The Poisson distribution

37.3

Introduction

In this block we introduce a probability model which can be used when the outcome of an experiment is a random variable taking on positive integer values and where the only information available is a measurement of its average value. This has widespread applications in analysing traffic flow, in fault prediction on electric cables and in the prediction of randomly occurring accidents for example. We shall look at the Poisson distribution in two distinct ways. Firstly, as a distribution in its own right. This will enable us to apply statistical methods to a set of problems which cannot be solved using the Binomial distribution. Secondly, as an approximation to the Binomial distribution $X \sim B(n, p)$ in the case where n is large and p is small. You will find that this approximation can often save the need to do much tedious arithmetic.

Prerequisites

Before starting this Section you should ...

① understand the concepts of probability.

② understand the concepts and notation used in Section 37.2, the binomial distribution.

Learning Outcomes

After completing this Section you should be able to ...

✓ recognise and use the formula for probabilities calculated from the Poisson model

✓ use the recurrence relation to generate a succession of probabilities

✓ use the Poisson model to obtain approximate values for binomial probabilities

1. The Poisson Approximation to the Binomial Distribution

The probability of the outcome $X = r$ of a set of Bernoulli trials can always be calculated by using the formula

$$P(X = r) = {}^nC_r q^{n-r} p^r$$

given above. Clearly, for very large values of n the calculation can be rather tedious, this is particularly so when very small values of p are also present.
In the situation when n is large and p is small and the product np is constant we can take a different approach to the problem of calculating the probability that $X = r$. In the table below the values of $P(X = r)$ have been calculated for various combinations of n and p under the constraint that $np = 1$.
You should try some of the calculations for yourself using the formula given above for some of the **smaller** values of n.

		Probability of X successes						
n	p	$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$X = 5$	$X = 6$
4	0.25	0.316	0.422	0.211	0.047	0.004		
5	0.20	0.328	0.410	0.205	0.051	0.006	0.000	
10	0.10	0.349	0.387	0.194	0.058	0.011	0.001	0.000
20	0.05	0.359	0.377	0.189	0.060	0.013	0.002	0.000
100	0.01	0.366	0.370	0.185	0.061	0.014	0.003	0.001
1000	0.001	0.368	0.368	0.184	0.061	0.015	0.003	0.001
10000	0.0001	0.368	0.368	0.184	0.061	0.015	0.003	0.001

Each of the Binomial distributions given has a mean given by $np = 1$. Notice that the probabilities that $X = 0, 1, 2, 3, 4, \ldots$ approach the values $0.368, 0.368, 0.184, \ldots$ as n increases.
If we have to determine the probabilities of success when large values of n and small values of p are involved it would be very convenient if we could do so without having to construct tables. In fact we can do such calculations by using the Poisson distribution which, under certain constraints, may be considered as an approximation to the Binomial distribution.

By considering simplifications applied to the Binomial distribution subject to the conditions

1. n is large 2. p is small 3. $np = \lambda$ (λ a constant)

we can derive the formula

$$P(X = r) = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!} \quad \text{as an approximation to } P(X = r) = {}^nC_r q^{n-r} p^r.$$

This is the Poisson distribution given previously. We now show how this is done.

We know that the Binomial distribution is given by

$$(q + p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}q^{n-r}p^r + \cdots + p^n$$

Condition (2) tells us that since p is small, $q = 1 - p$ is approximately equal to 1. Applying this to the terms of the Binomial expansion above we see that the right hand side becomes

$$1 + np + \frac{n(n-1)}{2!}p^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}p^r + \cdots + p^n$$

Applying condition (1) allows us to approximate terms such as $(n-1),(n-2),\ldots$ to n (mathematically, we are allowing $n \to \infty$) and the right hand side of our expansion becomes

$$1 + np + \frac{n^2}{2!}p^2 + \cdots + \frac{n^r}{r!}p^r + \ldots$$

Note that the term $p^n \to 0$ under these conditions and hence has been omitted.
We now have the series

$$1 + np + \frac{(np)^2}{2!} + \cdots + \frac{(np)^r}{r!} + \ldots$$

which, using condition (3) may be written as

$$1 + \lambda + \frac{(\lambda)^2}{2!} + \cdots + \frac{(\lambda)^r}{r!} + \ldots$$

