}$$

Therefore $\dfrac{dy}{dx} = 0$ when $x = 1$

Description of HELM Workbook layout

50.6

On the following three pages are reproduced from the Student's Guide explanatory pages concerning Workbook Layout.

Description of HELM Workbook layout

Complex Arithmetic 10.1

Introduction

Complex numbers are used in many areas of engineering and science. In this Section we define what a complex number is and explore how two such numbers may be combined together by adding, subtracting, multiplying and dividing. We also show how to find 'complex roots' of polynomial equations.

A **complex number** is a generalisation of an ordinary real number. In fact, as we shall see, a complex number is a pair of real numbers ordered in a particular way. Fundamental to the study of complex numbers is the symbol i with the strange looking property $i^2 = -1$. Apart from this property complex numbers follow the usual rules of number algebra.

Workbook introduction.

What you should learn.

What you should know before you start.

Prerequisites

Before starting this Section you should . . .

- be able to add, subtract, multiply and divide real numbers
- be able to combine algebraic fractions together
- understand what a polynomial is
- have a knowledge of trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- combine complex numbers together
- find the modulus and conjugate of a complex number
- obtain complex solutions to polynomial equations

Key Points.
Take especial note of these.

Key Point 1

The symbol i is such that

$$\mathrm{i}^2 = -1$$

Using the normal rules of algebra it follows that

$$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -\mathrm{i} \qquad \mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = (-1) \times (-1) = 1$$

and so on.

Task for you to try with space for your working.
Answer presented after solution box.

Task

If $z = -2 + \mathrm{i}$ and $w = 3 + 2\mathrm{i}$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

Your solution

(a)

Answer

$z + 2w = 4 + 5\mathrm{i}$

Your solution

(b) Hint: you should find that $z - w = -5 - \mathrm{i}$

Answer

$|z - w| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$

Your solution

(c)

Answer

$zw = -6 + 3\mathrm{i} - 4\mathrm{i} + 2\mathrm{i}^2 = -8 - \mathrm{i}$

Worked example.
Solution with explanation follows in box.

Example 2

Find $\dfrac{z}{w}$ if $z = 2 - 3\mathrm{i}$ and $w = 2 + \mathrm{i}$.

Solution

$$\begin{aligned} \frac{z}{w} = \frac{2-3\mathrm{i}}{2+\mathrm{i}} &= \frac{(2-3\mathrm{i})\times(2-\mathrm{i})}{(2+\mathrm{i})\times(2-\mathrm{i})} && \text{rationalising} \\ &= \frac{4-3+\mathrm{i}(-6-2)}{4+1} && \text{multiplying out} \\ &= \frac{1}{5} - \frac{8}{5}\mathrm{i} && \text{dividing through} \end{aligned}$$

Exercise for you to do.
Answers follow in box (usually no detailed solution).

Exercises

1. Find the roots of the equation $x^2 + 2x + 2 = 0$.
2. If i is one root of the cubic equation $x^3 + 2x^2 + x + 2 = 0$ find the two other roots.
3. Find the complex number z if $2z + z^* + 3\mathrm{i} + 2 = 0$.
4. If $z = \cos\theta + \mathrm{i}\sin\theta$ show that $\dfrac{z}{z^*} = \cos 2\theta + \mathrm{i}\sin 2\theta$.

Answers 1. $x = -1 \pm \mathrm{i}$ 2. $-\mathrm{i},\ -2$ 3. $-\dfrac{2}{3} - 3\mathrm{i}$

List of Sections in Workbooks 1 to 48

50.7

Workbook 1 - Basic Algebra (89 pages)	
1.1	Mathematical Notation and Symbols
1.2	Indices
1.3	Simplification and Factorisation
1.4	Arithmetic of Algebraic Fractions
1.5	Formulae and Transposition
Workbook 2 - Basic Functions (75 pages)	
2.1	Basic Concepts of Functions
2.2	Graphs of Functions and Parametric Form
2.3	One-to-one and Inverse Functions
2.4	Characterising Functions
2.5	The Straight Line
2.6	The Circle
2.7	Some Common Functions
Workbook 3 - Equations, Inequalities and Partial Fractions (71 pages)	
3.1	Solving Linear Equations
3.2	Solving Quadratic Equations
3.3	Solving Polynomial Equations
3.4	Solving Simultaneous Linear Equations
3.5	Solving Inequalities
3.6	Partial Fractions
Workbook 4 - Trigonometry (77 pages)	
4.1	Right-angled Triangles
4.2	Trigonometric Functions
4.3	Trigonometric Identities
4.4	Applications of Trigonometry to Triangles
4.5	Applications of Trigonometry to Waves
Workbook 5 - Functions and Modelling (49 pages)	
5.1	The Modelling Cycle and Functions
5.2	Quadratic Functions and Modelling
5.3	Oscillating Functions and Modelling
5.4	Inverse Square Law Modelling

Workbook 12 - Applications of Differentiation (63 pages)	
12.1	Tangents and Normals
12.2	Maxima and Minima
12.3	The Newton-Raphson Method
12.4	Curvature
12.5	Differentiation of Vectors
12.6	Case Study: Complex Impedance
Workbook 13 - Integration (62 pages)	
13.1	Basic Concepts of Integration
13.2	Definite Integrals
13.3	The Area Bounded by a Curve
13.4	Integration by Parts
13.5	Integration by Substitution and Using Partial Fractions
13.6	Integration of Trigonometric Functions
Workbook 14 - Applications of Integration 1 (34 pages)	
14.1	Integration as the Limit of a Sum
14.2	The Mean Value and the Root-Mean-Square Value
14.3	Volumes of Revolution
14.4	Lengths of Curves and Surfaces of Revolution
Workbook 15 - Applications of Integration 2 (31 pages)	
15.1	Integration of Vectors
15.2	Calculating Centres of Mass
15.3	Moment of Inertia
Workbook 16 - Sequences and Series (51 pages)	
16.1	Sequences and Series
16.2	Infinite Series
16.3	The Binomial Series
16.4	Power Series
16.5	Maclaurin and Taylor Series
Workbook 17 - Conics and Polar Coordinates (43 pages)	
17.1	Conic Sections
17.2	Polar Coordinates
17.3	Parametric Curves
Workbook 18 - Functions of Several Variables (40 pages)	
18.1	Functions of Several Variables
18.2	Partial Derivatives
18.3	Stationary Points
18.4	Errors and Percentage Change

Workbook 25 - Partial Differential Equations (42 pages)	
25.1	Partial Differential Equations
25.2	Applications of PDEs
25.3	Solution using Separation of Variables
25.4	Solutions using Fourier Series

Workbook 26 - Functions of a Complex Variable (58 pages)	
26.1	Complex Functions
26.2	Cauchy-Riemann Equations and Conformal Mappings
26.3	Standard Complex Functions
26.4	Basic Complex Integration
26.5	Cauchy's Theorem
26.6	Singularities and Residues

Workbook 27 - Multiple Integration (83 pages)	
27.1	Introduction to Surface Integrals
27.2	Multiple Integrals over Non-rectangular Regions
27.3	Volume Integrals
27.4	Changing Coordinates

Workbook 28 - Differential Vector Calculus (53 pages)	
28.1	Background to Vector Calculus
28.2	Differential Vector Calculus
28.3	Orthogonal Curvilinear Coordinates

Workbook 29 - Integral Vector Calculus (80 pages)	
29.1	Line Integrals Involving Vectors
29.2	Surface and Volume Integrals
29.3	Integral Vector Theorems

Workbook 30 - Introduction to Numerical Methods (64 pages)	
30.1	Rounding Error and Conditioning
30.2	Gaussian Elimination
30.3	LU Decomposition
30.4	Matrix Norms
30.5	Iterative Methods for Systems of Equations

Workbook 31 - Numerical Methods of Approximation (86 pages)	
31.1	Polynomial Approximations
31.2	Numerical Integration
31.3	Numerical Differentiation
31.4	Nonlinear Equations

Workbook 40 - Sampling Distributions and Estimation (22 pages)	
40.1	Sampling Distributions
40.2	Interval Estimation for the Variance
Workbook 41 - Hypothesis Testing (42 pages)	
41.1	Statistics Testing
41.2	Tests Concerning a Single Sample
41.3	Tests Concerning Two Samples
Workbook 42 - Goodness of Fit and Contingency Tables (24 pages)	
42.1	Goodness of Fit
42.2	Contingency Tables
Workbook 43 - Regression and Correlation (32 pages)	
43.1	Regression
43.2	Correlation
Workbook 44 - Analysis of Variance (57 pages)	
44.1	One-Way Analysis of Variance
44.2	Two-Way Analysis of Variance
44.3	Experimental Design
Workbook 45 - Non-parametric Statistics (36 pages)	
45.1	Non-parametric Tests for a Single Sample
45.2	Non-parametric Tests for Two Samples
Workbook 46 - Reliability and Quality Control (38 pages)	
46.1	Reliability
46.2	Quality Control
Workbook 47 - Mathematics and Physics Miscellany (70 pages)	
47.1	Dimensional Analysis in Engineering
47.2	Mathematical Explorations
47.3	Physics Case Studies
Workbook 48 - Engineering Case Studies (97 pages)	
	Engineering Case Studies 1 to 20

Commentaries on Workbooks 1 to 48

50.8

Workbook 1: Basic Algebra

The lack of fluency with basic algebra among engineering undergraduates is probably the area most commented upon by those who teach them mathematics and by engineering academics, who find that this deficiency hampers the student in solving problems in the wider engineering arena. To achieve the required level of fluency it is necessary to build the student's confidence by a careful programme of drill and practice examples. This idea may not be as fashionable as it once was, but effort invested with this material will reap rewards in later work.

This Workbook is therefore placed first in the series to emphasise the importance of the mathematics it covers. The five areas that it deals with are: mathematical notation and symbols, indices, simplification and factorisation (of expressions), algebraic fractions, and formulae and transposition.

Section 1.1 may seem elementary to the majority of students, but experience has shown that many of them have lacunae in their knowledge and understanding, even at this level. Particular attention needs to be paid to the understanding of, and correct use of, the modulus sign and the sigma notation. In the interpretation of index notation, care is needed with negative powers on the denominator of a fraction and the (obvious) fact that different powers of the same variable (e.g. $a^2 + a^3$) cannot be combined together. Correct use of a pocket calculator when fractional powers are involved also needs attention. In general it must be explained that using a pocket calculator successfully needs a good grasp of the underlying algebra.

The importance of Section 1.3 cannot be overstated. It is disappointing to find that so many undergraduates have difficulty in correct manipulation of algebraic expressions. To see in a student's attempted solution of a problem in reliability the 'mathematics' $1 - (1 - e^{-t}) \equiv e^{-t}$ demonstrates a lack of knowledge of the distributive law. Such errors are all too common and prevent the correct solution of straightforward problems in engineering. There are no short cuts: the basics must be thoroughly understood and students must be made aware of this. Factorisation should become second nature and this, too, demands time and effort.

Handling algebraic fractions is made more difficult these days when school students are not given a thorough grounding in the arithmetic of number fractions. It would be wise to give students some revision in dealing with number fractions as a precursor, so that the 'rules' for manipulating algebraic fractions make sense. Students who attempt to apply the rules without understanding them sometimes come a cropper, and subsequent stages of the attempted solution of an engineering problem are then incorrect.

The transposition of formulae is a skill whose lack is frequently commented on. To many academics this skill is too often taken for granted, but many students have at best a shaky grasp of what they are attempting to do and of the 'rules' that they are using. Once again, it is practice, which is required, and examples from their engineering studies should be used as much as possible so that they see the relevance of what they are doing.

The last point applies quite generally. Although the use of 'abstract' examples has merit, every set of examples should, where possible, include at least one framed in the context of the engineering discipline relevant to each student. To make R the subject of the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ is surely more interesting to an electrical engineering undergraduate than being asked to make z the subject of the formula $\dfrac{1}{z} = \dfrac{1}{x} + \dfrac{1}{y}$.

