The Exponential Form of a Complex Number 10.3

Introduction

In this Section we introduce a third way of expressing a complex number: the exponential form. We shall discover, through the use of the complex number notation, the intimate connection between the exponential function and the trigonometric functions. We shall also see, using the exponential form, that certain calculations, particularly multiplication and division of complex numbers, are even easier than when expressed in polar form.
The exponential form of a complex number is in widespread use in engineering and science.

Prerequisites

Before starting this Section you should . . .

① be able to convert from degrees to radians

② understand how to use the Cartesian and polar forms of a complex number

③ be familiar with the hyperbolic functions $\cosh x$ and $\sinh x$

Learning Outcomes

After completing this Section you should be able to . . .

✓ understand the relations between the exponential function e^x and the trigonometric functions $\cos x$, $\sin x$

✓ interchange between Cartesian, polar and exponential forms of a complex number

✓ understand the relation between hyperbolic and trigonometric functions

1. Series Expansions for Exponential and Trigonometric Functions

We have, so far, considered two ways of representing a complex number:

$$z = a + \mathrm{i}b \qquad \text{Cartesian form}$$

or

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad \text{Polar form}$$

In this Section we introduce a third way of denoting a complex number: the **exponential form**. If x is a real number then, as we shall verify in workbook 16, the exponential number e raised to the power x can be written as a series of powers of x:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

in which $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ is the factorial of the integer n. Although there are an infinite number of terms on the right-hand side, in any practical calculation we would only use a *finite* number. For example if we choose $x = 1$ (and taking only six terms) then

$$\begin{aligned} \mathrm{e}^1 &\approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \\ &= 2 + 0.5 + 0.16666 + 0.04166 + 0.00833 \\ &= 2.71666 \end{aligned}$$

which is 'close' to the accurate value of $\mathrm{e} = 2.71828$ (to 5d.p.)

We ask you to accept that e^x, for any value of x, is the *same as* $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and that if we wish to calculate e^x for a *particular value* of x we will only take a finite number of terms in the series. Obviously the more terms we take in any particular calculation the more accurate will be our calculation.

As we shall also see in Workbook 16, similar series expansions exist for the trigonometric functions $\sin x$ and $\cos x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

in which x is measured in radians.

The observant reader will see that these two series for $\sin x$ and $\cos x$ are similar to the series for e^x. Through the use of the symbol i $(=\sqrt{-1})$ we will examine this close correspondence.

In the series for e^x replace x on both left-hand and right-hand sides by $\mathrm{i}\theta$ to give:

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + (\mathrm{i}\theta) + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \cdots$$

Then, as usual, replace every occurrence of i^2 by (-1) to give

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{i}\frac{\theta^5}{5!} + \cdots$$

which, when re-organised into real and imaginary terms gives, finally:

$$\begin{aligned} e^{i\theta} &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Key Point

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Example Find complex number expressions, in Cartesian form, for (i) $e^{i\pi/4}$ (ii) e^{-i} (iii) $e^{i\pi}$

Solution

(i) according to our Key Point $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(ii) $e^{-i} = \cos(-1) + i\sin(-1) = 0.540 - i(0.841)$ don't forget: use *radians*

(iii) $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

Use four terms in the series representation for $\cos x$ to obtain an approximation to $\cos 45°$.

Your solution

You should obtain $\cos 45° \approx 0.707103$ since $\cos 45° = \cos(\pi/4)$ (using radians) and so:

$$\begin{aligned} \cos\frac{\pi}{4} &\approx 1 - \frac{(\pi/4)^2}{2!} + \frac{(\pi/4)^4}{4!} - \frac{(\pi/4)^6}{6!} \\ &= 1 - \frac{0.61685}{2} + \frac{0.38050}{24} - \frac{0.23471}{720} \\ &= 0.707103 \quad \text{(the accurate value to 6d.p. is 0.707107)} \end{aligned}$$

2. The Exponential Form

Since $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and since $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ we therefore obtain another way in which to denote a complex number: $z = r\mathrm{e}^{\mathrm{i}\theta}$, called the **exponential form**.

Key Point

The exponential form of a complex number is

$$z = r\mathrm{e}^{\mathrm{i}\theta} \qquad \text{in which} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

Example If $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $w = t\mathrm{e}^{\mathrm{i}\phi}$ then find expressions for (i) z^{-1} (ii) z^* (iii) zw

Solution

(i) If $z = r\mathrm{e}^{\mathrm{i}\theta}$ then $z^{-1} = \dfrac{1}{r\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$ using the normal rules for indices.
(ii) Working in polar form: if $z = r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$ then

$$z^* = r(\cos\theta - \mathrm{i}\sin\theta) = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\mathrm{e}^{-\mathrm{i}\theta}$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In fact this reflects the general rule: to find the complex conjugate of any expression simply replace i by $-$i wherever it occurs in the expression.
(iii) $zw = (r\mathrm{e}^{\mathrm{i}\theta})(t\mathrm{e}^{\mathrm{i}\phi}) = rt\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}(\theta+\phi)}$ which is again the result we are familiar with: when complex numbers are multiplied their moduli multiply and their arguments add.