You may recognise this as the expansion of e^{λ}.
If we are to be able to claim that the terms of this expansion represent probabilities, we must be sure that the sum of the terms is 1. We divide by e^{λ} to satisfy this condition. This gives the result

$$\frac{e^{\lambda}}{e^{\lambda}} = 1 = \frac{1}{e^{\lambda}}(1 + \lambda + \frac{(\lambda)^2}{2!} + \cdots + \frac{(\lambda)^r}{r!} + \ldots)$$

$$= e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + \cdots + e^{-\lambda}\frac{\lambda^r}{r!} + \cdots +$$

The terms of this expansion are very good approximations to the corresponding Binomial expansion under the conditions

1. n is large
2. p is small
3. $np = \lambda$ (λ constant)

The Poisson approximation to the Binomial distribution is summarized below.

Key Point

Assuming that n is large, p is small and that np is constant, the terms

$$P(X = r) = {}^nC_r(1-p)^{n-r}p^r$$

of a Binomial distribution may be closely approximated by the terms

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$$

of the Poisson distribution for corresponding values of r.

Example We introduced the Binomial distribution by considering the following scenario. A worn machine is known to produce 10% defective components. If the random variable X is the number of defective components produced in a run of 3 components, find the probabilities that X takes the values 0 to 3.

Suppose now that a similar machine which is known to produce 1% defective components is used for a production run of 40 components. We wish to calculate the probability that two defective items are produced. Essentially we are assuming that $X \sim B(40, 0.01)$ and are asking for $P(X = 2)$. We use both the Binomial distribution and its Poisson approximation for comparison.

Solution

Using the Binomial distribution we have the solution

$$P(X = 2) = \ {}^{40}C_2(0.99)^{40-2}(0.01)^2 = \frac{40 \times 39}{1 \times 2} 0.99^{38} 0.01^2 = 0.0532$$

Note that the arithmetic involved is unwieldy. Using the Poisson approximation we have the solution $\quad P(X = 2) = \mathrm{e}^{-0.4}\dfrac{0.4^2}{2!} = 0.0536$

Note that the arithmetic involved is simpler and the approximation is reasonable.

Practical Considerations

In practice, we can use the Poisson distribution to very closely approximate the Binomial distribution provided that the product np is constant with

$$n \geq 100 \qquad \text{and} \qquad p \leq 0.05$$

Note that this is not a hard-and-fast rule and we simply say that
'the larger n is the better and the smaller p is the better provided that np is a sensible size.'
The approximation remains good provided that $np < 5$ for values of n as low as 20.

Suppose mass-produced needles are packed in boxes of 1000. It is believed that 1 needle in 2000 **on average** is substandard. What is the probability that a box contains more than 2 defectives?
The correct model is the Binomial distribution with $n = 1000$, $p = \frac{1}{2000}$ (and $q = \frac{1999}{2000}$). Using the Binomial distribution calculate $P(X = 0)$, $P(X = 1)$ and hence calculate $P(\text{more than 2 defectives})$.

Your solution

$$P(X=0) = \left(\frac{1999}{2000}\right)^{1000} = 0.60645$$

$$P(X=1) = 1000\left(\frac{1999}{2000}\right)^{999} \times \left(\frac{1}{2000}\right) = \frac{1}{2}\left(\frac{1999}{2000}\right)^{999} = 0.30338$$

$$\therefore \quad P(X=0) + P(X=1) = 0.60645 + 0.30338 = 0.90983 \approx 0.9098 \quad \text{(4d.p.)}$$

Hence P(more than 2 defectives) $\approx 1 - 0.9098 = 0.0902$.

Now choose a suitable value for λ in order to use a Poisson model to approximate the probabilities.

Your solution

$\lambda =$

$\lambda = np = 1000 \times \frac{1}{2000} = \frac{1}{2}$

Now recalculate the probability that there are more than 2 defectives using the Poisson distribution with $\lambda = \frac{1}{2}$.