Workbook 2: Basic Functions

Functions are the verbs of mathematics and a firm grasp of their basic definitions and properties are crucial to later work. Many students are put off by the use of the notation $f(x) = x^3$ and this effect can be reduced by a gentle, notation-free introduction.

This Workbook covers the areas: the concept of a function, the graph of a function, functions in parametric form, classifying functions, linear functions, simple standard engineering functions, and the circle.

For students not versed in the idea of a function the box diagram (input - rule - output) has proved to be the most useful in discussing the topic, particularly with regard to the inverse of compound functions. By treating a few very simple functions in this way, it is then time to introduce the concept of a function as a mapping and the notation $f(x)$. Then compound functions can be studied and the box diagram approach helps to clarify the domain and range.

When presenting the representation of a function as a graph it is important to stress the word 'representation'; too many students think that the graph is the function. Now is the time to present a function as a box diagram, a rule and as represented by a graph in order to show the link between the three approaches.

At this early stage it is possible to give examples of functions described in parametric form and demonstrate how to calculate points on their graph. It is then the time to present an example of a function which would be very difficult, or impossible, to present in the standard $f(x)$ form. Incidentally, it is a good idea to now consider functions whose input variable is not x, say t .

The inverse of a function, its domain and range can be discussed first by box diagram, then by rule and then by graph, in that order. The use of the terms 'one-to-one' etc can now be safely mentioned provided that the treatment is gentle. The distinction between functions whose inverse is itself a function and those for which this is not true can be illustrated by examples such as $f(x) = x + 2,\ f(x) = 2x,\ f(x) = x^3$ and $f(x) = x^2$. Other examples can be presented via graphs.

With simple examples, and a liberal use of the graphical approach, the concepts of continuity, piecewise definition, periodicity, oddness and evenness can extend the portfolio of functions considered. Illustrating these concepts, however crudely, by examples from engineering is particularly important here, to prevent the presentation drifting into an apparently aimless academic exercise.

When students met the topic of straight lines at school it was almost certainly not as the graphical representation of a linear function. The advantage, for example, of calculating the exact distance between two points as opposed to a graphical estimation should be stressed, by asking students to discuss how they would find the distance between two points in three dimensions and why the problem is different from the two-dimensional case.

A circle in standard form is an example of an implicit function, but that may be too deep to discuss at this level. Rather it provides the opportunity to carry out some simple coordinate geometry where the visual checks are easy to perform. The use of some of these results in the relevant engineering discipline must be demonstrated.

Polynomial functions are well behaved and can act as a vehicle for looking at concepts of oddness and evenness, decreasing, increasing and stationary. Simple rational functions, including $f(x) = x^{-1}$ allow ideas of discontinuity and asymptotes to be introduced. The modulus function and the unit step function allow students to widen their horizons on the meaning of 'function' and can be (and must be) shown to have practical engineering applications.

Workbook 3: Equations, Inequalities and Partial Fractions

The solution of simple equations underpins much of engineering mathematics. The need to know whether the equation concerned has any solutions, and if so how many is essential to the satisfactory solution of problems. Much less well understood are inequalities and a thorough grounding is required at this level. The concept of the partial fractions of a rational function is useful in so many areas, for example inverting a Laplace transform.

The areas that this Workbook deals with are: the solution of linear equations, the 'solution' of quadratic equations and polynomial equations in general, simultaneous equations, simple inequalities and their solution, determination of partial fractions.

There are examples a-plenty of linear equations in engineering and the opportunity should be taken to introduce the topic via one from the students' discipline. Linking the solution to the graph of a straight line crossing an axis is recommended. Are there any cases therefore where no solution exists? Is the solution unique and why is it important to know this?

The solution of quadratic equations provides an opportunity to bring together several areas of mathematics. Looking at graphs of various quadratics suggests the possibility of two distinct solutions, a unique solution or no solutions; algebra can verify the suggestion. The usefulness of factorisation in determining the crossing points of the graph on the horizontal axis, and the location of the stationary point of the function can be stressed, but if no factors exist students may find the method wasteful of time; honesty is essential. Completing the square is clumsier in determining two crossing points but readily yields information about the stationary point when there are no solutions of the associated quadratic equation. The formula method then comes into its own as providing relevant information in all cases; this is the time to introduce it, but do stress the relevant strengths of all the approaches.

The introduction to higher degree polynomial equations should be gentle and, in the main, concentrating on general principles. It is a good moment in the course to emphasise how the complexity of the problem increases as the degree of the polynomial increases.

The solution of simultaneous equations allows the opportunity to show the interaction between graphical indication and algebraic determination. Start with the four graphical cases of single intersection (at almost right-angles), parallel lines, coincident lines, and almost-coincident lines. Then show how the graphs relate to the algebraic 'solution'. A simple introduction to ill-conditioned equations can be given here.

The relationship between a simple inequality and points on the real line helps students to visualise the meaning of the inequality; its solution algebraically can be related to the picture. The case of a quadratic inequality is only sensibly dealt with at this level by extensive use of a graphical argument.

The first point to emphasise with expressing a rational function as a sum of partial fractions is the need to choose suitably-shaped partial fractions; the process of finding the coefficients in the numerators can then, in principle, follow successfully. It is recommended that a single strategy is employed: combine the proposed partial fractions into a single fraction with the same denominator as the original fraction; substitute suitable values of the variable into the numerators to obtain, one by one, the unknown coefficients; if this does not determine all of them resort to comparing coefficients of powers of the variable. It is suggested that at least one example is given where one of the unknown coefficients is zero to warn the students that this can happen with a general strategy and ask whether they could have foreseen that zero value.

Workbook 4: Trigonometry

There are some colleagues who believe that the sine, cosine and tangent functions should be introduced via right-angled triangles and then extended to their application to oscillatory motion. Others believe that the application to oscillatory motion should precede their link to triangles. This Workbook follows the first approach, but with suitable guidance students following the second approach can also use it successfully.

This Workbook deals with the areas of right-angled triangles and their solution, trigonometric functions of any angle, including their graphs, simple trigonometric identities, the solution of triangles without a right angle, the application of trigonometry to the study of wave motion.

In this age of the pocket calculator many students are not familiar with the exact values of the trigonometric ratios for angles such as $30°, 45°, 60°$. Does it matter? The view taken here is that it does. It is also important to note that the inverse trigonometric functions found by pressing calculator buttons may not provide the answer required. It is worth emphasising that checking whether a triangle is right-angled before solving it via sine and cosine rules can save much effort.

These days it might be argued that there is little value in knowing, or using, throwbacks such as the CAST rule; on the contrary, we argue that such knowledge and usage leads to a greater 'feel' for the trigonometric functions than mere blind button-pressing. The ability to sketch graphs of the basic trigonometric functions adds to their safe application to problem solving.

How far one should delve into trigonometric identities is a matter for debate but surely everyone should know the identity $\sin^2\theta + \cos^2\theta \equiv 1$ and its close relatives. An awareness of some of the addition formulae allows the 'double angle' identities to be seen as plausible.

A strategy for the solution of triangles in general, using angle sum, sine rule and cosine rule should be presented. This is a good situation in which to explain the idea of adapting the strategy to different input information. The fact that the information given does not allow a triangle to be drawn illustrates the principle that 'if you can draw the triangle then you can calculate the unknown sides and angles'. The ambiguous case, presented pictorially first, gives emphasis to recognising that for an angle $< 180°$, knowing its sine does not define it uniquely (except, of course, for $90°$).

This Workbook assumes no previous knowledge of the subject and begins with a discussion of the trigonometry of the right-angled triangle. More general definitions are then given of the trigonometric (or circular) functions and the main trigonometric identities are obtained.

The sine rule and cosine rule for triangles are introduced and a careful discussion given of which is the appropriate rule to use in a given situation.

Detailed worked examples and a generous selection of problems (with answers) are provided at all stages.

As an application a full section is devoted to properties of sinusoidal waves whilst other smaller optional engineering examples utilising trigonometry (for example in the diffraction of sound, brake cables, amplitude modulation and projectile motion) are interspersed at appropriate intervals.

Workbook 5: Functions and Modelling

One of the difficulties found frequently by engineering students is the progression from the statement of a problem in words to a mathematical solution and its interpretation. In this context, a reasonable proficiency in a range of mathematical techniques is not sufficient. Students need to be able to abstract salient features from a real problem, make suitable assumptions, assign sensible symbols to the variables, state the mathematical problem (the *mathematical model*) and devise a strategy for solving it. Having solved it, the student needs to be able to interpret the result either in terms of the predicted behaviour or numerically (using consistent units). If the interpretation is not satisfactory the student needs to be able to refine the initial model.

This Workbook is intended to introduce this idea of the 'Modelling Cycle'. The emphasis is on 'choosing a function for a model'. Since Workbook 5 is early in the HELM series, it deals only with models that involve linear, quadratic and trigonometric functions. (The last section in Workbook 6 includes modelling examples that use the logarithmic and exponential functions.) An attempt has been made to use the 'modelling' format in all of the engineering examples in the HELM series.

Following a section on the use of linear functions as models we turn our attention to the use of quadratic functions. An obvious example is that of projectile motion. Given the relative simplicity of the quadratic function it is worth examining the assumptions made by using it to model such situations; how accurate are the predictions from the model and does it provide an exact model, or merely a very good approximation to reality? To avoid confusion in the case of two-dimensional projectile motion, it is worth spending time with the example of a stone thrown vertically upwards distinguishing between the graph of the vertical displacement against time and the actual motion of the stone.

Then we model oscillatory motion by means of the sine and cosine functions, reinforcing the introduction provided in Workbook 4. Finally, we take an example of the inverse square law model; this is such a widely-applied model that it needs an early mastery.

The definition of the 'Modelling Cycle' and the approach adopted are similar to those used in a series of Open University courses (TM282 *Modelling by Mathematics* and MST121 *Using Mathematics*). These courses cite some engineering examples but, since the intention was to avoid contexts that required a lot of prior knowledge, several other contexts are used here. For example, the Open University materials for TM282 use three contextual themes: Vehicle Safety, Town Planning and Population Growth. Further illustrations of modelling are to be found in the texts for those courses.

More advanced modelling examples may be found in later HELM Workbooks (particularly 34, 47 and 48) and in the following references:

The Open University MST207 Mathematical Methods and Models,

J. Berry and K. Houston, *Mathematical Modelling*, publ. Edward Arnold, 1995,

K. Singh, *Engineering Mathematics through Applications*, Palgrave, 2003.

Workbook 6: Exponential and Logarithmic Functions

Two of the most hard-worked functions in models of engineering problems are the exponential and logarithmic functions. As with all work-horses, a thorough knowledge of their behaviour is essential to their efficient use; effort spent now will reap rewards over and over again.

After some examples of functions from the family a^2 , the exponential function is introduced by defining e as the limit of a sequence. A study of the slope of the tangent to curves of the family can be followed by stating that in the case of the exponential function that slope is equal to the value of the function at the point of contact. This then allows a generalisation to the slope being proportional to the value and hence to models of growth and decay, for example Newton's law of cooling.

The hyperbolic functions can be introduced at this point without too much fuss. Given the contrast between some of their properties and those of their trigonometric near-namesakes, it might be wise to hint at their use in parametric coordinates on a hyperbola (hence the trailing h).

The use of the logarithmic function in solving model equations that employ the exponential function reinforces the concept of an inverse function. The properties of logarithmic functions need treating with care: the law $\log_a(AB) = \log_a A + \log_a B$ is not true for A and B both negative, for example.

To an extent the material in Section 6.6 on the log-linear transformation duplicates that in earlier sections of the Workbook. However this section provides an alternative approach and illustrates its use in various modelling contexts. Indeed the emphasis in Section 6.6 is on the use of exponential and logarithm functions for modelling.

Workbook 7: Matrices

The algebra of matrices can be a dry topic unless the student sees that matrices have a vital role to play in the solution of engineering problems. Of course, one obvious application is in the solution of simultaneous linear equations, but there are others, for example in the description of communication networks, and at least one such example should be introduced at the outset.