We see that in some circumstances the exponential form is even more convenient than the polar form since we need not worry about cumbersome trigonometric relations.

Express the following complex numbers in exponential form: (i) $z = 1 - \mathrm{i}$
(ii) $z = 2 + 3\mathrm{i}$ (iii) $z = -6$.

Your solution

(i)

$z = \sqrt{2}\mathrm{e}^{\mathrm{i}7\pi/4}$ (or, equivalently, $\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$)

Your solution

(ii)

$z = \sqrt{13}e^{i(0.9828)}$.

Your solution

(iii)

$z = 6e^{i\pi}$.

3. Hyperbolic and Trigonometric Functions

We have seen in Section 1 that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It follows from this that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Now if we add these two relations together we obtain

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

whereas if we subtract the second from the first we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

These new relations are reminiscent of the hyperbolic functions introduced in Section 4.2. There we defined $\cosh x$ and $\sinh x$ in terms of the exponential function:

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

In fact, if we replace x by $i\theta$ in these last two expressions we obtain

$$\cosh(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \equiv \cos\theta \qquad \text{and} \qquad \sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} \equiv i\sin\theta$$

Although, by our notation, we have implied that both x and θ are real quantities in fact these expressions for cosh and sinh in terms of cos and sin are more general.

Key Point

If z is *any* complex number then

$$\cosh(\mathrm{i}z) = \cos z \qquad \text{and} \qquad \sinh(\mathrm{i}z) = \mathrm{i}\sin z$$

or, equivalently, if we replace z by $\mathrm{i}z$

$$\cosh z = \cos(\mathrm{i}z) \qquad \text{and} \qquad \mathrm{i}\sinh z = \sin(\mathrm{i}z)$$

If $\cos^2 z + \sin^2 z = 1$ for all z then, utilising complex numbers, obtain the equivalent identity for hyperbolic functions.

Your solution

You should obtain $\cosh^2 z - \sinh^2 z = 1$ since, if we replace z by iz in the given identity then $\cos^2(\mathrm{i}z) + \sin^2(\mathrm{i}z) = 1$. But as noted above $\cos(\mathrm{i}z) = \cosh z$ and $\sin(\mathrm{i}z) = \mathrm{i}\sinh z$ so the result follows.

Further analysis similar to this leads to **Osborne's rule**:

Key Point

Osborne's rule:

Hyperbolic function identities are obtained from trigonometric identities by replacing $\sin\theta$ by $\sinh\theta$ and $\cos\theta$ by $\cosh\theta$ except that every occurrence of $\sin^2\theta$ is replaced by $-\sinh^2\theta$.

Example Use Osborne's rule to obtain the hyperbolic equivalent of $1+\tan^2\theta = \sec^2\theta$.

Solution

Here $1+\tan^2\theta = \sec^2\theta$ is equivalent to $1 + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$. Hence if

$$\sin^2\theta \to -\sinh^2\theta \qquad \text{and} \qquad \cos^2\theta \to \cosh^2\theta$$

then we obtain

$$1 - \frac{\sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta} \qquad \text{or, equivalently,} \qquad 1 - \tanh^2\theta = \text{sech}^2\theta$$

Exercises

1. Use four terms of the series expansions for e^x and $\sin x$ to obtain approximate values for $e^{1.2}$ and $\sin 57°$. Compare with the accurate values.
2. Two standard relations in trigonometry are $\sin 2z = 2\sin z\cos z$ and $\cos 2z = \cos^2 z - \sin^2 z$. Use Osborne's rule to obtain the corresponding relations for hyperbolic functions.
3. Express the following complex numbers in Cartesian form (i) $3e^{i\pi/3}$ (ii) $e^{-2\pi i}$ (iii) $e^{i\pi/2}e^{i\pi/4}$.
4. Express the following complex numbers in exponential form (i) $z = 2 - i$ (ii) $z = 4 - 3i$ (iii) z^{-1} where $z = 2 - 3i$.
5. Obtain the real and imaginary parts of $\sinh(1 + \dfrac{i\pi}{6})$

Answers

1. $e^{1.2} \approx 3.208$ (accurate: 3.32012 to 5d.p.)

   $\sin 57° \approx 0.83865$ (accurate: 0.83867 to 5d.p.)

2. $\sinh 2z = 2\sinh z\cosh z$, $\cosh 2z = \cosh^2 z + \sinh^2 z$.
3. (i) $1.5 + i(2.598)$ (ii) 1 (iii) $-0.707 + i(0.707)$
4. (i) $\sqrt{5}e^{i(5.820)}$ (ii) $5e^{i(5.6397)}$ (iii) $2 - 3i = \sqrt{13}e^{i(5.300)}$ therefore $\dfrac{1}{2-3i} = \dfrac{1}{\sqrt{13}}e^{-i(5.300)}$
5. $\sinh(1 + \dfrac{i\pi}{6}) = \dfrac{\sqrt{3}}{2}\sinh 1 + \dfrac{i}{2}\cosh 1 = 1.0178 + i(0.7715)$