Your solution

$P(X=0) =$ $\qquad\qquad$ $P(X=1) =$

$\therefore \quad P$(more than 2 defectives)$=$

$P(X=0) = e^{-\frac{1}{2}}, \quad P(X=1) = \frac{1}{2}e^{-\frac{1}{2}}$

$\therefore \quad P(X=0) + P(X=1) = \frac{3}{2}\, e^{-\frac{1}{2}} = 0.9098$ (4 d.p.)

Hence P(more than 2 defectives) $\simeq 1 - 0.9098 = 0.0902$.

We have obtained the same answer to 4 d.p., as the exact Binomial calculation, essentially because p was so small. We shall not always be so lucky.

Example In the manufacture of glassware, bubbles can occur in the glass which reduces the status of the glassware to that of a 'second'. If, on average, one in every 1000 items produced has a bubble, calculate the probability that exactly six items in a batch of three thousand are seconds.

Solution

Suppose that X = number of items with bubbles, then

$$X \sim B(3000, 0.001)$$

Since $n = 3000 > 100$ and $p = 0.001 < 0.005$ we can use the Poisson distribution with $\lambda = np = 3000 \times 0.001 = 3$. The calculation is:

$$P(X=6) = \mathrm{e}^{-3}\frac{3^6}{6!} \approx 0.0498 \times 1.0125 \approx 0.05$$

The result means that we have about a 5% chance of finding exactly six seconds in a batch of three thousand items of glassware.

Example A manufacturer produces light-bulbs that are packed into boxes of 100. If quality control studies indicate that 0.5% of the light-bulbs produced are defective, what percentage of the boxes will contain:

(a) no defective.

(b) 2 or more defectives?

Solution

As n is large and p, the P(defective bulb), is small, use the Poisson approximation to the Binomial probability distribution.

If X = number of defective bulbs in a box, then

$$X \sim P(\mu) \text{ where } \mu = n \times p = 100 \times 0.005 = 0.5$$

Hence,

(a) $P(X = 0) = \dfrac{e^{-0.5}(0.5)^0}{0!} = \dfrac{e^{-0.5}(1)}{1} = 0.6065 \approx 61\%$

(b) $P(X = 2 \text{ or more}) = P(X = 2) + P(X = 3) + P(X = 4) + \ldots$

But easier to consider,

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$P(X = 1) = \frac{e^{-0.5}(0.5)^1}{1!} = \frac{e^{-0.5}(0.5)}{1} = 0.3033$$

i.e. $P(X \geq 2) = 1 - [0.6065 + 0.3033] = 0.0902 \approx 9\%$

2. The Poisson distribution

The Poisson distribution is a probability model which can be used to find the probability of a single event occurring a given number of times in an interval of (usually) time. The occurrence of these events must be determined by chance alone which implies that information about the occurrence of any one event cannot be used to predict the occurrence of any other event. It is worth noting that only the *occurrence* of an event can be counted; the *non-occurrence* of an event cannot be counted. This contrasts with Bernoulli trials where we know the number of trials, the number of events occurring and therefore the number of events not occurring.

The Poisson distribution has widespread applications in areas such as analysing traffic flow, fault prediction in electric cables, defects occurring in manufactured objects such as castings, email messages arriving at you computer and in the prediction of randomly occurring events or accidents. One well known series of accidental events concerns Prussian cavalry who were killed by horse kicks. Although not discussed here (death by horse kick is hardly an engineering application of statistics!) you will find accounts in many statistical texts. One example of the use of a Poisson distribution where the events are not necessarily time related is in the prediction of fault occurrence along a long weld - faults may occur anywhere along the length of the weld. A similar argument applies when scanning castings for faults - we are looking for faults occurring in a volume of material, not over an interval if time.

The following definition gives a theoretical underpinning to the Poisson distribution.