Of necessity, some groundwork must be carried out and this can be tedious; where possible, emphasise the 'natural' definitions, for example addition of two matrices of the same shape; then ask how they would add together say

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. In new territory nothing can be taken for granted.

Matrix multiplication is a strange process. The multiplication of two-by-two matrices can be illustrated as representing two successive rotations (use simple angles). The generalisation to three-by-three matrices is a hard one, so it is important to emphasise the scalar product basis of deriving the two-by-two entries e.g. for the product $C = AB,\ c_{11} = a_{11}b_{11} + a_{12}b_{21}$ etc. And, of course, the shock to the system of non-commutativity and results such as $AB = 0$ does not necessarily imply that either $A = 0$ or $B = 0$ (or both) should be exploited. Again it is important to give simple examples of where multiplication is not possible then give them back their confidence by giving them simple guidelines on how to negotiate the minefield.

A simple introduction to the concept of the determinant of a square matrix and some simple properties of determinants should be accompanied by an illustration of how tedious it can be to evaluate a determinant with even four rows. Please emphasise that there is a different notation for determinants than that for matrices and that they are not interchangeable.

The inverse of a two-by-two matrix is best introduced by the idea of reversing a rotation. The fact that even some square matrices do not have an inverse must be justified. That we say "*the* inverse matrix, when one exists," is a useful vehicle to explain the vital importance of existence and uniqueness theorems. Emphasise that division of two matrices is NOT an operation. Build up confidence by finding inverses of two-by-two matrices before embarking on the three-by-three examples; these might be postponed until simultaneous linear equations are introduced. (Workbook 9). The contrast between the Gauss elimination and the determinant method should be drawn. Whilst students should be made aware of both methods they should be allowed to use the method which they prefer.

Workbook 8: Matrix Solution of Equations

This Workbook introduces the methods of solution by Cramer's rule, by inverse matrices and by Gauss elimination. The relative strengths of these methods should be highlighted, in particular how they cope with systems of equations that have no unique solution.

Cramer's rule provides a simple approach to understanding the nature of solution of two equations in two unknowns and is easily extended to three equations in three unknowns. It should be noted how much more arithmetic is involved and that extension to larger systems is really impracticable.

The use of the inverse matrix provides an alternative approach, which has useful links to the underlying theory but is not much help in the cases where no unique solution exists.

In contrast, the method of Gauss elimination can, properly handled, lead to useful information when there is no unique solution, especially if there are an infinite number of solutions. These days, students are not fluent in handling arithmetic fractions and care must be taken to avoid examples, which lead to anything but the simplest fractions. The method provides a variety of routes to the solution and this fact may cause students to be wary of using it. A warning could be usefully issued by showing the exact solution to a system of ill-conditioned equation alongside a Gauss 'solution'.

Workbook 9: Vectors

In this Workbook we provide a careful introductory account of vectors together with a few basic applications in science and engineering. The approach is relatively conventional, starting with the elementary ideas, approached graphically, of vector addition and subtraction and the multiplication of a vector by a scalar. Applications involving the addition and resolution of forces and a discussion of the forces on an aeroplane in steady flight are provided. (Some elementary knowledge of trigonometry is required here and later in the Workbook.) An opportunity can be taken to discuss the contrast between displacement of an object (vector) and the distance it travels (scalar).

We underline vectors, which better reflects how students and lecturers write them (rather than emboldening them as is done in textbooks). Note that we write $d\underline{r}/dt$ (underlining r only) but $\underline{dr}$ (underlining the expression dr).

Cartesian representation of vectors, firstly in two and then in three dimensions is covered including position vectors. The basic unit vectors are denoted by $\underline{i}, \underline{j}$ and $\underline{k}$. It is important not to rush through the two-dimensional case: once this is well understood the three-dimensional case can be sold as a straightforward extension (and the reason for 'the adding of an extra term' allowing us to move into three dimensions can be explained).

A careful coverage of the scalar (or dot) product of two vectors and its properties, together with some applications in basic electrostatics, is followed by the corresponding vector (or cross) product. Carefully explain why $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$; it is a result to frighten the faint-hearted. In the latter the determinant form for evaluation in Cartesian coordinates is used and simple applications to the torque of a force are discussed. Triple products are covered only in exercises. Many students find difficulty with problems in three-dimensional statics in their engineering modules and the use of the vector product in tackling these problems can be a good selling-point here (case study or coursework, perhaps).

The final section involves some geometrical aspects of vectors particularly direction cosines and direction ratios and vector equations of lines and planes. The simplicity of approach using vectors can be highlighted by contrasting it with the alternative approach.

Workbook 10: Complex Numbers

Complex numbers are a basic tool in engineering and science and this Workbook gives a gentle introduction to their properties and uses. A non-rigorous approach is adopted with a complex number being denoted by $z = a + \mathrm{i}b$ in Cartesian coordinates and by $z = r(\cos\theta + \mathrm{i}\sin\theta)$ in polar coordinates. The complex exponential form is also covered.

The Argand diagram provides a useful means of visualisation of complex numbers but care must be taken to stress that a point on the diagram *represents* a complex number and is not the number itself. The same point can be made when representing a complex number as a vector.

The Workbook initially covers the algebra of complex numbers in Cartesian form and glances briefly at the solution of polynomial equations with real coefficients. Given the poor grasp of real number algebra among many students, it is wise not too assume too much algebraic fluency, so take it slowly. Assuming some knowledge of trigonometric identities allows the introduction of the polar form and its advantage over the Cartesian form when *carrying* out multiplication and division (and taking powers). A simple application to rotations about a point provides a gentle illustration.

Use of the exponential form leads to a discussion of the relations between trigonometric and hyperbolic functions and identities and the Workbook concludes with De Moivre's theorem for finding powers and roots of complex numbers. It is worth remarking that the nth roots of a given complex number share the same modulus and therefore lie on a circle in the Argand diagram and are spaced $\left(\frac{360}{n}\right)^{\circ}$ apart along the circumference.

Workbook 11: Differentiation

At the outset it should be remarked that a sensible ordering of the sections in Workbooks 11 and 12 enhances the presentation of the material to students. After a new technique a reinforcing application example aids the learning process. There was a time when virtually all students embarking on an engineering degree course were well versed in the rudiments of differentiation; that is no longer the case. When the topic was covered in secondary education much time was devoted to it; it is not a topic that can be skimmed over.

Many of today's students seem to have difficulty understanding the concept of rate of change and this needs careful attention. The average rate of change of a function over a given interval should be introduced and the increment in the independent variable denoted by h: the use of δx can come later - to make plausible the notation $\dfrac{dy}{dx}$. Linking the instantaneous rate of change to the gradient of the tangent should be done at the same time.

It is important to define carefully the terms 'differentiable', derived function' and 'derivative' and to distinguish clearly between them. It is also straightforward to give examples where a function is not differentiable at a point: the modulus function, ramp function and unit step function are good examples.

Familiarisation with the table of basic derivatives (derived functions) should be encouraged and the results interpreted on graphs of the appropriate functions. Do stress that angles in this context MUST be measured in radians.

A graphical explanation of how the derivatives of $f(kx), kf(x), f(x+k), f(x)+k$ are related to that of $f(x)$ has been found to help. Should it be necessary to explain why the derivative of a constant function is zero? Probably it is.

The rule for differentiating a linear combination of two functions (and the special cases of sum, difference and scalar multiple) is reasonably intuitive and need not be overstated. Confidence is the keyword here.

The second derivative could be presented in terms of displacement, velocity and acceleration - and a good time to explain that x is not the only notation for an independent variable. The notation $\dfrac{d^2y}{dx^2}$ etc. does seem clumsy and needs justification.

The remaining sections of the Workbook deal with the techniques of differentiating products and quotients, the chain rule and parametric and implicit differentiation. The examples chosen must be simple illustrations; don't lose the wood of the technique in the trees of awkward differentiation. A useful approach is to illustrate some applications of one technique before moving onto more advanced techniques; in particular the last three techniques and especially the last two can be postponed until earlier techniques have been mastered and time allowed for their absorption.

Workbook 12: Applications of Differentiation

The application of differentiation to the finding of the tangent and (indirectly) the normal to a curve should be straightforward but it will help if the results are interpreted on a sketch graph of the relevant function. This suggests the use of simple functions until the student is confident in using the strategy.

Stationary Points and Points of Inflection

Most students have an imperfect understanding of the definitions of local maximum, local minimum and point of inflection. Simple graphs can be used to illustrate these features.

Of the following three statements only the first two are known with any certainty by most students:

Given a twice differentiable function f for which $f'(a) = 0$

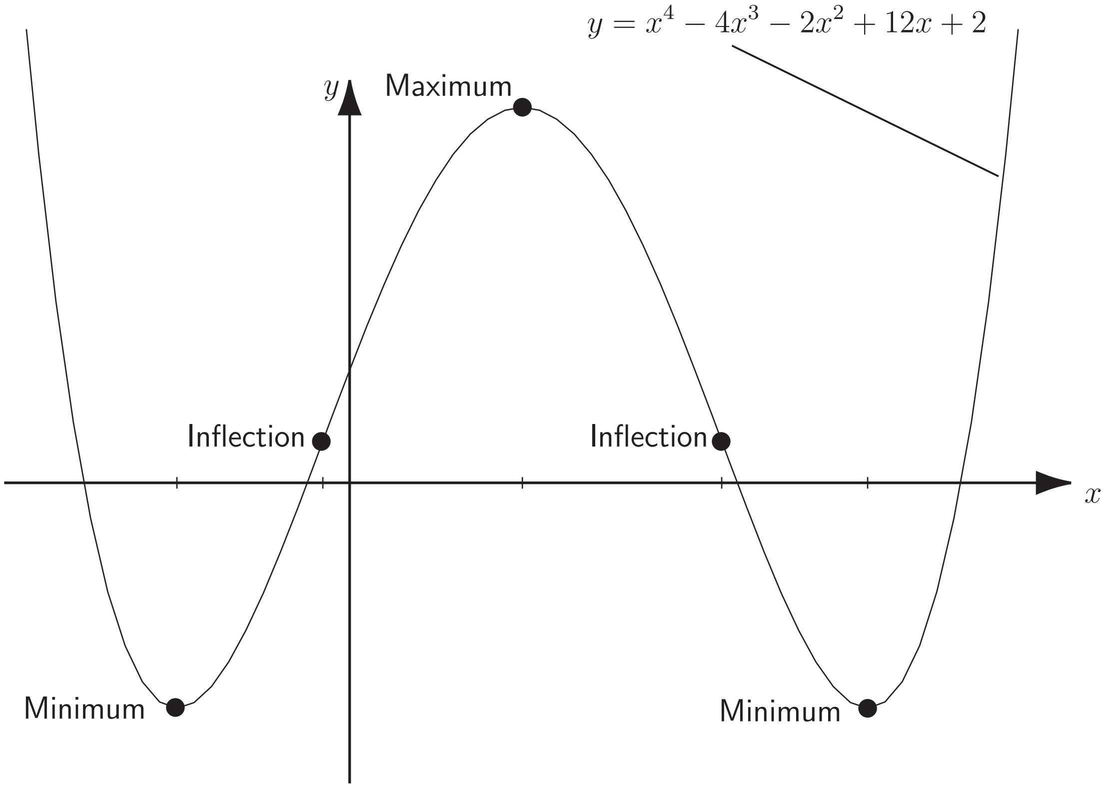
(1) If $f''(a) > 0$, then $f(x)$ has a minimum when $x = a$,

(2) If $f''(a) < 0$, then $f(x)$ has a maximum when $x = a$,

(3) If $f''(a) = 0$, then $f(x)$ has minimum or a maximum or a point of inflection when $x = a$.

Many students think (3) **always** leads to a point of inflection but the graph of $f(x) = x^4$ clearly shows this to be untrue when $x = 0$.