Definition

Suppose that events occur at random throughout an interval. Suppose further that the interval can be divided into subintervals which are so small that:

1. the probability of more than one event occurring in the subinterval is zero
2. the probability of one event occurring in a subinterval is proportional to the length of the subinterval
3. an event occurring in any given subinterval is independent of any other subinterval.

then the random experiment is known as a Poisson process. The word 'process' is used to suggest that the experiment takes place over time which is the usual case. If the average number of events occurring in the interval (not subinterval) is $\lambda(> 0)$ then the random variable X representing the actual number of events occurring in the interval is said to have a Poisson distribution and it can be shown (we omit the derivation) that

$$P(X = r) = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!} \qquad r = 0, 1, 2, 3, \ldots$$

The following key point may be taken as a summary.

Key Point

The Poisson Probabilities

If X is the random variable

'number of occurrences in a given interval'

for which the average rate of occurrence is λ then, according to the **Poisson** model, the probability of r occurrences in that interval is given by

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!} \qquad r = 0, 1, 2, 3, \ldots$$

Using the Poisson distribution $P(X = r) = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!}$ write down the formulae for $P(X = 0)$, $P(X = 1)$, $P(X = 2)$ and $P(X = 6)$, noting that $0! = 1$.

Your solution

$P(X = 0) =$ $\qquad$ $P(X = 1)$

$P(X = 2) =$ $\qquad$ $P(X = 6)$

$$P(X=0) \;=\; e^{-\lambda} \times \frac{\lambda^0}{0!} = e^{-\lambda} \times \frac{1}{1} \equiv e^{-\lambda} \qquad P(X=1) = e^{-\lambda} \times \frac{\lambda}{1!} = \lambda e^{-\lambda}$$

$$P(X=2) \;=\; e^{-\lambda} \times \frac{\lambda^2}{2!} = \frac{\lambda^2}{2} e^{-\lambda} \qquad P(X=6) = e^{-\lambda} \times \frac{\lambda^6}{6!} = \frac{\lambda^6}{720} e^{-\lambda}$$

Calculate $P(X = 0)$ to $P(X = 5)$ when $\lambda = 2$ and present the results to 4d.p. in a table.

Your solution

r	0	1	2	3	4	5
$P(X = r)$	0.1353	0.2707	0.2707	0.1804	0.0902	0.0361

Notice how the values for $P(X = r)$ increase and then decrease relatively rapidly (due to the significant increase in $r!$ with increasing r). In this example two of the probabilities are equal and this will always be the case when λ is integral.

In the exercise we only went up to $P(X = 5)$. Each probability need not be calculated directly. We can use the following relations (which can be checked from the formulae for $P(X = r)$) to get the next probability from the previous one:

$$P(X = 1) = \frac{\lambda}{1}\,P(X = 0)\;, \quad P(X = 2) = \frac{\lambda}{2}\,P(X = 1) \qquad P(X = 3) = \frac{\lambda}{3}\,P(X = 2)\;, \text{ etc.}$$

Key Point

In general, for ease of calculation we use the **recurrence relation**

$$P(X = r) = \frac{\lambda}{r}\, P(X = r - 1) \qquad \text{for } \; r \geq 1.$$

Example Calculate the value for $P(X = 6)$ in the Table above using the recurrence relation and the value for $P(X = 5)$.

Solution

The recurrence relation gives the formula

$$\begin{aligned} P(X = 6) &= \frac{2}{6} P(X = 5) \\ &= \frac{1}{3} \times 0.0361 \\ &= 0.0120 \end{aligned}$$

We now look further at the Poisson distribution by considering an example based on traffic flow.

Example Suppose that it has been observed that, on average, 180 cars per hour pass a specified point on a particular road in the morning rush hour. Due to impending roadworks it is estimated that congestion will occur closer to the city centre if more than 5 cars pass the point in any one minute. What is the probability of congestion occurring?

Firstly, note that we cannot use the Binomial model since we have no values of n and p. Essentially we are saying that there is no fixed number (n) of cars passing the specified point and that we have no way of estimating p. The only information available is the average rate at which cars pass the specified point.

Solution

Let X be the random variable $X =$ number of cars arriving in any minute. We need to calculate the probability that more than 5 cars arrive in any one minute. Note that in order to do this we need to convert the information given on the average rate (cars arriving per hour) into a value for λ (cars arriving per minute). This gives the value $\lambda = 3$. Using $\lambda = 3$ to calculate the required probabilities gives:

r	0	1	2	3	4	5	Sum
$P(X = r)$	0.04979	0.149361	0.22404	0.22404	0.168031	0.10082	0.91608

To calculate the required probability we note that

$$P(\text{more than 5 cars arrive in one minute}) = 1 - P(\text{5 cars or less arrive in one minute})$$

Thus

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) - P(X = 4) - P(X = 5) \end{aligned}$$

Then $P(\text{more than } 5) = 1 - 0.91608 = 0.08392 = 0.0839$ (4 d.p).

Example The mean number of bacteria per millilitre of a liquid is known to be 6. Find the probability that in 1 ml of the liquid, there will be:

(a) 0, (b) 1, (c) 2, (d) 3, (e) less than 4, (f) 6 bacteria.

Solution

Here we have an *average rate of occurrences* but no estimate of the probability so it looks as though we have a Poisson distribution with $\lambda = 6$. Using the formula above we have:

(a) $P(X = 0) = e^{-6}\frac{6^0}{0!} = 0.00248$. That is the probability of having no bacteria in 1 ml of liquid is 0.00248

(b) $P(X = 1) = \frac{\lambda}{1} \times P(X = 0) = 6 \times 0.00248 = 0.0149$. That is the probability of having 1 bacteria in 1 ml of liquid is 0.0149

(c) $P(X = 2) = \frac{\lambda}{2} \times P(X = 1) = \frac{6}{2} \times 0.01487 = 0.0446$. That is the probability of having 2 bacteria in 1 ml of liquid is 0.0446

(d) $P(X = 3) = \frac{\lambda}{3} \times P(X = 2) = \frac{6}{3} \times 0.04462 = 0.0892$. That is the probability of having 3 bacteria in 1 ml of liquid is 0.0892

(e) $P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 0.1512$

(f) $P(X = 6) = e^{-6}\dfrac{6^6}{6!} = 0.1606$

Note that in working out the first 6 answers, which link together, all the figures were kept in the calculator to ensure accuracy. Answers were rounded off when written down. Never copy down answers correct to, say, 4 decimal places and then use those rounded figures to calculate the next figure as rounding-off errors will become greater at each stage. In the example above you would get answers 0.0025, 0.0150, 0.0450, 0.9000 and $P(X < 4) = 0.1525$. The difference is not great but could be very significant.

A Council is considering whether to base a recovery vehicle on a stretch of road to help clear incidents as quickly as possible. The road concerned carries over 5000 vehicles during the peak rush hour period. Records show that, on average, the number of incidents during the morning rush hour is 5. The Council won't base a vehicle on the road if the probability of having more than 5 incidents in any one morning is less than 30%. Based on this information should the Council provide a vehicle?

Your solution

We need to calculate the probability that more than 5 incidents occur i.e. $P(X > 5)$. To find this we use the fact that $P(X > 5) = 1 - P(X \leq 5)$. Now, for this example:

$$P(X = r) = e^{-5}\frac{5^r}{r!}$$

Writing answers to 5 d.p. gives:

$$\begin{aligned}
P(X = 0) &= e^{-5}\frac{5^0}{0!} = 0.00674 \\
P(X = 1) &= 5 \times P(X = 0) = 0.03369 \\
P(X = 2) &= \frac{5}{2} \times P(X = 1) = 0.08422 \\
P(X = 3) &= \frac{5}{3} \times P(X = 2) = 0.14037 \\
P(X = 4) &= \frac{5}{4} \times P(X = 3) = 0.17547 \\
P(X = 5) &= \frac{5}{5} \times P(X = 4) = 0.17547
\end{aligned}$$

$$\begin{aligned}
P(X \leq 5) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\
&= 0.61596
\end{aligned}$$

The probability of more than 5 incidents is $P(X > 5) = 1 - P(X \leq 5) = 0.38403$. That is, the probability of having more than 5 incidents is 38.4% (to 3 s.f.) so the Council should provide a vehicle.