Another misconception is that a point of inflection **requires** $f'(a) = 0$. This is not true as can easily be seen, for example, on the sine curve. This raises another point - for any continuous function there is always a point of inflection between every local minimum and local maximum. The graph below highlights these features.



Maxima and Minima without Calculus

Students all too readily turn to the calculus when needing to find maxima and minima. There are, however, cases when alternative approaches are simpler, quicker or more informative:

Example 1

Find the minimum value of $f(x) \equiv x^2 + 2x + 3$.

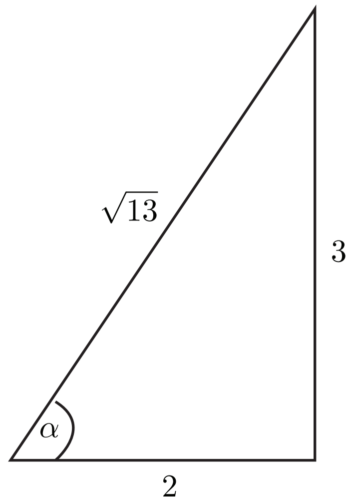
Completing the square gives $f(x) = (x+1)^2 + 2$.

This clearly is a minimum when $x = -1$ and there $f(x)$ has value 2.

Example 2

Find the maximum value of $f(x) \equiv 2\sin(x) + 3\cos(x)$.

Using the trigonometric identity $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ and utilising the triangle in the diagram we have



$$\begin{aligned} f(x) &= \sqrt{13}\left[\frac{2}{\sqrt{13}}\sin x + \frac{3}{\sqrt{13}}\cos x\right] \\ &= \sqrt{13}[\cos\alpha \sin x + \sin\alpha\cos x] \\ &= \sqrt{13}\sin(x+\alpha) \end{aligned}$$

This clearly has a maximum value of $\sqrt{13}$ at $x = \dfrac{\pi}{2} - \alpha$ (for example).

One point of caution should be stated. When applying differentiation to find for example the dimensions of a rectangle of fixed perimeter and maximum area students often get confused between the graph of the function being maximised and the geometry of the problem. It is also useful to remember that differentiation can find a local maximum and physical considerations can be brought in to justify the term "absolute" maximum. Also, in some physical problems, with the knowledge of the behaviour of the function at the physical extremes of the independent variable the nature of the single stationary point can be decided without the need to differentiate a second time, or use the first derivative test.

The Newton-Raphson method is straightforward to derive and relatively easy to apply. Mention should be made of its proneness to failure under certain circumstances. The presentation of curvature should be illustrated by simple examples so that the student can relate algebraic results to the graph of each function. The differentiation of vectors in Cartesian component form follows an intuitive path and this should not be clouded by too much rigour.

Workbook 13: Integration

The concept of an indefinite integral is introduced in the traditional way, as the reverse of differentiation. It is helpful to illustrate this by integrating the velocity function to obtain the displacement function and noting that we cannot say where an object is unless we knew where it was at an earlier time - hence the need for the arbitrary constant. Using the table of derivatives 'back to front' and the rule for linear combinations of functions, which parallels that for derivatives ought not to present many difficulties.

The extension to definite integrals by itself is not too abstruse but the notation in the statement $\displaystyle\int_1^2 3x^2\,dx = \left[x^3\right]_1^2 = 8 - 1 = 7$ is quite deep and needs a gentle presentation.

Surely one of the great moments in learning mathematics is to realise that the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ can be found exactly by evaluating the definite integral $\int_a^b f(x)dx$. Make the most of that moment. Simple examples, please. The area bounded by the sine curve over a quarter-period has a satisfyingly simple value.

Further techniques can be postponed until some further simple applications such as those in Workbook 14 have been looked at.

The techniques of integration by parts, by substitution and by partial fractions are best treated in two stages; first use really simple examples in order to allow students to gain confidence in the basics of each technique, preferably with illustrative examples from engineering to demonstrate that their use is not just 'pure mathematics', then revisit with more advanced versions of the techniques.

As a final phase, integration of functions involving more complicated trigonometric expressions can be tackled.

Workbooks 14 & 15: Applications of Integration 1 & 2

These two Workbooks present a selection of the most common applications of definite integration. Workbook 14 first deals with definite integration as the limit of a sum, then considers mean value and root-mean-square value of a function, volumes of revolution and the lengths of arcs and the area of surfaces of revolution. Workbook 15 considers the integration of vectors, finding centres of mass, and finding moments of inertia.

The importance of recognising a definite integral as the limit of a sum cannot be over-emphasised: it allows the correct establishment of a definite integral whose value is that of the physical quantity concerned. The remainder of the solution is the use of techniques of integration met in Workbook 13. The first examples of each application should lead to very simple integrands. The results should always be examined to see whether they are sensible in the light of the problem.

The mean value of the sine function over a quarter-period, over a half-period and over a full period can be compared and the need for a more meaningful measure of the 'average' of the sine function suggested. Of course, the root-mean-square value is more tricky to evaluate and the double angle formula should be provided as a tool to be picked up and used.

When finding volumes of solids of revolution the cylinder, cone and sphere are the obvious examples and the placing of an axis of symmetry along the x-axis should be highlighted. Careful explanation of the division of the solid into the union of a set of non-overlapping discs should be followed by remarking how an approximating sum can be transformed into a definite integral yielding the exact answer - one of the glories of the calculus.

Approach the problem of finding the length of arc by first using a line parallel to the x-axis, then a line inclined to the x-axis and thus make plausible the length of a general arc. The resulting formula can then be shown to yield the same results as in the first two cases. The formulae for this application and that of surface area only yield simple integrations in a few cases and it is worthwhile pointing out that other examples may require much more complicated methods of integration, or perhaps need to employ numerical approximate methods.

The integration of vectors in Cartesian coordinate form is relatively straightforward and can be dealt with immediately after differentiating them - use velocity goes to displacement as a suitable vehicle for discussion. When discussing the centre of mass of a plane uniform lamina the use of symmetry is a vital ingredient; sensible positioning of the axes may simplify the resulting integrations, not least finding the value of the area itself. Many students do not have a grasp of the idea of a moment and a matchstick model is not too elementary a teaching aid. Finally, the calculation of moment of inertia is not a great leap mathematically but does need a careful foundation of the physical concept. Many students have only a rudimentary knowledge of mechanical principles and very little can be assumed.

Workbook 16: Sequences and Series

This Workbook gives a comprehensive introduction (adequate for most science and engineering students) of these topics. Apart from basic algebra there are no significant prerequisites. Even sigma notation is introduced as a new topic.

In the first section the concept of sequences and their convergence or otherwise is introduced. Arithmetic and geometrical sequences (progressions) are covered and formulae obtained for the sum of n terms in each case.

We then move on to more general infinite series. The meaning of convergence is carefully explained in terms of partial sums. The ratio test for the convergence of series of positive terms is explained together with a brief mention of other convergence tests and the ratio test is also used to briefly consider conditional or absolute convergence for more general series.

A brief section follows on the binomial series and the binomial theorem for positive integers. This is followed by a section on the convergence of power series in which the idea of the radius of convergence is carefully explained. Various properties of convergent power series are also discussed such as their differentiation and integration.

The most useful aspect of convergent power series for applications (their ability to represent functions) is the subject of the final section in which both Maclaurin series and, briefly, Taylor series are discussed.

Workbook 17: Conics and Polar Co-ordinates

This Workbook offers a brief but, for most purposes, adequate introduction to the Co-ordinate Geometry of the three conic sections - the ellipse, parabola and hyperbola - with the circle also being introduced as a special case of the ellipse. The basic Cartesian equations of the conics with axes parallel to one or other of the co-ordinate axes are obtained in the first section and mention made of more general cases.

In the second section polar co-ordinates are introduced and simple curves whose equations are given in polars are described while the equations of the standard conics in polars are also explained.

Finally parametric descriptions of curves are covered including those for the standard conic sections.

The prerequisites for this Workbook are relatively modest, with some basic algebra including completion of the square and algebraic fractions and knowledge of trigonometric functions being required for the first two sections. Knowledge of hyperbolic functions and simple differentiation is required for the final section but detailed discussion of parametric differentiation is covered in Workbook 12.

The Workbook contains a good number of fully worked examples and many exercises for all of which answers are provided.

Notes relating to Workbook 17 page 20:

The general conic

Although every conic is of the form

$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ (with not all A, B, C zero),

not every equation of that form represents a conic. (The equation $x^2 + 1 = 0$ is an obvious counter-example!)

Meaningful cases are:

Ellipse: $AC - B^2 > 0$ [eccentricity $e < 1$]

Possibilities: Ellipse, Circle, Single point.

Parabola: $AC - B^2 = 0$ [eccentricity $e = 1$]

Possibilities: Parabola, Two parallel lines, Two coincident lines.

Hyperbola: $AC - B^2 < 0$ [eccentricity $e > 1$]

Possibilities: Hyperbola, Rectangular hyperbola, Two intersecting lines.

Definition of conic

A conic is the locus of a point, which moves in a plane so that its distance from a fixed point called the **focus** bears a constant ratio called the **eccentricity** to its distance from a fixed straight line in the plane called the **directrix**.

If (p, q) are the coordinates of the focus, e is the eccentricity, and $ax + by + c = 0$ is the equation of the directrix then the equation of the conic is

$$(x - p)^2 + (y - q)^2 = e^2(ax + by + c)^2/(a^2 + b^2)$$

which can be rearranged into a biquadratic of the form

$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ (as stated at the beginning) where

$$
\begin{aligned}
A &= a^2 + b^2 - a^2e^2 \\
B &= -abe^2 \\
C &= a^2 + b^2 - b^2e^2 \\
D &= -p(a^2 + b^2) - ace^2 \\
E &= -q(a^2 + b^2) - bce^2 \\
F &= (a^2 + b^2)(p^2 + q^2) - c^2e^2
\end{aligned}
$$

Workbook 18: Functions of Several Variables

This Workbook is in four sections. We introduce first of all functions of several variables and then discuss graphical representation (surfaces) for the case of two independent variables x and y.

The heart of the text is the section on partial differentiation (for which, naturally) a basic knowledge of differentiation in the case of one independent variable is needed. The material is actually quite straightforward and proceeds only to second partial derivatives including mixed derivatives.

The third section covers stationary points for functions of two variables with computer drawn diagrams to illustrate the various possibilities. The location of stationary points using partial differentiation is carefully discussed and the second derivative tests to determine the nature of these points clearly set out but not proved.

The final section covers errors and percentage changes using partial differentiation. Many worked examples including applications are provided at appropriate intervals.

Note relating to Workbook 18 page 18:

It is difficult to give a watertight yet straightforward explanation of the tangent surface in 3-D although the concept is readily understandable. We have therefore not attempted to be rigorous here.

Workbook 19: Differential Equations

This topic is one of the most useful areas of mathematics for the scientist and engineer in view of the extensive occurrence of differential equations as mathematical models. This Workbook provides a sound introduction to first and second order ordinary differential equations (ODEs). No previous knowledge of the topic is required but a basic competence at differentiation and integration is needed.

The first section introduces ODEs through a model and then discusses the basic concepts of order, type and general solution together with the need for additional condition(s) to obtain a unique solution.

The section on first order ODEs covers separation of variables and the conversion of linear ODEs into exact equations of the form

$$\frac{d}{dx}(y \times f(x)) = g(x)$$

via the use of an integrating factor.

As would be expected in an introductory treatment, second order ODEs are restricted to the linear constant coefficient type but these of course have many applications. Both homogeneous and non-homogeneous types are fully discussed. Knowledge of complex numbers is useful here for dealing with forcing functions involving sinusoids. A number of engineering application examples from electric circuit theory are fully worked out.

The final consolidating section of the Workbook returns to the applications aspect with, among other topics, a reasonably full discussion of mechanical oscillations.

Workbook 20: Laplace Transforms

This Workbook is in many ways a counterpart to Workbook 21 on z-transforms but the two books are independent. Workbook 20 contains a thorough introduction to the Laplace Transform and its main properties together with various applications. Only the one-sided Laplace Transform is considered. A facility with integration is a valuable prerequisite.

The first section introduces causal functions of the form

$$h(t) = f(t)u(t)$$

where $u(t)$ is the unit step function. Simple properties of causal functions are studied as are delayed functions of the form

$$f(t-a)u(t-a)$$

and signals of finite duration of the form

$$f(t)[u(t-a) - u(t-b)]$$

The definition of the Laplace Transform is used to transform various common functions including delayed functions. Inversion of transforms is demonstrated using partial fractions (an outline knowledge of which is assumed). Introduction of the shift theorems both in t and the Laplace variable s allows a wider range of transforms and inversions to be dealt with.

The Laplace Transform of derivatives is covered and this leads on naturally to the solution of ordinary differential equations and systems of these. Specific electrical and mechanical engineering examples are demonstrated. A section on convolution integrals and the time convolution theorem is provided and the concept of transfer or system function for linear systems follows naturally. The negative feedback system is investigated. These application topics enhance the value of the Workbook in a specifically engineering context.

Workbook 21: Z-Transforms

This topic does not appear in older classical texts covering mathematical methods in science and engineering and hence the detailed coverage from first principles provided in this Workbook should be of considerable value particularly to those involved in discrete signal processing and digital control.

The Workbook begins with a basic discussion of sequences and simple difference equations before moving on to a full discussion of the z-transform and derivations of the transforms of important sequences. The transform is introduced ab initio with no previous knowledge of Laplace transforms required.

The shift properties of the z-transform are fully discussed and applied to the solution of linear constant coefficient difference equations. Inversion of z-transforms by partial fractions and by the use of residues is covered although no detailed knowledge of Complex Variable Theory is needed. A detailed application of the solution of difference equations to obtain currents in a ladder network provides a good consolidating example.

A full section is devoted to the concept of transfer (or system) function of discrete systems and the time convolution theorem is also discussed in some detail.

A final (optional) section briefly extends the theory to sampled functions and the relation between the z and Laplace transforms is finally obtained.

Workbook 22: Eigenvalues and Eigenvectors

This Workbook assumes a knowledge of basic matrix algebra and an outline knowledge of determinants although brief revision notes are provided on the latter topic. After a brief rsum of the various situations that can arise in solving systems of linear equations of the form

$$CX = K$$

a full discussion is given of the eigenvalue problem

$$AX = \lambda X$$

and calculation of the eigenvalues λ and the eigenvectors X is demonstrated both using the characteristic equation and later by numerical methods.

A detailed section on applications considers diagonalisation of matrices with distinct eigenvalues and this leads on to solving systems of differential equations (such as coupled spring systems) by the 'decoupling' method.

Further applicable theory deals with the pleasant properties of symmetric matrices in this context and an outline is given of the situations that can arise with general matrices possessing repeated eigenvalues.

A student who has mastered the material of this Workbook is in a strong position to study specific applications such as advanced dynamics and modern control.

Workbook 23: Fourier Series

Fourier series are a well-known tool for analysing engineering and physical systems involving periodic functions (signals). They also arise in solving partial differential equations analytically by the separation of variables method. The basic idea of course is to represent a periodic signal in terms of sinusoids and co-sinusoids. Full details as to how to do this are given in a plausible rather than rigorous fashion in this Workbook. Facility with integration is the main prerequisite of the reader.

An introductory section outlines the basic jargon of the topic (frequency, harmonics etc) and then, after a discussion of the relevant orthogonality properties, we show how to obtain Fourier series for functions of period 2π and then of more general period.

The particular form of the Fourier series for functions which are odd or are even is covered and this leads on, after a brief discussion of convergence, to obtaining Fourier sine or cosine series for functions defined over a limited interval.

The final section of the Workbook gives a detailed coverage of the complex exponential form of Fourier series together with Parseval's theorem and a brief discussion of electrical engineering applications.

Fourier transforms are covered separately in Workbook 24, and the use of Fourier series in connection with partial differential equations is introduced in Workbook 25.

Workbook 24: Fourier Transforms

In this Workbook a brief but adequate introductory account is given of the complex exponential form of the Fourier transform. An informal derivation of the transform as a limiting case of Fourier series is given and mention made of the different ways of writing the transform and its inverse. Our choice is to put the factor $\frac{1}{2\pi}$ in the formula for the inverse and to use the angular frequency ω in the complex exponential factors.

The Fourier transforms of common signals such as exponential and rectangular are derived from the definition after which the main properties of the transform are obtained. The time and frequency shift (or translation) properties, the frequency and time differentiation properties and the form of the transform for even and for odd functions are among the topics discussed here.

Brief plausible discussions of the Fourier transforms of the unit impulse function (also known as the Dirac delta function) and of the Heaviside unit step function are given in the final section.

A generous selection of worked examples and of exercises (with outline answers) is provided. There is little specific discussion of applications of Fourier transforms but a student who has mastered the contents of the Workbook would be in a good position to use them in specific fields.

Workbook 25: Partial Differential Equations

This topic is of course a very broad one. This Workbook is restricted to consideration of a few special PDEs but they are ones which model a wide range of applied problems. No previous knowledge of PDEs is required but a basic knowledge of partial differentiation and the ability to solve constant coefficient ordinary differential equations is assumed although brief revision of the latter topic is given. Numerical methods of solution are not dealt with.

The main PDEs studied - the two-dimensional Laplace equation, the one-dimensional wave equation and the one-dimensional diffusion (or heat conduction) equation - are fully discussed (although not derived in detail) so that the student knows how the detailed mathematics to follow may be applied. Various types of initial and boundary conditions are mentioned.

The separation of variables method of solution for PDEs is the core of the Workbook and this is developed carefully, initially looking at problems where Fourier Series do not arise. Emphasis is given to the fact that the method is a logical sequence of steps relevant to a wide range of problems. The Workbook concludes with problems where Fourier Series are required, the latter being the subject of Workbook 23 in this series.

Workbook 26: Functions of a Complex Variable

This Workbook builds especially on the work on complex numbers covered in Workbook 10. The six areas that this Workbook deals with are: the basic ideas of a function of a complex variable and its derivatives, the Cauchy-Riemann equations, the standard complex functions, basic complex integration, Cauchy's theorem, and singularities and residues.

Starting with some simple complex functions and their evaluation, the idea of the limit of a function as the input variable tends to a special value is best tackled by choosing examples where the limit is path-dependent, followed by one where it is not; in the latter case the fact that the limit does not depend on the two or three paths chosen does not imply *per se* that the limit is path-independent, and raises the question as to how to be sure that it *is* path-independent. The derivative of a complex function should follow, with particular emphasis being placed on the idea of a singular point and on the similarity between processes of differentiating a complex function and a real function at a regular point.

The use of the Cauchy-Riemann equations to answer the question of analyticity of a function should be a demonstration of the elegance of this area of mathematics, and leads nicely into the idea of conjugate harmonic functions. The explanation of the role of these functions in an application of Laplace's equation should be taken. The usefulness of a mapping being conformal will have more impact if related to an example from the appropriate engineering discipline without the need to attempt the solution of that example. With what could be a relatively abstract area of mathematics, the aim should be first to show relevance, in the hope of producing sufficient interest to go through the necessary algebra.

When introducing the standard functions of a complex variable, the similarities to and differences from the real counterparts need careful treatment. The case of multiple solutions to the equation $e^z = 1$ contrasts nicely with the unique solution to the equation $e^x = 1$.

Section 4 gives the reader a first glance at complex integration. The message is 'keep the integrands simple'. Again, it is useful to give an example where the integral is path-dependent and one where it appears to be path-independent, and raise the question of how we could verify the path-independence in any particular case. This area of mathematics is awash with elegant theorems and may appear daunting to students not weaned on theorems at school; a gentle approach is recommended.

The beauty of the simplicity of Cauchy's theorem and its wide applicability has the potential to excite the imagination of even weak students, if sensitively presented. A gentle gradation of examples is especially called for here. The extension to Cauchy's Integral Formula should follow immediately to link the two results at the earliest opportunity.

The classification of singularities of a complex function is a natural consequence of its Laurent series, which in turn is a neat extension of its Taylor series. Experience shows that Taylor series can be particularly daunting and the use of simple functions and a relatively low expectation of the depth of coverage at a first run-through is advisable.

The residue theorem can be introduced by means of simple examples and forms a suitable point at which to conclude the first venture into functions of a complex variable.

The aim throughout should be to enthuse the student by the elegance and simplicity of the topic and not go too deeply into each sub-topic. Harder examples are best tackled when the imagination has been fired and this is done by keeping the examples simple; far better not to tackle harder examples at this stage than to turn the students off.

Workbook 27: Multiple Integration

Multiple Integration is a topic following on from definite integration and is often taught in the second year of University courses. The approach taken here has been moulded by modules taught to Mechanical and Aerospace Engineers in recent years.

The comparison with "single" integration (i.e. volume under a surface plays an analagous role to area under a curve) is quite deliberate as are the use of the terms "inner" and "outer" integrals and the demonstration of double integration over a rectangular region (27.1) before going onto non-rectangular regions (27.2). The majority of the examples given in a rectangular geometry are in terms of variables x and y, reflecting the fact that most of the applications given involve integration over an area or volume. However, at least one example (in 27.1) draws attention to the use of alternative notation in terms of s and t.

Similarly, most examples in a circular geometry make use of r (or ρ) and θ (or in spherical geometry, r, θ and ϕ). However, the attention drawn to s and t in 27.1 reinforces to the student that there is nothing "sacred" about the variable names used. Similarly, while the notation dA or dV is used to represent an area element or a volume element, it should be appreciated that other notation could be used. While 27.4 is concerned with the Jacobian and change of variables, it was felt the transformation to plane polar coordinates was sufficiently universal to allow this special case in 27.2

and let 27.4 show this case in context and look at other possible transformations.

Note relating to Workbook 27 page 64:

Note that here in discussing Jacobians the notation of two vertical bars is used in close proximity for two quite different concepts: modulus and determinant.

Workbook 28: Differential Vector Calculus

The main thrust of this Workbook is in 28.2. The introduction of the vector differential operators div, grad and curl with a necessary background to the subject in terms of scalar and vector fields is given in 28.1, and 28.3 developes the ideas in alternative coordinate systems. The underline notation (e.g. $\underline{u}$) has been used for vectors and the cartesian unit vectors expressed as $\underline{i}, \underline{j}$ and $\underline{k}$, giving consistency with Workbook 9.

However, two sets of notation are used for the vector differential operator i.e. the use of the words grad f, div $\underline{F}$ and curl $\underline{F}$ are sometimes used rather than the ∇ notation ($\underline{\nabla} f, \underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$). It was felt important to introduce students to both notational forms as the word notation often puts the mathematics in more of a physical or engineering context while the operator often simplifies calculation.

Originally, 28.3 on vector derivatives in orthogonal curvilinear coordinates was intended to be more general i.e. to mention cylindrical polar coordinates and spherical polar coordinates in the context of being special cases and to look at a few more coordinate systems and how to calculate relevant vector derivatives. However, pressure on space did not allow this and the Section was revised to include only cylindrical and spherical polar coordinate systems. [For these other cases, refer to *Handbook of Mathematical Formulas and Integrals*, Alan Jeffrey, Academic Press.]

In 28.3, unit vectors $\underline{\hat{\rho}}, \underline{\hat{\phi}}$ and $\underline{\hat{z}}$ (cylindrical polar coordinates) and $\underline{\hat{r}}, \underline{\hat{\theta}}$ and $\underline{\hat{\phi}}$ (spherical polar coordinates) are used. Of course, these are equivalent to $\underline{\hat{\rho}}, \underline{\hat{\phi}}$ and $\underline{i}, \underline{j}$ and $\underline{k}$ for Cartesian coordinates. In fact, the vectors $\underline{\hat{z}}$ and $\underline{k}$ are the same vector but the notational form chosen fits the form of the other vectors under consideration i.e. $\underline{k}$ for cartesian coordinates and $\underline{\hat{z}}$ for cylindrical polar coordinates.