3. Expectation and Variance of the Poisson Distribution

The expectation and variance of the Poisson distribution can be derived directly from the definitions which apply to any discrete probability distribution. However, the algebra involved is a little lengthy. Instead we derive them from the Binomial distribution from which the Poisson distribution is derived.

Intuitive Explanation

One way of deriving the mean and variance of the Poisson distribution is to consider the behaviour of the Binomial distribution under the following conditions:

1. n is large 2. p is small 3. $np = \lambda$ (a constant)

Recalling that the expectation and variance of the Binomial distribution are given by the results

$$E(X) = np \qquad \text{and} \qquad V(X) = np(1-p) = npq$$

it is reasonable to assert that condition (2) implies, since $q = 1 - p$, that q is approximately 1 and so the expectation and variance are given by

$$E(X) = np \qquad \text{and} \qquad V(X) = npq \approx np$$

In fact the algebraic derivation of the expectation and variance of the Poisson distribution shows that these results are in fact *exact*.

Note that the expectation and the variance are equal.

Key Point

The Poisson Distribution

If X is the random variable {number of occurrences in a given interval}

for which the average rate of occurrences is λ and X can assume the values $0, 1, 2, 3, \ldots$ and the probability of r occurrences in that interval is given by

$$P(X = r) = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!}$$

then the expectation and variance of the distribution are given by the formulae

$$E(X) = \lambda \qquad \text{and} \qquad V(X) = \lambda$$

For a Poisson distribution the Expectation and Variance are equal.

Exercises

1. Large sheets of metal have faults in random positions but on average have 1 fault per 10m^2.

   What is the probability that a sheet $5\text{m}\times 8\text{m}$ will have at most one fault?

2. If 250 litres of water are known to be polluted with 10^6 bacteria what is the probability that a sample of 1cc of the water contains no bacteria?

3. Suppose vehicles arrive at a signalised road intersection at an average rate of 360 per hour and the cycle of the traffic light is set at 40 seconds. In what percentage of cycles will the number of vehicles arriving be (a) exactly 5, (b) less than 5? If, after the lights change to green, there is time to clear only 5 vehicles before the signal changes to red again, what is the probability that waiting vehicles are not cleared in one cycle?

4. Previous results indicate that 1 in 1000 transistors are defective on average.

   (a) Find the probability that there are 4 defective transistors in a batch of 2000.

   (b) What is the largest number, N, of transistors that can be put in a box so that the probability of no defectives is at least 1/2?

5. A manufacturer sells a certain article in batches of 5000. By agreement with a customer the following method of inspection is adopted: A sample of 100 items is drawn at random from each batch and inspected. If the sample contains 4 or fewer defective items, then the batch is accepted by the customer. If more than 4 defectives are found, every item in the batch is inspected. If inspection costs are 75p per hundred articles, and the manufacturer normally produces 2% of defective articles, find the average inspection costs per batch.

6. A book containing 150 pages has 100 misprints. Find the probability that a particular page contains (a) no misprints, (b) 5 misprints, (c) at least 2 misprints, (d) more than 1 misprint.

7. For a particular machine, the probability that it will break down within a week is 0.009. The manufacturer has installed 800 machines over a wide area. Calculate the probability that (a) 5, (b) 9, (c) less than 5, (d) more than 4 machines breakdown in a week.

8. At a given university, the probability that a member of staff is absent on any one day is 0.001. If there are 800 members of staff, calculate the probabilities that the number absent on any one day is (a) 6, (b) 4, (c) 2, (d) 0, (e) less than 3, (f) more than 1.

9. The number of failures occurring in a machine of a certain type in a year has a Poisson distribution with mean 0.4. In a factory there are ten of these machines. What is

   (a) the expected total number of failures in the factory in a year?

   (b) the probability that there are fewer than two failures in the factory in a year?

Exercises Continued

10. A factory uses tools of a particular type. From time to time failures in these tools occur and they need to be replaced. The number of such failures in a day has a Poisson distribution with mean 1.25. At the beginning of a particular day there are five replacement tools in stock. A new delivery of replacements will arrive after four days. If all five spares are used before the new delivery arrives then further replacements cannot be made until the delivery arrives. Find

(a) the probability that three replacements are required over the next four days.