Workbook 29: Integral Vector Calculus

This Workbook uses the same notation for vectors and vector derivatives as Workbook 28. Subsection 1 of 29.1 on Line Integrals was a candidate for being 27.5 (i.e. in the Workbook on Multiple Integrals) but it was decided to include it here in condensed form (i.e. at the beginning of the Section on Vector Line Integrals).

A possible challenge when carrying out integrals over a sloping area is of expressing dS correctly in terms of dx and dy (or dz and one of dx or dy). This point has been emphasized in the text and in a couple of examples.

The convention used was that the mathematical expressions were referred to as theorems e.g. Gauss' theorem while the term law was reserved for the application to electromagnetism (see Engineering Example).

Clearly, these topics (particularly Green's theorem) could be mentioned in greater depth.

Workbook 30: Introduction to Numerical Methods

The Workbook assumes familiarity with basic matrix theory. Although more can be said to those who know more about matrices, what is assumed is the realistic minimum.

Rounding error and Conditioning
The key here is to ensure students have an awareness that errors due to rounding (and indeed to any other source) can build up, and that injudicious choice of operations can exaggerate the error.

Gaussian elimination and partial pivoting
Solving simultaneous equations needs care if it is to be effective. Although to many Gaussian elimination is common sense (and the impressive name may cause some not to notice that this is what you might do if nobody had told you how to solve equations!) there are the issues of errors building up. Pivoting is a way to tackle this, effectively by avoiding dividing by small numbers.

LU decomposition
In principle one could solve simultaneous equations by inverting the matrices involved, and there is an explicit algorithm for finding inverses. However, while this is fine in theory, the algorithm involves many determinants and as the determinant of an $n \times n$ matrix can have $n!$ terms, this is not really practical for even modest values of n. The L-U decomposition is much simpler, and you could if you wished ask students to estimate the number of operations involved and compare it with inverting.

Matrix norms
The condition number gives an estimate of the accuracy of the solution process of a matrix equation, in that, very roughly you might expect the errors in the solution to be the size of those on the right hand side multiplied by the condition number. For square symmetric real matrices, the choice of norm which gives the smallest condition number will make it the ratio of the eigenvalues of largest and smallest modulus: this will make sense only to students who know something about eigenvalues.

Iterative methods
Jacobi and Gauss Seidel iteration: see Engineering Example 1 for a case where these take a long time to converge.

Workbook 31: Numerical Methods of Approximation

Polynomial approximations
Approximation by polynomials is common sense in that we want to approximate the unknown functions by something simple, and polynomials are simple functions. Lagrange interpolation produces an approximation that is exact at certain points. It may be worth exploring with the students just how certain an engineer is going to be about the data that are being interpolated, and to look at the effects of changes, especially if two of the x-values are close.

For students whose background is suitable one can mention approximation by other types of functions (e.g. trigonometric polynomials), or functions, which share properties which the solution is known to have. Least squares approximation of straight lines is another approach: a polynomial of degree 1 is being made to fit the data as well as possible. One could use the best least squares approximation of N points by polynomials of some chosen degree, to make the answer less susceptible to changes in the data. Exploring the effects here with a real set of data relevant to the students might be useful.

Numerical integration
Consider the size of the errors for the composite trapezium (ask the students to estimate it). In many cases this is small enough that nothing more sophisticated is needed, although there are more accurate ways of doing numerical integration: they are also more complicated.

Nonlinear equations
The bisection methods, although slow, will always work. Newton-Raphson is much more efficient, if the starting point is "close enough", but requires the derivative to be calculated. It can be shown (and Figure 11 almost does this) that Newton-Raphson will work provided the second derivative does not change sign in the area considered. There are more complicated ways of guaranteeing convergence if the derivative does change sign.

For students and teachers willing to investigate this, try Newton-Raphson for solving $x^3 - x = 0$, with various starting points. Which root does it converge to? This is quite sensitive near 0, where the second derivative does change sign.

The last word . . .
Student comment: "I am amazed that you can base a whole module around guessing and still make it sound credible."

Workbook 32: Numerical Initial Value Problems

Much of this Workbook deals with PDEs and, although most Engineering students will not have encountered this, the Workbook reflects the mathematical difficulties involved in analysing PDEs. It is important that students appreciate that stability and convergence are important, even though the analysis may be hard.

General linear multistep methods
It's worth making the analogy with solving ordinary equations, that some of the apparently more awkward formulations may be easy to use as iterative methods.

Runge-Kutta method, predictor corrector methods
This gives a way of finding a reasonable initial estimate of the solution, which can be used in the corrector method: this effectively turns an implicit method into an explicit iteration. As that can be iterated, the result may be used to obtain good accuracy, provided we use something to get it started.

The sections on parabolic and hyperbolic PDEs, both of which usually arise in the form of initial-value problems, illustrate the care needed to make the methods work on particular PDEs, and that the same method will not necessarily work on other types of PDE.

Workbook 33: Numerical Boundary Value Problems

Two point boundary value problems
These link well with some of the earlier work on Workbook 30, and it is worth emphasising that the numerical schemes, which are reasonably intuitive, give rise to potentially large sets of simultaneous equations, with all the difficulties discussed in Workbook 30. The methods there, however, will allow solutions to be found.

Elliptic PDEs
These normally arise in the form of boundary-value problems, and the Workbook gives methods for Laplace's and Poisson's equations, and that it is known that these converge. The matrices involved are all "sparse" in that they have a structure and most of the entries are zero. Finding ways of exploiting the structure of the matrices can lead to improved numerical methods for their solution. Again the techniques of Workbook 30 are important, and the students should be able to see that further progress may need more attention to the methods of solving set of equations, or of matrix iteration.

Workbook 34: Modelling Motion

This Workbook relates to the modelling ideas in Workbook 5 and Workbook 6 Section 6.6. Workbook 34 would be best used to consolidate the material in Workbooks 9 to 15 (i.e. vectors and differential and integral calculus). It does so in the context of the dynamics of rigid body dynamics (projectiles, friction forces, motion on an inclined plane, motion in a circle, 3D examples of motion and resisted motion). The level is most appropriate to first year engineering mathematics. However it might form the basis for further trips around the modelling cycle in later years. Further developments might include the numerical solution of the differential equation for resisted motion that is not linear in velocity, and more realistic modelling of vehicle motion on bends.

Workbooks 35-46: Probability and Statistics

Probability and statistics deal with uncertainty and variation. Engineering students do not always easily feel comfortable with these ideas, being perhaps more at home with the apparent certainties of physical laws. In the world in which engineers work, of course, variation does exist, in materials, in the environment, in the wear and tear to which products are subjected and so on. Therefore there is also uncertainty.

Engineering students often need help in coming to terms with these concepts and with the idea that the "answer" to questions is often a statement, of one type or another, about the remaining uncertainty in a situation rather than a fixed value for some physical quantity. Physical analogy or "feel" does not always come so easily with such ideas. In students' efforts to deal with this, motivation is often of great importance and motivation is easier when the student sees the material as relevant. This is one reason why we have attempted to provide examples and explanations, which relate to engineering. It would, of course, be impractical to provide examples of every topic drawn from every specialised branch of engineering but students should be able to be less narrow than this and tutors may be able to provide additional, more specialised, examples by changing the "stories" in some of ours. The second reason for trying to use relevant engineering examples and illustrations is to help students to attain that "feel" which will help them to grasp what is going on. This is made easier if they can relate probability statements, for example, to frequencies of events of familiar types. So, for example, it may be easier at first to think of 100 light bulbs, or whatever, of which "on average" one in twenty, that is 5, might be expected to fail before a certain time, rather than talk more abstractly of a probability of 0.05. Similarly, it might help to think of a product with two components, A and B, where one in ten products "on average" has a defect in component A and one in five products "on average" has a defect in component B, regardless of the state of component A. Then, out of 100 products, we might expect on average 'one in five of one in ten products', that is one in 50, to have defects in both components.

There is more than one school of thought in probability and statistics but presenting both of the major approaches in introductory material such as this would risk causing confusion. Although the Bayesian approach has made major advances in recent years, it is probably still true that the approach which engineering graduates will be expected to have learned is the more traditional frequentist approach. Therefore probability is presented in terms of limiting relative frequency and statistical inference is presented in terms of the familiar ideas of estimators, confidence intervals and significance tests. Thus properties can be explained in terms of long-run frequencies. This is not too difficult in the context of a mass-production industry, for example, although care is still required in explaining exactly what is meant by a confidence interval or the interpretation of a non-significant result in a test. In branches of engineering concerned with one-off constructions, the application of relative-frequency ideas requires more careful thought and abstraction although, even here, it is often the case that, for example, large numbers of similar components are used.

Students should be encouraged at an early stage to distinguish between a population and a sample, between the (usually unknown) value of a population parameter and its estimate and, of course, between an estimator, that is a random function, and an estimate, that is a particular value. This becomes particularly important from Workbook 40 onwards. Experience shows that students do not always find this easy and often fall into the trap of believing that the data tell us that $\mu = 1.63$ or that the null hypothesis is (definitely) false.

This is, of course, an introductory service course in probability and statistics and it is neither possible nor desirable to attempt to achieve either the breadth or depth that would be given in a course for specialist statistics students. Inevitably there are topics which we do not cover but which may be encountered by engineering students at some time. We feel that the important thing to be achieved in a course such as this is a grasp of the basic ideas and concepts. Once these are understood, coping with new distributions and procedures, perhaps with the aid of suitable computer software and, where appropriate, specialist advice, can often be relatively straightforward.

In the modern world, statistical calculations are usually done with a computer and we expect that students will learn how to do some calculations this way. However it is also useful from a learning point of view to do some, relatively simple, calculations without the aid of a computer program. When a computer is used, it is best to use special statistical software. There are many packages available and many of these are easy to use for beginners, for example Minitab. Some excellent software, e.g. R, is available free of charge on the Web though non-specialist students might not find this quite as convenient to use. Having said this, many students will have access to a spreadsheet program and may prefer to use this. Most of the calculations described in the HELM Workbooks are possible without too much difficulty using popular spreadsheet software. Some engineering students may well be using, other software such as Matlab in other parts of their courses. Matlab also provides statistical and graphics facilities and, if the students are using it anyway, it may be a sensible option. Few engineering students should have difficulty with the *mathematics* in the statistics Workbooks (especially if they have the benefit of the HELM mathematics Workbooks!). Experience suggests that difficulty is more common with the underlying statistical and probabilistic concepts.

Workbook 35: Sets and Probability

This Workbook serves to underpin all of the probability and statistics covered in the HELM series of Workbooks 35-46. For this reason it is essential that the students fully understand and become conversant with the notation used and the concepts explored. The notation used is standard set notation as found in a myriad of standard textbooks on the subject. Particular attention should be paid to Section 4 - Total Probability and Bayes' Theorem. It is the authors' experience that effort put in to learning the concepts described and fully studying the worked Examples and then completing all of the Tasks and Exercises will pay handsome dividends when later Workbooks concerned with probability and statistics are studied.

Some of the Figures and Examples are in terms of electrical items such as switches and relays wired together in series and/or parallel. Even though the diagrams are straightforward, care should be exercised to ensure that the student is familiar with and understands these diagrams. No knowledge of circuit theory as such is assumed but it is possible that some students may not understand the ideas involved in series and parallel circuit wiring, for example. It is hoped that this will not be a problem for engineering students!

Workbook 36: Descriptive Statistics

The first part of this Workbook, entitled "Describing Data", looks at the basics of calculating means and standard deviations from tabulated data. Many students feel initially that in the age of electronic calculators and computers this is unnecessary. However, it should be born in mind that a familiarity with these basic formulae will pay dividends when later studying Workbook 37: Discrete Probability Distributions and Workbook 38: Continuous Probability Distributions. The first section of Workbook 36 lays the necessary ground for a proper understanding of the formulae and methods used to calculate means and variances for discrete and continuous distributions.

The second section of Workbook 36 is aptly titled "Exploring Data". This is important. The material enables the student to avoid the twin pitfalls of simply taking data either at their face value or accepting them simply as a mass of meaningless figures. By using the techniques outlined in this section of Workbook 36, the student should begin to develop the ability to "look into" a data set and draw tentative conclusions prior to any attempt to analyse the data. This also has the benefit of allowing the student to comment on the possible validity of any conclusions reached using statistical analysis.

Workbook 37: Discrete Probability Distributions

This Workbook looks at the basics of discrete distributions in general and then turns its attention to three distributions in very common use. The binomial distribution is followed by work concerning the Poisson distribution. This latter distribution is looked at in two distinct ways, firstly as a viable approximation to the binomial distribution, a role in which it can save the user a large amount of tedious arithmetic normally without significantly reducing the accuracy of the answers obtained. The second way in which the Poisson distribution is considered is as a distribution in its own right used to describe the occurrence of events by the so-called Poisson process.

The Workbook concludes with a short section concerning the hypergeometric distribution. This has applications in, for example, acceptance sampling where, because we are sampling without replacement from a finite batch, the results of individual trials, e.g. of whether an item is defective, are not independent. This is in contrast to the usual assumptions underlying the binomial distribution. The distribution of the numbers of defective and acceptable items in a sample from a batch may seem to be of little direct interest. However, when we turn the question round from "Given certain numbers within the batch, what is the probability of observing x?" to the inferential question, "Given observation x, what can we learn about the rest of the batch?" then the hypergeometric distribution turns out to be very relevant. It will also reappear in Workbook 42 in the context of the test for no association in contingency tables.

The notations E for expectation and V for variance are introduced.

Workbook 38: Continuous Probability Distributions

This Workbook follows the same general format as Workbook 37 "Discrete Probability Distributions". Firstly, a look at the basics of continuous distributions in general confirms that the same general pattern may be discerned in the development and use of, for example, the formulae used to calculate means and variances.

The Workbook then turns its attention to two distributions in very common use, the uniform distribution and the exponential distribution. In introducing the exponential distribution, we also begin the development of applications to reliability and lifetimes. The statistical analysis of the nature of the expected lifetime of a product has huge relevance to the industrial applications of statistics and for this reason the exponential distribution is worthy of close study. The concept of a distribution function can be explained in terms of a lifetime distribution and this can then lead to the idea of a probability density function. Analogy with physical mass density can also be helpful.

Workbook 39: The Normal Distribution

The normal distribution is extremely important to all students of statistics, and since a very large number of practical experiments result in measurements, which are closely modelled by this distribution, it is certainly essential that engineers fully understand and can apply this distribution to practical situations. A great many problems can be formulated and solved by using the so-called standard normal distribution. This Workbook will guide the student through a number of applications sufficient to lay a firm basis for future competence in problem solving using the normal distribution. The usual notation of $N(\mu, \sigma^2)$ for a normal distribution with mean μ and variance σ^2 is used.

The second section of the Workbook builds on the foundations laid in Workbook 37 concerning the binomial distribution. Here we look at how the normal distribution can by used to approximate the binomial distribution provided that the conditions are appropriate. This application can save an enormous amount of arithmetic without loss of accuracy when applied properly.

Finally, this Workbook looks at practical situations whose description involves sums and differences of random variables. Without a working knowledge of the behaviour of these quantities, unacceptable restrictions have to be placed on the range of applications, which an engineer (or anyone else!) is able to consider.

Workbook 40: Sampling Distributions and Estimation

Having looked at probability distributions in Workbooks 37-39, we now move on to inferential statistics. The approach taken to inference is the traditional frequentist approach. Experience shows that attempting to teach *both* frequentist inference *and* the alternative Bayesian inference in one introductory statistic course can lead to confusion and, whether we like it or not, frequentist inference still predominates in much of industry and graduates will be expected to know about significance tests and confidence intervals.

Concepts of inference can often be difficult for students to grasp at first since students are often unused to dealing with uncertainty and questions where the answer is a statement about something which we do not know for certain. For this reason it is important to be careful with the distinctions between, for example, population parameters and sample statistics.

As the title of this Workbook implies, it is concerned with two crucially important aspects of statistics, namely samples, their properties and their use in estimating population parameters. The student should realise that it is often impractical or impossible to deal with a complete population. It is usually the case that the parameters of a population - the mean and variance for example - have to be estimated by using information which is available from a sample or samples taken from the population. Properties of estimators: bias, consistency and efficiency, are introduced. It is important at this stage to make sure that students understand the distinction between population and sample and between estimator and estimate and the concept of a sampling distribution.

It is reasonable to enquire as to the degree of accuracy likely to be involved when a population parameter is estimated. This question is addressed in the Workbook by the construction of confidence intervals for the mean of a population and the variance of a population. The idea of a confidence interval is subtle and not always well understood. Again the concept of properties in repeated sampling is important.

Constructing a confidence interval for the variance of a normal population involves the use of the Chi-squared distribution. An introduction in sufficient detail is given to enable the student to gain a good level of understanding of this distribution and its application to finding confidence intervals involving estimates of the variance of a normal population. Given that a great many practical situations involve statistical estimation it will be appreciated that this Workbook deserves the very close attention of the student.

Workbook 41: Hypothesis Testing

A solid understanding of the theory and techniques contained in this Workbook is essential to any student wishing to apply statistics to real-world engineering problems. Many students find the material challenging, one of the reasons being that work on hypothesis testing is often attempted on the basis of half-understood and half-remembered theory and techniques which necessarily underpin the subject. The student should already be familiar with the material in the HELM statistics series concerning the following topics:

- sampling;
- the normal distribution;
- the binomial distribution;
- the chi-squared distribution.

In this Workbook it is assumed that students have some understanding and a working familiarity with all the above topics. Remember that no attempt is made in this Workbook to teach those topics and that coverage of them will be found elsewhere in the HELM statistics series.

Student's t-distribution and the F-distribution are introduced in this Workbook.

Since many students find some difficulty with this topic, it is strongly suggested that the worked examples are given particular attention and that *all* of the student Tasks are fully worked. The students should not allow themselves to fall into the trap of simply looking up the solution to a problem if they cannot successfully complete it. Suggest that the students keep a record of their attempts to solve problems in order that any lack of understanding may be identified and rectified. Even published scientific literature often contains suggestions of misinterpretation of the results of significance tests. It should be emphasised that a significance test measures (in a particular sense) the evidence against a null hypothesis. A significant result means that we have "strong" evidence against a null hypothesis but not usually certainty. A non-significant result simply means that we do not have such strong evidence. This may be because the null hypothesis is true or it may be that we have just not obtained enough data.

Workbook 42: Goodness of Fit and Contingency Tables

In order that the student understands the first section of this Workbook ("Goodness of Fit") it is essential that the principles of hypothesis testing contained in Workbook 41 are properly understood. In particular, the student should be recommended to revise, if necessary, any work done previously concerning the Chi-squared distribution.

Essentially, the first section of the Workbook is concerned with making decisions as to whether or not a given set of data follow a given distribution (say normal or Poisson) sufficiently closely to be regarded as a sample taken from such a population. If an underlying distribution can be identified, then clearly certain assumptions follow. For example, if a data set is normally distributed then we know that in general we may calculate parameters such as the mean and variance by using certain standard formulae. If a data set follows the Poisson distribution then we may assume that the population mean and variance are numerically identical.

The second section of the Workbook ("Contingency Tables") is concerned with situations in which members of samples drawn from a population can be classified by more than one method, for example the failure of electronic components in a system installed in a machine and the positions in a machine in which they are mounted. The Workbook discusses how such information may be presented and follows this by a discussion involving hypothesis tests to decide (using the example outlined above) whether or not there is sufficient evidence to conclude that failure of a component is related to position in the machine.

Workbook 43: Correlation and Regression

Section 43.1 introduces the study of regression analysis, that is, the study of possible predictive or explanatory relationships between variables. The student should note that the real work of this Section deals with linear regression on one variable only. In the real world, regression may be multiple (i.e. with several explanatory variables) and may be non-linear. A very short introduction to non-linear regression is given at the end of 43.1 and is introduced for the sake of completeness only - simply to signal to the student that non-linear regression does exist and that the real world is often anything but linear!

Modern computer software makes the calculations for multiple regressions and many nonlinear regressions readily available but an understanding of the basic ideas, gained through looking at simple cases, is essential to interpret the results.

The main technique used to study linear regression is the method of least squares. In order that this topic be fully understood, it is necessary to use some multi-variable calculus. The Workbook deliberately ignores this and simply presents the equations resulting from the use of this technique.

Section 43.2 is concerned with the topic of correlation. Two common methods of measuring the degree of a possible relationship between two random variables are considered, these are Pearson's r and Spearman's R. The methods of calculation given will result in a numerical value being obtained. The strength of evidence for the existence of a relationship may be measured by performing a significance test. The importance of the normality assumption for the usual test of Pearson's r should be emphasised. In cases where this assumption cannot be made or where a relationship may be monotonic but not linear Spearman's R should be used. It is worth noting that significance tests involving both measures of correlation involve the use of Student's t-distribution and that it may be worth revising this topic before a study of correlation is attempted.

Workbook 44: Analysis of Variance

This Workbook covers three topics, one-way analysis of variance, two-way analysis of variance and an introduction to experimental design. In order to obtain the maximum benefit from this Workbook, the students should ensure that they are familiar with the general principles of hypothesis testing. In particular, a working knowledge of the F-distribution is essential. This groundwork is covered in Workbook HELM 41-Hypothesis Testing.

The Workbook starts with one-way ANOVA and the introduction makes clear the advantages accruing from a technique enabling us to *simultaneously* compare several means. On reading the introduction, it should be clear to the student that not only is the amount of work needed to compare several means drastically reduced but also that use of ANOVA gives the required significance level whereas use of a collection of pairwise tests would distort the significance level. When working through the learning material, students should ensure that they clearly understand the difference between the phrases "variance *between* sample" and "variance *within* samples".

While one-way ANOVA considers the effect of only one factor on the values taken by a variable, two-way ANOVA considers the simultaneous effect of two or more factors on a variable. Two distinct cases are considered, firstly when possible interaction between the factors themselves is ignored and secondly when such interaction is taken into account. The Workbook concludes with a short introduction to experimental design. This section is included to encourage students to consider the design of any engineering based experimental work that they may be required to undertake and to appreciate that the application of statistical methods to engineering begins with the design of an experiment and not with the analysis of the data arising from the experiment.

Workbook 45: Non-parametric Statistics

This Workbook looks at hypothesis testing under conditions, which are such that we do not, or cannot, because of a lack of evidence, make the claim that the distributions we are required to deal with follow a specific distribution. Commonly, we may assume that data follow a normal distribution and apply say, a t-test under appropriate conditions. Such assumptions are not always possible and we have to use distribution-free tests. Such tests are usually referred to as non-parametric tests since they do not refer to distribution parameters, for example the mean of a normal distribution. This Workbook assumes a familiarity with the techniques of hypothesis testing covered in Workbook HELM 41, in particular the t-test. A working knowledge of the binomial distribution (HELM 37) is also essential background reading. The Workbook considers the application of non-parametric testing to situations involving single samples and situations involving two samples.

A common error is to suppose that there are such things as "parametric data" and "non-parametric data". It is not the data, which are "parametric" or "non-parametric", of course, but the procedures, which are applied, and the assumptions underlying them. It is hoped that this Workbook will convey an understanding of the different assumptions which are made, just what it is that is being tested and when it is appropriate to use a nonparametric test. The tests which are specifically addressed, the sign test, the Wilcoxon signed-rank test and the Wilcoxon rank-sum test (which is equivalent to the Mann-Whitney U-test) are straightforward in themselves but the assumptions underlying them and their interpretation are not always well grasped.