(b) the expected number of replacements actually made over the next four days.

Answers

1.

Poisson Process. In a sheet size $40m^2$ **we expect** 4 faults

$$\therefore \qquad \lambda = 4 \qquad P(X = r) = \lambda^r e^{-\lambda}/r!$$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-4} + 4e^{-4} = 0.0916$$

2. In 1cc **we expect** 4 bacteria$(= 10^6/250000)$ $\quad \therefore \qquad \lambda = 4$

$$P(X = 0) = e^{-4} = 0.0183$$

3. In 40 seconds **we expect** 4 vehicles $\quad \therefore \qquad \lambda = 4$

(a) P (exactly 5) $= \lambda^5 e^{-\lambda}/5! = 0.15629$ i.e. in 15.6% of cycles

(b) P (less than 5) $= e^{-\lambda}\left[1 + \lambda^4 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right]$

$\qquad = e^{-4}\left[1 + 4 + 8 + \frac{32}{3} + \frac{32}{3}\right] = 0.6288$

Vehicles will not be cleared if more than 5 are waiting.

P (greater than 5) $= 1 - P$ (exactly 5) $-P$ (less than 5)

$$= 1 - 0.15629 - 0.6288 = 0.2148$$

4(a). Poisson approximation to Binomial

$$\lambda = np = 2000.\frac{1}{1000} = 2$$

$$P(X = 4) = \lambda^4 e^{-\lambda}/4! = 16e^{-2}/24 = 0.09022$$

(b) $\lambda = Np = N/1000; \quad P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = e^{-N/1000}$

$$e^{-N/1000} = 0.5 \qquad \therefore \quad \frac{-N}{1000} = \ln(0.5)$$

$$\therefore \qquad N = 693.147 \qquad \text{choose } N = 693 \text{ or less.}$$

5. P (defective) $= 0.02$. Poisson approx. to Binomial $\lambda = np = 100(0.02) = 2$

P (4 or fewer defectives in sample of 100)

$$= \quad P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \quad e^{-2} + 2e^{-2} + \frac{2^2}{2}e^{-2} + \frac{2^3}{3!}e^{-2} + \frac{2^4}{4!}e^{-2} = 0.947347$$

Inspection costs

Cost c	75	75×50
$P(X = c)$	0.947347	0.0526

$$E(\text{Cost}) = 75(0.947347) + 75 \times 50(0.0526) = 268.5p$$

Answers 6. (a) 0.51342m (b) 0.00056, (c) 0.14430, (d) 0.14430

7. (a) 0.12038, (b) 0.10698, (c) 0.15552, (d) 0.84448

8. (a) 0.00016, (b) 0.00767, (c) 0.14379, (d) 0.44933, (e) 0.95258, (f) 0.19121

9. Let X be the total number of failures.

(a) $\mathrm{E}(X) = 10 \times 0.4 = 4$.

(b)

$$\begin{aligned} \Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1) \\ &= e^{-4} + 4e^{-4} \\ &= 5e^{-4} = 0.0916. \end{aligned}$$

10. Let the number required over 4 days be X. Then $\mathrm{E}(X) = 4 \times 1.25 = 5$ and $X \sim \text{Poisson}(5)$.

(a)

$$\Pr(X = 3) = \frac{e^{-5}5^3}{3!} = 0.1404.$$

(b) Let R be the number of replacements made.

$$\mathrm{E}(R) = 0 \times \Pr(X = 0) + \cdots + 4 \times \Pr(X = 4) + 5 \times \Pr(X \geq 5).$$

$$\Pr(X \geq 5) = 1 - [\Pr(X = 0) + \cdots + \Pr(X = 4)]$$

So

$$\begin{aligned} \mathrm{E}(R) &= 5 - 5 \times \Pr(X = 0) - \cdots - 1 \times \Pr(X = 4) \\ &= 5 - e^{-5}\left[5 \times \frac{5^0}{0!} + 4 \times \frac{5^1}{1!} + \cdots + 1 \times \frac{5^4}{4!}\right] \\ &= 5 - 0.8773 \\ &= 4.123. \end{aligned}$$