Workbook 46: Reliability and Quality Control

This Workbook is split into two distinct sections, the first covering the topic of reliability and the second giving an introduction to the crucially important topic of quality control. Necessary background reading for this Workbook consists of general probability (HELM 35) and continuous probability distributions (HELM 38). In Workbook 46 the study of reliability begins with a look at lifetime distributions and progresses to work concerning system reliability with some surprising results. For example while it may be taken to be intuitively obvious that a set of electronic components wired in series will be as reliable as the least reliable member, this turns out not to be the case - the set turns out to be less reliable than its least reliable member!

The section concerning quality control starts with a very brief history of the subject and introduces the student to some essential elementary control techniques including simple quality control charts, R-charts and Pareto charts. Elementary trend detection is considered via the use of "standard" checks, which the student may apply to, given data sets.

The material in this Workbook will have a clear relevance for many engineering students. Tutors might like to consider using some of the material earlier in the sequence of Workbooks if this seems appropriate to maintain interest and motivation.

Workbook 47: Mathematics and Physics Miscellany

Section 47.1 "Dimensional Analysis in Engineering" sets out several examples, which show the use of dimensional analysis in the analysis of physical systems. The standard examples such as the simple pendulum are treated in a detailed manner, so as to emphasize that dimensional analysis has several "blind spots"; in particular the unknown "constant" which appears in standard simple treatments must actually be regarded as a **dimensionless quantity** and so could be an arbitrary function of an angle, of a Mach number, etc. In several cases the usual simple approach is compared with the Buckingham approach, which makes the dimensionless quantities the dominant quantities in the description of a physical system. The examples treated gradually increase in complexity, with the theory of fluids providing examples for which the set of relevant physical variables is sufficiently rich to provide two different dimensionless quantities in the Buckingham approach.

[See the references at the end of this Section in the Workbook, which includes suggested Google search keywords.]

Section 47.2 "Mathematical Explorations" provides an introduction to a range of techniques, which provide useful alternatives to those of the other Workbooks. One theme, which appears repeatedly, is that of the use of power series methods in tackling problems, which are often treated, by the use of ordinary algebra. The tutor might note that there is an implicit underlying theme here which is worth emphasizing in discussions: when we find the Maclaurin series for $\tan(x)$ by studying the equation $dy/dx = 1 + y^2$ what we are actually doing is finding a power series solution of a differential equation. For functions, which are already familiar, this aspect is not obtrusive but these "known" examples can be used to motivate an approach to the study of power series solutions of more general differential equations, especially for new cases in which we do not know the solution before we start. For example, the power series approach is a useful tool in finding the energy levels associated with the Schrödinger equation in quantum mechanics, particularly for cases in which the equation cannot be solved by the simple analytic methods set out in simple textbooks.

Section 47.3 "Physics Case Studies" has eleven items. Many engineering problems are based upon fundamental physics and require mathematical modelling for their solution. This Section contains a compendium of case studies involving physics (or related topics) as an additional teaching and learning resource beyond those included in the previous HELM Workbooks.

Each case study will involve several mathematical topics; the relevant HELM Workbooks are stated at the beginning of each case study.

Table 1: Physics topics, related Mathematical topics and Workbooks

	Physics Case Study Title	Mathematical Topics	Related Workbooks
1	Black body radiation 1	Logarithms and Exponentials; Numerical integration	6, 31
2	Black body radiation 2	Logarithms and Exponentials; Numerical solution of equations	6, 31
3	Black body radiation 3	Logarithms and Exponentials; Differentiation	6, 11
4	Black body radiation 4	Logarithms and Exponentials; Integration; Numerical integration	6, 13, 31
5	Amplitude of a monochromatic optical wave passing through a glass plate	Trigonometric functions; Complex numbers; Sum of geometric series	4, 10, 16
6	Intensity of the interference field due to a glass plate	Trigonometric functions; Complex numbers	4, 10
7	Propagation time difference between two light rays	Trigonometric functions	4
8	Fraunhofer diffraction through an infinitely long slit	Trigonometric functions; Complex numbers; Maxima and minima	4, 10, 12
9	Fraunhofer diffraction through an array of parallel infinitely long slits	Trigonometric functions; Exponential functions; Complex numbers; Maxima and minima; Geometric series; Fourier transform	4, 6, 10, 12, 16, 24
10	Interference fringes due to two parallel infinitely long slits	Trigonometric functions; Complex numbers; Maxima and minima; Maclaurin series	4, 10, 12, 16
11	Acceleration in polar coordinates	Vectors; Polar coordinates	9, 17

Time allocation

To work through the whole Workbook will require at least twenty hours of independent study. However it would be more normal (and preferable) to use it to 'dip-in-and-out', and to follow up additional examples of modelling using particular techniques for an hour or two at a time.

Format

The eleven Physics Case Studies in Workbook 47 have a common format (with rare minor variations), This is the same as that used for the Engineering Examples in Workbooks 1 to 34. The section headings are:

Introduction
This consists of a paragraph or two of background information, setting the context and stating essential engineering information, definitions and fundamental concepts.

Problem in words
This consists of a statement of the problem in words including the purpose of the model.

Mathematical statement of problem
Here the problem is expressed in mathematical form including notation, assumptions and strategy.

Mathematical analysis
This part gives the solution to the problem or explains how it can be solved using techniques in the indicated Workbooks.

Interpretation
This interprets the mathematical result in engineering terms and includes at least a statement of the result in words. If appropriate, there are comments about whether results make sense, mathematical points, indications of further extensions or applications and implications.

Workbook 48: Engineering Case Studies

This Workbook offers a compendium of twenty 'Engineering Case Studies' providing additional teaching and learning resources to the sixty-seven 'Engineering Examples' included in Workbooks 1 to 34.

Table 2 below summarises the engineering contexts, the mathematical topics and the relevant HELM mathematics Workbooks for these Case Studies. It should be possible to use this Workbook to reinforce notions of modelling using a wide cross section of mathematical techniques. The more elementary mathematical topics and relevant Workbooks are not usually mentioned - for example basic functions (Workbook 2) pervades every case Study and so is omitted. However, where there is significant algebraic manipulation (Workbook 1) or equation rearrangement (Workbook 3) this has been reflected in the table.

The Case Studies have been grouped together by broad engineering topic and not by mathematical topic or difficulty or length. (The shortest and most straightforward is 11.)

Table 2: Engineering topics, related Mathematical topics and Workbooks

	Engineering Topic	Mathematical Topics	Related Workbooks
1	Adding sound waves	Trigonometry; Second order ODEs	4, 19
2	Complex representations of sound waves and sound reflection	Complex numbers; Trigonometry	4, 10
3	Sensitivity of microphones	Definite integrals (function of a function)	13, 14
4	Refraction	Algebra; Equations; Trigonometry; Differentiation	1, 3, 4, 11
5	Beam deformation	Algebra; Equations; Inequalities; Trigonometry; Definite integrals; First order ODEs	1, 3, 4, 13, 19
6	Deflection of a beam	Equations; Definite integrals; ODEs; Laplace transforms	3, 13, 19, 20
7	Buckling of columns	Trigonometric identities; Determinants; Matrices; Integration; Fourth order ODEs	4, 7, 8, 13, 19
8	Maximum bending moment for a multiple structure	Algebra; Equations; Inequalities; Maxima and minima (differentiation)	1, 3, 12
9	Equation of the curve of a cable fixed at two endpoints	Trigonometric identities; Hyperbolic functions; Differentiation; Integration; First order ODEs	4, 6, 11, 13, 19
10	Critical water height in an open channel	Algebra; Equations; Derivative of polynomials; Maxima and minima (differentiation); Integration	1, 3, 11, 12, 13
11	Simple pendulum	Algebra - rearranging formulae	1
12	Motion of a pendulum	Trigonometric functions; Differentiation; Maclaurin's expansion; Contour plotting for a function of two variables; Second order ODEs	4, 11, 16, 18, 19
13	The falling snowflake	Integration; First order ODEs; Second order ODEs; Vector differentiation	13, 19, 28
14	Satellite motion	Trigonometry; Equations of conics in polar coordinates; Second order ODEs; Vector differential calculus; Polar coordinate system	4, 17, 19, 28
15	Satellite orbits	Trigonometry; Equations of conics in polar coordinates; Second order ODEs; Vector differential calculus; Polar coordinate system	4, 17, 19, 28

16	Underground railway signals location	Trigonometry; Differentiation of inverse trigonometrical functions; Geometry (arc length); Partial differentiation	4, 11, 17, 18
17	Heat conduction through a wine cellar roof	Trigonometry; First order ODEs; Second order ODEs; Fourier series; Partial differential equations	4, 19, 23, 25
18	Two-dimensional flow past a cylindrical obstacle	Algebra - rearranging formulae; Trigonometry; Polar coordinates	1, 4, 17
19	Two-dimensional flow of a viscous liquid on an inclined plate	Trigonometry; Integration; Polar coordinates; Partial differentiation; Partial differential equations; Vector differential calculus; Dimensional analysis	4, 13, 17, 18, 19, 28, 47
20	Force on a cylinder due to a two-dimensional streaming and swirling flow	Algebra; Equations; Trigonometry; Logarithmic functions; Integration; Orthogonality relations of trigonometric functions; Partial differentiation; Polar coordinates; Surface integrals	1, 3, 4, 6, 13, 17, 18, 28, 29

Time allocation

To work through the whole Workbook will require at least fifty hours of independent study. However it would be more normal (and preferable) to use it to 'dip-in-and-out', and to follow up additional examples of modelling using particular techniques for an hour or two at a time.

Format

The twenty Engineering Case Studies in Workbook 48 have a common format (with rare minor variations). This is the same as that used for the Engineering Examples in Workbooks 1 to 34. The section headings are:

Introduction
This consists of a paragraph or two of background information, setting the context and stating essential engineering information, definitions and fundamental concepts.

Problem in words
This consists of a statement of the problem in words including the purpose of the model.

Mathematical statement of problem
Here the problem is expressed in mathematical form including notation, assumptions and strategy. Mathematical analysis This part gives the solution to the problem or explains how it can be solved using techniques in the indicated Workbooks.

Interpretation
This interprets the mathematical result in engineering terms and includes at least a statement of the result in words. If appropriate, there are comments about whether results make sense, mathematical points, indications of further extensions or applications and implications.

Where possible there are comments on the sensitivity of the analysis to input data and the necessary numerical accuracy of the outputs.

Index of Engineering Contexts in Workbooks 1 to 48

50.9

Engineering Topic	Workbook	Page Number
Acceleration in polar coordinates	Wbk 47	67
Admittance of an electronic circuit	Wbk 3	69
Aerofoil	Wbk 26	14, 19
Aircraft	Wbk 9	13
Aircraft wings	Wbk 42	4
Aircraft wings	Wbk 45	26
Airline booking	Wbk 39	29
Alloy impurities	Wbk 41	18
Alloy spacers	Wbk 44	4, 16
Alloy stretching	Wbk 45	27
Alloy-twisting resistance	Wbk 45	26
Aluminium alloy tensile strength	Wbk 44	45
Aluminium sheet faults	Wbk 42	4
Amplifier	Wbk 10	26
Amplitude	Wbk 3	67, 74
Amplitude modulation	Wbk 4	47
Amusement rides	Wbk 34	6, 43-50
An LC circuit with sinusoidal input	Wbk 19	48
An RC circuit with single frequency input	Wbk 19	26
Angular velocity of Earth	Wbk 34	40
Anti-lock brakes	Wbk 45	11
Arrhenius' law	Wbk 6	32
Assembly machines	Wbk 44	33
Asteroid	Wbk 17	22
Atomic theory	Wbk 47	13, 14
Ball bearing diameters	Wbk 40	19
Banked tracks	Wbk 34	49
Basketball	Wbk 34	26
Battery lifetime	Wbk 41	15
Beam	Wbk 19	65, 67
Beam	Wbk 20	52
Beam deflection	Wbk 48	20
Beam deformation	Wbk 48	15
Beats	Wbk 19	64

NOTES

Workbook 50

HELM®

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk